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Semidynamical Systems with Non Unique Global Backward Extensions

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§ 1. Introduction.

The theory of semidynamical system has been developed by Bhatia and Hajek [3]. Numerous authors contributed to the theory and we mention here the work in [2], [6] and [10] which is closely related to this paper. One of the principal motivations behind the study of semidynamical systems is that they describe some classes of functional differential equations, partial differential equations of evolution, Volterra integral equations and control systems [10].

Roughly speaking, a semidynamical system is a dynamical system which is defined uniquely and has a global existence in one direction of time (either positive or negative) and lacks uniqueness or existence in the other direction. We will restrict ourselves here to those systems that possess global existence in both directions of time but lack uniqueness in atmost one direction of time. Such systems will be called *semidynamical systems with nonunique global backward extensions.* In [10] Hajek showed that such systems describe normal control systems (see Example 3.6).

Formally, a (positive) semidynamical system is a pair (X, π), where X is a topological space and $\pi: X\times R^+\to X$ is a continuous map such that (1) $a\pi 0=a$ for $a\in X$ and (2) $(a\pi t)\pi s=a\pi(t+s)$ for each $a\in X$ and $t, s\in R^+$ (R^+ denotes the set of nonnegative reals). Let I be any interval of the form $(-\infty, 0]$, $(-a, 0]$ or $[-a, 0]$. Then following [3] a function $\sigma: I\to X$ is called a negative solution of (X, π) if $\sigma(-t+s)=\sigma(-t)\pi s$ whenever $s\in R^+$, $-t\in I$ and $-t+s\in I$. If domain $\sigma=(-\infty, 0]$, then σ is called a principal negative solution. A subset of X is called a negative trajectory through a point $a\in X$ if it is the range of a left maximal solution σ with $\sigma(0)=a$; a principal negative trajectory if σ is a principal negative solution [3]. For a classification of various types of negative trajectories, the reader may consult [3]. The following sets are of special interest.

(1) The funnel through $a\in X$ [3];

$F(a)=\{b\mid b\pi t=a \text{ for some } t\in R^+\}$,

(2) The attainability set of $a\in X$ at $-t_0\in R^-$ or a cut of the funnel through a [3];

$F(a, -t_0)=\{b \mid b\pi t_0=a\}$,

(3) The attainability set of $A \subset X$ at $S \subset R^-$ or a section of the funnel through A [3];

$F(A, S)=\{b \mid b\pi s=a,\ a \in A,\ -s \in S\}$.

A point $a \in X$ is said to be a start point [3] if $F(a)=\phi$.

Our techniques are based on the theory of inverse limits which will be further developed here. To each semidynamical system (X, π) we associate a dynamical system (X_∞, π_∞) which is the inverse limit of an inverse spectrum induced by (X, π) and such that the points in X_∞ corresponds to principal negative trajectories in X. Hence instead of dealing with negative trajectories in X we will be dealing with more simpler objects; the points in X_∞. It was Brown [5] who first studied the inverse limit of a family of discrete semidynamical systems. He defined an inverse limit as the least upper bound of an inverse system of discrete semidynamical systems, a deviation from the classical definition, in order to avoid the problem of existence of an inverse limit. In this paper we will not follow Brown's definition and we will adopt the classical definition of an inverse limit [14]. This is because the classical or the explicit definition is what we need to study dynamical properties and notions in semidynamical systems. Thus care should be taken when comparing our results with Brown's.

In section 2 the inverse limit (X_∞, π_∞) of a semidynamical system (X, π) is constructed. It is shown that (X, π) has a global backward extension iff the maps $\pi^t: X \to X$, $t \in R^+$ are surjective iff (X^*, π^*) is a semidynamical system, where $X^*=X \cup \{\infty\}$ is the one point compactification of X and π^* is an extension of π to X^* (Theorem 2.6). Then McCann's Theorem [11] on extensions of semidynamical systems is recast and slightly extended to say that (X^*, π^*) is a semidynamical system iff the attainability sets $F(A, P)$ are compact whenever A and P are compact in their respective spaces (Theorem 2.5). In section 3, attainability functions $F: X \times R^- \to X$ are considered as setvalued maps which take $(a, -t)$ to the attainability set $F(a, -t)$. Using Michael's result on hyperspaces [12], it is shown that F is continuous in t, upper semicontinuous in x and upper semicontinuous in $(x, -t)$ (Theorem 3.2). Then we show that the attainability set $F(A, [-t, 0])$ is compact and connected whenever A is compact and connected (Theorem 3.3). An example is given to show that this result is false if $[-t, 0]$ is replaced by a compact connected subset of R^- which does not contain 0 (Example 3.5).

In Section 4 we list some of the dynamical properties that if possessed by either (X, π) or (X_∞, π_∞), then it is possessed by the other. Among those listed properties are (1) Liapunov stability, (2) equicontinuity, (3) distality (4) almost periodicity, (5) recurrence, (6) weakly equicontinuity, (7) strongly characteristic 0^+. This list is not intended to be complete and it will be a good exercise to add more properties to the above list.

Throughout this paper (X, π) will denote a semidynamical system with the phase space X always assumed to be locally compact and Hausdorff, unless otherwise specified. For a net $\{a^i\}$ we say that $a^i \to \infty$ if $\{a^i\}$ has no convergent subnets.

§ 2. Construction of inverse limits.

Let (X, π) be a semidynamical system and let $\tilde{X} = \prod_{t \in R^+} X_t$ be the product space over the index set R^+ directed by the usual order relation on the reals and $X_t = X$ for all $t \in R^+$. Define the map $\tilde{\pi}: \tilde{X} \times R^+ \to \tilde{X}$ by $\tilde{\pi}(x, s) = (x_t \pi s)$, where $x = (x_t) \in \tilde{X}$ and $s \in R^+$. Then $(\tilde{X}, \tilde{\pi})$ is a semidynamical system [2] which is usually called the direct product or just the product of the semidynamical systems $\{(X_t, \pi) \mid X_t = X,\ t \in R^+\}$. Let $X_\infty = \{x = (x_t) \in \tilde{X} \mid x_t = x_s \pi(s-t)$ for $s, t \in R^+$ with $s > t\}$. Define a map $\pi_\infty: X_\infty \times R \to X_\infty$ in the following manner:

for $r \in R^+$ and $x = (x_t) \in X_\infty$,

$x\pi_\infty r = (y_t \mid y_t = x_0 \pi(r-t)$, for $0 \leq t \leq r$ and $y_t = x_{t-r}$ if $t \geq r)$;

and $x\pi_\infty(-r) = (z_t \mid z_t = x_{t+r})$.

For $t, s \in R^+$ with $t < s$ let $\pi_{ts} = \pi^{s-t}: X_s \to X_t$ defined by $\pi_{ts}(b) = b\pi(s-t)$ for $b \in X_s = X$. It is clear that π_{ts} is surjective for each $t < s$ if π^t is surjective for each $t \in R^+$, where $\pi^t(b) = \pi(b, t)$.

Lemma 2.1. *The set X_∞ is closed in $\tilde{X}$ and positively invariant in $(\tilde{X}, \tilde{\pi})$.*

Proof. The proof is simple and will be omitted.

Lemma 2.2 *The maps $\pi_{ts}: X_s \to X_t$ for all $s, t \in R^+$ with $t < s$ are surjective iff the maps $p_l: X_\infty \to X_l$ are surjective for each $l \in R^+$, where p_l is the canonical projection on the l-coordinate ($p_l(x_t) = x_l$).*

Proof. Assume that the maps π_{ts} are surjective for $t < s$. Let $a \in X_l = X$. We now construct a point $x = (x_t) \in X_\infty$ such that $p_l(x) = a = x_l$. For $t \leq l$ put $x_t = a\pi(l-t)$. We choose a point from the set $\{b \in X \mid b\pi 1 = a\}$ and call it x_{l+1}. After choosing x_{l+1} we choose a point from the set $\{b \in X \mid b\pi 1 = x_{l+1}\}$ and call it x_{l+2}. Inductively, we choose x_{l+n} from the set $\{b \in X \mid b\pi 1 = x_{l+n-1}\}$ for $n = 1, 2, 3 \cdots$. Let $r \in R^+$, $r > l$ such that $r \neq l+n$. Then there exists a positive integer k such that $l+k-1 < r < l+k$. Let $x_r = x_{l+k}\pi(l+k-r)$. Hence the point $x = (x_t) \in \tilde{X}$ has been completely defined. Furthermore, it is easy to see that $x \in X_\infty$ and that $p_l(x) = x_l = a$. This proves that p_l is surjective. The converse is straightforward.

Recall that a map $\psi: (X, \pi) \to (Y, \nu)$ is called a homomorphism if ψ is a continuous map from X onto Y and $\psi(a)\nu t = \psi(a\pi t)$ for all $a \in X$ and $t \in R^+$.

Theorem 2.3. *If the maps $\pi^t: X \to X$ are surjective for each $t \in R^+$, then the following statements are valid.*

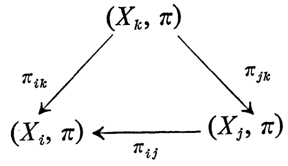
(1) *The triple $(R^+, (X_t, \pi), \pi_{ts})$ is an inverse spectrum* [5], [14].

(2) *The pair* (X_∞, π_∞) *is a dynamical system*

(3) $(X_\infty, \pi_\infty)=\text{inv}\lim_{t\in R^+}\{(X_t, \pi)\}$ [14].

Proof. (1) Let $i, j, k \in R^+$ such that $i<j<k$. Consider the digram (A).

(A)

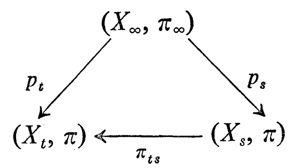


Since π^t is surjective for each $t \in R^+$, the maps π_{ik}, π_{jk} and π_{ij} are surjective. It is clear that the maps π_{ik}, π_{jk} and π_{ij} are homomorphisms. Furthermore, the diagram (A) commutes. Hence according to [5] or [14] the system $(R^+, (X_t, \pi), \pi_{ts})$ is an inverse spectrum.

(2) The proof of this part follows immediately from the definition of X_∞ and π_∞.

(3) Let $t, s \in R^+$ with $t<s$. Consider the following diagram (B).

(B)

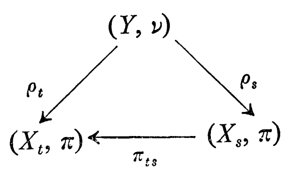


To show that $(X_\infty, \pi_\infty)=\text{inv}\lim\{(X_t, \pi)\,|\, t \in R^+\}$ we need to show that

(i) all diagrams (B) commute and

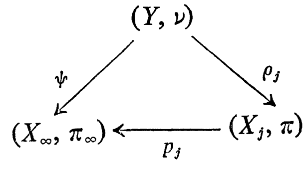
(ii) if (Y, ν) is another dynamical system with homomorphisms ρ_i, $i \in R^+$ for which all diagrams (C) commute, then

(C)



there exists a homomorphism ψ from (Y, ν) onto (X_∞, π_∞) such that all diagrams (D) commute.

(D)



Part (i) can be easily verified. To prove (ii) we need to define ψ. For each $y \in Y$, let $\psi(y)_j = \rho_j(y)$ for each $j \in R^+$. Then $p_j \psi(y) = \rho_j(y)$ and consequently the diagram (D) commutes. This shows that $(X_\infty, \pi_\infty) = \text{inv lim } \{(X_t, \pi) \mid t \in R^+\}$.

Remark 2.4. Let us define an ordering $<$ on the class of all dynamical systems in the following way. We say that $(X_1, \pi_1) < (X_2, \pi_2)$ if there exists a homomorphism ψ from (X_2, π_2) onto (X_1, π_1). In this case (X_1, π_1) is called a factor of (X_2, π_2) and is usually denoted by $(X_1, \pi_1) \mid (X_2, \pi_2)$. Using part (3) of Theorem 2.3 one could say that (X_∞, π_∞) is the smallest dynamical system that has (X, π) as a factor in the sense that if (Y, ν) is another dynamical system which has (X, π) as a factor, then (X_∞, π_∞) would be a factor of (Y, ν). Henceforth, (X_∞, π_∞) will be called the inverse limit of (X, π) and this will be denoted by inv lim $(X, \pi) = (X_\infty, \pi_\infty)$.

Theorem 2.5. *Let (X, π) be a semidynamical system with no start points. Then the following statements are pairwise equivalent.*

(1) *(X, π) can be extended to the semidynamical system (X^*, π^*).*

(2) *The negative escape time (in McCann's sense) $N(a) = \infty$ for each $a \in X$.*

(3) *The attainability set $F(A, P)$ is compact if both $A \subset X$ and $P \subset R^-$ are compact in their respective spaces.*

Proof. (1) implies (2) McCann's definition of the negative escape time $N(a)$ of $a \in X$ [14] in the absence of start points in X is as follows: $N(a) = \inf\{t \in R^+$ there exist nets $\{t_i\}$ in R^+ and $\{a^i\}$ in X such that $t_i \to t$ with each $t_i \leq t$, $a^i \pi t_i = a$, $a^{i+1} \in F(a^i, R^-)$, $a^i \to \infty\}$. Suppose that for some $b \in X$, $N(b) = r < \infty$. Then there are nets $\{b^i\}$ in X and $\{t_i\}$ in R^+ such that $t_i \to r'(r' > r)$ with each $t_i \leq r'$, $b^i \pi t_i = b$ and $b^i \to \infty$. Since $\pi^* : X^* \times R^+ \to X^*$ is continuous, $b = b^i \pi^* t_i \to \infty \pi^* r' = \infty$ and we then have a contradiction. Thus $N(b) = \infty$ for each $b \in X$. (2) implies (3). Let $A \subset X$ and $P \subset R^-$ be compact. Then from [14; 3.3] it follows that the attainability set $F(A, P) = \{b \in X \mid b\pi t = a$ for some $a \in A$ and $t \in P\}$ is closed. Since P is compact, there exists $s \in R^+$ such that $P \subset [-s, 0]$. Hence $F(A, P) \subset F(A, [-s, 0])$. By [14; 3.8] we have $\infty = \inf\{N(a) \mid a \in A\} = N(A) = \sup\{s \in R^+ \mid F(A, [-s, 0])$ is compact$\}$. Consequently, the set $F(A, [-s, 0])$ is compact. Thus $F(A, P)$ is compact. (3) implies (1). To show that (X^*, π^*) is a semidynamical system, it suffices to show that the map $\pi^* : X^* \times R^+ \to X^*$ is continuous at ∞. Let $\{b^i\}$ and $\{t_i\}$ be nets in X^* and R^+, resepectively, such that $b^i \to \infty$ and $t_i \to t$. Assume that $b^i \pi^* t_i \to a \in X$. Put $b^i \pi^* t_i = b^i \pi t_i = a^i$ for each i. If $A = \{a^i\} \cup \{a\}$ and $P = \{t_i\} \cup \{t\}$, then both A and P are compact in their respective spaces. Thus $F(A, P)$ is compact. Since $\{b^i\} \subset F(A, P)$, it follows that $b^i \not\to \infty$ and thus we have a contradiction. This completes the proof of the theorem.

We remark here that the equivalence of parts (1) and (2) was proved by McCann [11]. However our proof is different from his.

We now give an equivalent definition of the negative escape time using the terminology of negative solutions. It can be shown easily that for $a \in X$, $N(a) = \inf\{t \mid$ there exists a negative solution σ with $\sigma(0) = a$ and a net $\{t_i\}$ in R^+ with $t_i \leq t$, $t_i \rightarrow t$ such that $\sigma(-t_i) \rightarrow \infty\}$.

The following theorem is crucial in the study of semidynamical systems with nonunique global backward extensions.

Theorem 2.6 *Let (X, π) be a semidynamical system. Then the following are equivalent.*

(1) *The maps π^t: $X \rightarrow X$ are surjective for all $t \in R^+$.*

(2) *All negative trajectories in X are principal and for each $a \in X$ there exists at least one principal negative trajectory passing through a.*

(3) *(X, π) can be extended to the semidynamical system (X^*, π^*) and X has no start points.*

Proof. The equivalence of statements (1) and (2) is easy to verify. We now show that (1) and (3) are equivalent.

(1) implies (3): Since the maps π^t are surjective for all $t \in R^+$, it is clear that X has no start points. Thus to show (3), it suffices, according to theorem 2.5, to prove that $N(a) = \infty$ for each $a \in X$. Suppose that for some $b \in X$, $N(b) = r \neq \infty$. Then there exists a negative solution σ passing through b and a net $\{t_i\}$ in R^+, $t_i \leq r'$ $(r' > r)$ with $t_i \rightarrow r'$ and $\sigma(-t_i) \rightarrow \infty$. Since all negative solutions are principal, $\sigma(-r')$ exists as a point in X. Since σ is continuous [3], $\sigma(-t_i) \rightarrow \sigma(-r')$ which is a contradiction. Thus $N(a) = \infty$ for each $a \in X$. The proof of this part is now complete.

(3) implies (1): Suppose that for some $t \in R^+$, π^t: $X \rightarrow X$ is not surjective. Then there exists $a \in X$ such that $a \notin X\pi t$. Consider the set $S = \{s \in R^+ \mid a \notin X\pi s\}$. Then clearly $S \neq \phi$ and is bounded below. Hence $r = \inf(S)$ exists. If $r \notin S$, then $a \in X\pi r$. Thus $c\pi r = a$ for some $c \in X$. Furthermore, c is a start point. For if c is not a start point, then there exists $m > 0$ and $d \in X$ such that $d\pi m = c$. Hence $d\pi(m+r) = c\pi r = a$. Furthermore, for $r < t < m+r$, $(d\pi(m+r-t))\pi t = a$. Thus $r >$ inf (S) and we then have a contradiction. This shows that c is a start point which contradicts the assumptions of (3). We now conclude that $r \in S$. There exists a positive integer k such that $r > 1/k$. Then for each $n \geq k$, choose $a^n \in X$ such that $a^n\pi(r-(1/n)) = a$. Assume that $a^n \rightarrow e \in X^*$. Then $a^n\pi^*(r-(1/n)) \rightarrow e\pi^* r$. This implies that $e\pi^* r = a$. This is false since $r \in S$. This completes the proof of the theorem.

Remark 2.7 Assuming that for each $t \in R^+$, the map π^t: $X \rightarrow X$ is surjective, then there is a one to one correspondence between X_∞ and the set Γ of all principal negative solutions in X. To prove this we define the function f: $X_\infty \rightarrow \Gamma$ by letting $f(x) = f((x_t)) = \sigma$, where $\sigma(-t) = x_t$ for each $t \in R^+$. Since each principal negative solution gives rise to a principal negative trajectory and vice versa, a one to one correspondence can be established between X_∞ and the set of all principal negative trajectories in X.

This remark reveals the whole mystery about using inverse limits as a major tool in our study of semidynamical systems. For instead of dealing with the rather ambiguous notion of negative trajectories we deal with more simpler objects; the points of X_∞.

Standing terminology. A semidynamical system which satisfies any one of the statements (1), (2), (3) in theorem 2.6 will be called a semidynamical system *with global backward extension* (*g. b. e*).

Lemma 2.8 *The family* $B=\{p_t^{-1}(U)\,|\,U$ *is an open set in* X *and* $t\in R^+\}$ *is an open base for the topology on* X_∞ *inherited from the product space* $\tilde{X}$.

Proof. The proof is straightforward and will be thus omitted.

Theorem 2.9 *Let* (X, π) *be a semidynamical system with g. b. e. and* (X_∞, π_∞) *be its inverse limit. If* X *is locally compact, then* X_∞ *is locally compact.*

Proof. Assume that X is locally compact. Let $x=(x_t)\in X_\infty$. Let U be an open neighborhood of x_0 with compact closure $\bar{U}$. Then $p_0^{-1}(U)$ is an open neighborhood of $x\in X_\infty$. For each $t\in R^+$, $p_t(p_0^{-1}(\bar{U}))=F(\bar{U}, -t)$. It follows from theorem 2.5 that $p_t(p_0^{-1}(\bar{U}))$ is compact in X. Since $p_0^{-1}(\bar{U})\subset\prod\{p_t(p_0^{-1}(\bar{U}))\,|\,t\in R^+\}$, it follows that $p_0^{-1}(\bar{U})$ is compact. Hence $\overline{p_0^{-1}(U)}$ is compact. This shows that X_∞ is locally compact.

Standing hypothesis. Throughout the rest of the paper we will consider only semidynamical systems with g.b.e..

§ 3. Attainability functions.

The following scheme of topologizing a collection of closed sets is due to Michael [12]. Let 2^X be the set of all nonempty closed subsets of X. For any subset U of X, let $L(U)=\{A\in 2^X\,|\,A\cap U\neq\phi\}$ and $M(U)=\{A\in 2^X\,|\,A\subset U\}$. Then the topology generated by all sets $L(U)$, where U is an open set in X, as a subbase is called the lower semifinite topology T_l on 2^X and the topology generated by all sets $M(U)$, where U is an open set in X, as a base is called the upper semifinite topology T_u on 2^X. The smallest topology on 2^X containing both T_l and T_u is the finite topology T_f. The topology T_f can also be generated by the sets

$$\langle U_1, U_2, \cdots, U_n\rangle=\left\{E\in 2^X\,|\,E\subset\bigcup_{i=1}^{n}U_i,\quad E\cap U_i\neq\phi\quad\text{for } i=1, 2, \cdots, n\right\},$$

where all U_i are open sets in X. A setvalued function $g: X \to Y$ is said to be {*lower semicontinuous* (*l. s. c.*)} {*upper semicontinuous* (*u. s. c.*)} {*continuous*} if the corresponding function $\hat{g}: X \to 2^Y$ defined by $\hat{g}(x) = g(x)$ is {T_l-continuous} {T_u-continuous} {T_f-continuous}.

Theorem 3.1 (Michael [12]). *A setvalued map f: $X \to Y$ is*

(1) *u. s. c. if the set $\{x \in X \mid f(x) \cap A \neq \phi\}$ is closed in X whenever A is closed in Y;*

(2) *l. s. c. if the set $\{x \in X \mid f(x) \cap A \neq \phi\}$ is open in X whenever A is open in Y and*

(3) *continuous if it is both u. s. c. and l. s. c..*

Note that the attainability sets $F(a, -t)$, defined in the introduction, can be regarded as images of a setvalued map F: $X \times R^- \to X$. The map F is called the attainability function. It is clear that the function F is surjective.

Theorem 3.2. *The setvalued map $F(a, -t)$ is*

(i) *continuous in $-t$,*

(ii) *u. s. c. in a and*

(iii) *u. s. c. in $(a, -t)$ for every $a \in X$ and $t \in R^+$.*

Proof. (i) Let B be an open subset of X and let $S = \{(a_0, -t) \mid F(a_0, t) \cap B \neq \phi\}$, where a_0 is a fixed point in X. We will show that S is open in the subspace $\{a_0\} \times R^-$ of $X \times R^-$. Let $\{(a_0, -t_i)\}$ be a net converging to $(a_0, -t_0)$ and such that $(a_0, -t_i) \notin S$ for all i. Then $F(a_0, -t_i) \cap B = \phi$ for all i. Claim that $F(a_0, -t_0) \cap B = \phi$. To prove this claim assume the contrary; that is there exists $b_0 \in F(a_0, -t_0) \cap B$. This implies that $b_0 \pi t_0 = a_0$. Choose a point $x = p_{t_0}^{-1}(b_0) \in X_\infty$. Then clearly $x_{t_0} = b_0$ and $x_0 = a_0$. If the net $\{x_{t_i}\}$ in X does not have any convergent subnet; that is $x_{t_i} \to \infty$, then $a_0 = x_0 = x_{t_i} \pi^* t_i \to \infty \pi^* t_0 = \infty$ and hence we have a contradiction. We may assume that $x_{t_i} \to c$. Without loss of generality, we may either assume that all $t_i \leq t_0$ or all $t_i \geq t_0$. Suppose first that $t_i \leq t_0$. Then $x_{t_i} = x_{t_0} \pi(t_0 - t_i) \to x_{t_0} \pi 0 = x_{t_0} = b_0$. This implies that $F(a_0, -t_i) \cap B = F(x_0, -t_i) \cap B \neq \phi$ for all $i \geq i_0$ and hence we have a contradiction. On the other hand, if $t_i \geq t_0$, then $x_{t_0} = x_{t_i} \pi(t_i - t_0) \to c\pi 0 = c$. Consequently, we have $x_{t_0} = c$ and $x_{t_i} \to x_{t_0} = b_0$. Again by the same argument above one could obtain a contradiction. Hence $F(a_0, -t_0) \cap B = \phi$ and consequently S is open in $\{a_0\} \times R^+$. Hence, according to Theorem 3.1, $F(a, -t)$ is l. s. c. in $-t$. To show that $F(a, -t)$ is u. s. c. in $-t$, let A be a closed set in X and let $T = \{(a_0, -t) \mid F(a_0, -t) \cap A \neq \phi\}$. We need to show that the set T is closed in $\{a_0\} \times R^+$. Let $\{(a_0, -t_i)\}$ be a net in T which converges to $(a_0, -t_0)$. Let $d^i \in F(a_0, -t_i) \cap A$. Then $d^i \pi t_i = a_0$ for all i. Let $P = \{-t_i\} \cup \{-t_0\}$. Then P is a compact subset of R^-. Furthermore, $\{d^i\} \subset F(a_0, P)$. Since $F(a_0, P)$ is compact, we may assume, without loss of generality, that $d^i \to d \in A$. Hence $d \pi t_0 = a_0$. Consequently $(a_0, -t) \in P$

and P is thus closed. This implies by Theorem 3.1 that $F(a, -t)$ is l. s. c. in $-t$. Thus $F(a, -t)$ is continuous in $-t$.

The proofs of the remaining parts of the theorem are quiet similar to the one given above and are thus omitted.

Theorem 3.3. *The attainability set $F(A, [-t, 0])$ is compact, connected whenever A is compact and connected.*

Proof. We first remark that according to Theorem 2.5 the sets $F(A, [-t, 0])$ and $F(a, [-t, 0])$, $a \in A$, are compact. Claim that $F(a, [-t, 0])$ is connected for each $a \in A$. To prove the claim assume that for some $a \in A$, $F(a, [-t, 0])=B_1 \cup B_2$ is a separation of $F(a, [-t, 0])$ into two nonempty disjoint compact sets. Suppose that $F(a, 0)=a \in B_1$. Let

$$P_1=\{-s \in [-t, 0] \mid F(a, -s)\subset B_1\} \quad \text{and} \quad P_2=\{-s \in [-t, 0] \mid F(a, -s)\cap B_2 \neq \phi\}.$$

Then it can be easily seen that $[-t, 0]=P_1 \cup P_2$ is a separation of $[-t, 0]$ and hence we obtain a contradiction. This complete the proof of the claim. Now suppose that $F(A, [-t, 0])$ is disconnected. Then $F(A, [-t, 0])=F_1 \cup F_2$, where F_1 and F_2 are two nonempty disjoint compact sets. Let $A_1=\{a \in A \mid F(a, [-t, 0]) \cap F_1 \neq \phi\}$ and $A_2=\{a \in A \mid F(a, [-t, 0]) \cap F_1 \neq \phi\}$. Then A_1 and A_2 are nonempty. Furthermore, it follows from Theorems 2.2 and 2.1 that A_1 and A_2 are closed. Since $F(a, [-t, 0])$ is connected for each $a \in A$,

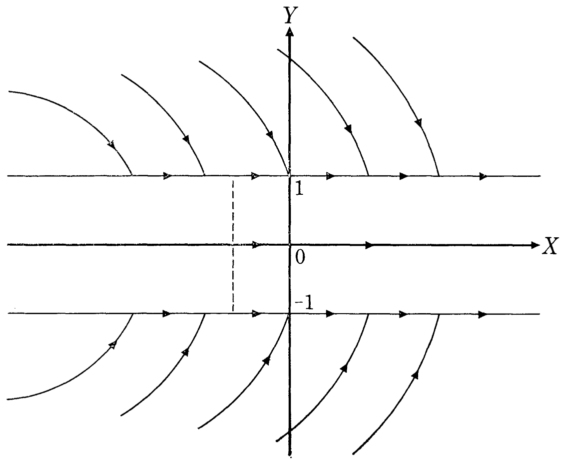
$$A_1=\{a \in A \mid F(a, [-t, 0])\subset F_1\} \quad \text{and} \quad A_2=\{a \in A \mid F(a, [-t, 0])\subset F_2\}.$$

This implies that $A_1 \cap A_2=\phi$. Thus $A_1 \cup A_2=A$ is a separation of A and consequently we have a contradiction. This completes the proof of the theorem.

Remark 3.4. The above Theorem is the best thing we could hope for. The set $F(A, P)$, with both $A\subset X$ and $P\subset R^-$ are compact and connected ($P=[-t, -t_0]$) may fail to be connected if $t_0 \neq 0$. This is in contrast to the situation in a dynamical system in which $F(A, P)$ is always connected whenever A and P are connected. The following example will demonstrate this phenomenon.

Example 3.5. Let $X=\{(x, y) \mid -1 \leq y \leq 1\} \cup \{(x, y) \mid y=x+n,\ n \in Z,\ y \leq -1\} \cup \{(x, y) \mid y=-x+n,\ n \in Z,\ y \geq 1\}$ be a subspace of R^2. Let $(x, y)\pi t$ be the point at distance t from (x, y) along the trajectory through (x, y) as shown in the diagram. For any point $(n, \pm 1)$, $n \in Z$, the sets $F((n, \pm 1), [-t, -t_0])$ is not connected if $t_0 \neq 0$. (Z is the set of integers.)

We now give an example of a semidynamical system with a global backward extension that arises from a control system.



Example 3.6. (Hajek [10]) Consider the control system

(S) $x^{\cdot}=Ax+Bu$,

where A, B are constant $n\times n$ and $n\times m$ matrices respectively, and the controls u are restricted by $|u_i|\leq 1$. Assume that (S) is normal. Let

$$R(t)=\left\{\int_0^t e^{-As}Bu(s)ds\,\middle|\,|u_i(s)|\leq 1,\ u\in L_1[0,t]\right\}$$

be the attainability set at time $t\geq 0$ and let $R=\cup\{R(t)\,|\,t\geq 0\}$ be the attainability set. Then to every point $x\in R$ we associate an optimal solution y: $[0,\infty)\to R^n$ of (S) through $y(o)=x$. Define π: $R\times[0,\infty)\to R$ by $\pi(x,t)=y(t)$ for $t\geq 0$. Then (X,π) is a semidynamical system with nonunique global backward extension.

§ 4. Properties of inverse limits.

Due to some technical difficulties in dealing with iterated limits in nets we will assume throughout this section that the phase space X is first countable and hence nets may be replaced by sequences.

Lemma 4.1. *If $\{a^i\}$ is a sequence in X converging to $a\in X$, then there exists a subsequence $\{a^{n_i}\}$ in X and a corresponding sequence $\{x^{n_i}\}$ in X_∞ which converges to $x\in X_\infty$ such that $x_0^{n_i}=a^{n_i}$ and $x_0=a$.*

Proof. Let $\{a^i\}$ be a sequence in X converging to a. For each i, choose $x^i\in p_0^{-1}(a^i)$. Then $x_0^i=a^i$. Let $B_1=\{x_0^i\}\cup\{x_0\}$, where $x_0=a$. Then B_1 is compact. Hence by Theorem 1.10 the attainability set $F(B_1,-1)$ is compact. Since $\{x_1^i\}\subset$

$F(B_1, -1)$, there exists a subsequence $\{x^{i_{j,1}}\} \in X_\infty$ of $\{x^i\}$ such that $\{x_1^{i_{j,1}}\}$ converges to a point which will be denoted by x_1. Let $B_2 = \{x_1^{i_{j,1}}\} \cup \{x_1\}$. Then $F(B_2, -1)$ is compact. Since $\{x_2^{i_{j,1}}\} \cup F(B_2, -1)$, there exists a subsequence $\{x^{i_{j,2}}\}$ of $\{x^{i_{j,1}}\}$ such that $\{x_2^{i_{j,2}}\}$ converges to a point which will be denoted by x_2. Similarly, for each $n \in Z^+$ there exists a sequence $\{x^{i_{j,n}}\}$ which is a subsequence of $\{x^{i_{j,n-1}}\}$ and such that $\{x_n^{i_{j,n}}\}$ converges to $x_n \in X$. Then, according to the well known procedure, the diagonal sequence $\{x^{i_{j,j}}\}$ is a subsequence of $\{x^i\}$ and for each $n \in Z^+$, $x^{i_{j,j}} \to x_n$. If $t \in R^+ - Z^+$, then there exists $k \in Z^+$ such that $k-1 < t < k$. Then $x_t^{i_{j,j}} = x_k^{i_{j,j}}\pi(t-k)$ $\to x_k\pi(t-k)$. Let us denote $x_k(t-k)$ by x_t. Thus a point $x = (x_t) \in X$ has been constructed in such a way that for each $t \in R^+$, $x_t^{i_{j,j}} \to x_t$. The proof of the Lemma is now complete.

Corollary 4.2. *Let $\{\sigma_i\}$ be a sequence of principal negative solutions for which $\{\sigma_i(0)\}$ converges to $a \in X$. Then there exists a principal negative solution σ with $\sigma(0) = a$ and a subsequence $\{\sigma_{i_j}\}$ of $\{\sigma_i\}$ such that $\sigma_{i_j}(t) \to \sigma(t)$ for all $t \in R^+$.*

Proof. This follows immediately from Remark 2.9 and Lemma 4.1.

Definition 4.3. For a point $a \in X$, we have the following [3] and [13]

(1) The positive limit set of a;
$L^+(a) = \cap \{\overline{a\pi(t+R^+)} \mid t \in R^+\}$,

(2) The first positive prolongation set of a;
$D^+(a) = \{\overline{V\pi R^+} \mid V$ is a neighborhood of $a\}$

(3) The first positive limit of a;
$J^+(a) = \cap \{D^+(a\pi t) \mid t \in R^+\}$.

The corresponding notions in X^* will be denoted by $L_*^+(a)$, $D_*^+(a)$ and $J_*^+(a)$, respectively. We say that $a \in X$ is of strong characteristic 0^+ [8] if whenever there exist nets $\{a^i\}$ in X and $\{t_i\}$ in R^+ with $a^i \to a$ and $a^i\pi t_i \to b$, then $a\pi t_i \to b$. A point $a \in X$ is (positively) weakly equicontinuous [7] if $D_*^+((a, a)) \subset \Delta_*$, where Δ_* denotes the diagonal in $X^* \times X^*$. A semidynamical system is said to have a property if every point in its phase space has that property.

The following Lemma is crucial in lifting properties that may be characterized by prolongations.

Lemma 4.4. *Let $a, b \in X$ with $b \in D^+(a)$. Then there are points $x, y \in X_\infty$ with $x_0 = a$, $y_0 = b$ and $y \in D^+(x)$.*

Proof. Since $b \in D^+(a)$, there exist sequences $\{a^+\}$ in X and $\{t_i\}$ in R^+ such that $a^i \to a$ and $a^i\pi t_i \to b$. Put $y_0 = b$. Then according to Lemma 3.1 there exists a sequence $\{x^{i_{j,j}}\}$, which will be denoted for simplicity by $\{x^i\}$, in X_∞ converging to $x \in X_\infty$ such that $x_t^i \to x_t$, $x_0^i = a^i$ and $x_0 = a$. Since $x_0^i\pi t_i \to b$, it follows by Theorem 2.10 (as in the proof of Lemma 4.1) that $\{x^i\pi_\infty t_i\}$ has a subsequence $\{x^{i_{j,1}}\pi_\infty t_{i_{j,1}}\}$ such that

$\{x_1^{i_j,1}\pi t_{i_{j,1}}\}$ converges to a point which will be denoted by y_1. Repeating this process one can find, as in Lemma 4.1, a diagonal sequence $\{x^{i_j,j}\pi t_{i_{j,j}}\}$ such that $x_t^{i_j,j}\pi t_{i_{j,j}} \to y_t$ for all $t \in R^+$. Consequently, $y \in D^+(x)$.

Theorem 4.5. *(X, π) is of strong characteristic 0^+ iff (X_∞, π_∞) is of strong characteristic 0^+.*

Proof. Assume that (X, π) is of strong characteristic 0^+. Let $x, y \in X_\infty$ such that $x^i \to x$ and $x^i\pi_\infty t_i \to y$ for some sequences $\{x^i\}$ in X_∞ and $\{t_i\}$ in R^+. Hence for each $t \in R^+$, $x_t^i \to x_t$ and $x_t^i\pi t_i \to y_t$. Consequently, $x_t\pi t_i \to y_t$ for all $t \in R^+$. Thus $x\pi_\infty t_i \to y$ and therefore x is of strong characteristic 0^+.

$\Leftarrow$Conversely, assume that (X_∞, π_∞) is of strong characteristic 0^+. Let $a, b \in X$ such that $a^i \to a$ and $a^i\pi t_i \to b$ for some sequences $\{a^i\}$ in X and $\{t_i\}$ in R^+. Then it follows from [3] that $b \in D^+(a)$. Then by Lemma 4.4 there are sequence $\{x^{i_j}\}$ in X_∞, $\{t_{i_j}\}$ in R^+ and points x and y in X_∞ such that $x^{i_j} \to x$, $x^{i_j}\pi_\infty t_{i_j} \to y$, $x_0^{i_j} = x^{i_j} = a^{i_j}$, $x_0 = a$, $y_0 = b$ and $\{a^{i_j}\}$ is a subsequence of $\{a^i\}$. This implies that $x\pi_\infty t_{i_j} \to y$. Consequently, $a\pi t_{i_j} = x_0\pi t_{i_j} \to y_0 = b$. This proves that a is of strong characteristic 0^+ and the proof of the theorem is now complete.

Theorem 4.6. *(X, π) is (positively) weakly equicontinuous iff (X_∞, π_∞) is positively weakly equicontinuous.*

Proof. This follows immediately from Lemma 4.4.

Since X is locally compact T_2, it is completely regular. It is well known from general topology that this implies that X is uniformizable [14]: that is there exists a uniform structure $\mathcal{U}$ which is compatible with the topology on X. This will induce a uniform structure $\mathcal{U}_\infty$ on X_∞ generated by the sets $\{p_t^{-1} \times p_t^{-1}(\alpha) \mid \alpha \in U\}$, $t \in R^+$ as a base. Furthermore, $\mathcal{U}_\infty$ is compatible with the topology on X_∞.

Definition 4.7. A semidynamical system (X, π) is said to be positively equicontinuous [9] relative to a uniformity $\mathcal{U}$ on X if the family of maps $\{\pi^t \mid t \in R^+\}$ is *pointwise equicontinuous* on X; that is for each index $\alpha \in U$ and $a \in X$ there exists an index $\beta \in U$ such that whenever (a, b) then $(a\pi t, b\pi t) \in \alpha$ for all $t \in R^+$. The system (X, π) is *positively distal* [9] if given $a, b \in X$ with $a \neq b$, there exists an index $\alpha \in \mathcal{U}$ with $(a\pi t, y\pi t) \notin \alpha$ for all $t \in R^+$.

Theorem 4.8. *(X, π) is (positively) equicontinuous iff (X_∞, π_∞) is positively equicontinuous.*

Proof. $\Rightarrow$Assume that (X, π) is (positively) equicontinuous. Suppose that the family $\{\pi^t \mid t \in R^+\}$ is not equicontinuous at $x = (x_t) \in X_\infty$. Then there are sequences $\{x^i\}$, $\{y^i\}$ in X_∞ and $\{t_i\}$ in R^+ and an index $P_s^{-1} \times p_s^{-1}(\alpha) \in U_\infty$, for some $\alpha \in U$ and

$s \in R^+$, such that $x^i \to x$, $y^i \to x$ and $(x^i \pi_\infty t_i, y^i \pi_\infty t_i) \notin p_s^{-1} \times p_s^{-1}(\alpha)$. This implies that $(x_s^i \pi t_i, y_s^i \pi t_i) = p_s \times p_s(x^i \pi_\infty t_i, y^i \pi_\infty t_i) \notin \alpha$ for all i. There are indices $\beta, \gamma \in \mathcal{U}$ with $\gamma^2 \subset \alpha$ and such that whenever $(x_s, b) \in \beta$ then $(x_s \pi t, b \pi t) \in \gamma$ for all $t \in R^+$. There exists i_0 such that $(x_s, x_s^i) \in \beta$ and $(x_s, y_s^i) \in \beta$ for all $i \geq i_0$. Thus $(x_s^i \pi t_i, y_s^i \pi t_i) = (x_s^i \pi t_i, x_s \pi t_i)(x_s \pi t_i, y_s^i \pi t_i) = \gamma \circ \gamma \subset \alpha$ for all $i \geq i_0$ and hence we have a contradiction. This shows that (X_∞, π_∞) is positively equicontinuous. Conversely, assume that (X_∞, π_∞) is positively equicontinuous. Then (X_∞, π_∞) is positively weakly equicontinuous [7]. Suppose that the family $\{\pi^t \mid t \in R^+\}$ is not equicontinuous at a point $a \in X$. Then there are sequences $\{b^i\}$, $\{c^i\}$ in X and $\{t_i\}$ in R^+ and an index $\alpha \in U$ such that $b^i \to a$, $c^i \to a$ and $(b^i \pi t_i, c^i \pi t_i) \notin \alpha$ for all i. Apply Lemma 4.1 on the space $X \times X$ and the corresponding inverse limit $(X \times X)_\infty = X_\infty \times X_\infty$. Then there exists a subsequence $\{(b^{n_i}, c^{n_i})\}$ of $\{(b^i, c^i)\}$, a corresponding sequence $\{(x^{n_i}, y^{n_i})\}$ in $X_\infty \times X_\infty$ and $x \in X_\infty$ such that $(x^{n_i}, y^{n_i}) \to (x, x)$, $x_0^{n_i} = b^{n_i}$, $y_0^{n_i} = c^{n_i}$, $x_0 = a$. There exists an index $\beta \in U$ such that whenever $(x, z) \in p_0^{-1} \times p_0^{-1}(\beta)$ then $(x \pi_\infty t, z \pi_\infty t) \in p_0^{-1} \times p_0^{-1}(\gamma)$, where $\gamma \in U$ with $\gamma^2 \subset \alpha$. Hence $(x^{n_i} \pi t_{n_i}, y^{n_i} \pi t_{n_i}) = (x^{n_i} \pi t_{n_i}, x \pi t_{n_i})(x \pi t_{n_i}, y^{n_i} \pi t_{n_i}) \in p_0^{-1} \times p_0^{-1}(\gamma) \circ p_0^{-1} \times p_0^{-1}(\gamma)$ for all $n_i \geq i_0$. This implies that

$$(b^i \pi t_{n_i}, c^i \pi t_{n_i}) = (y_0^{n_i} \pi t_{n_i}, x_0^{n_i} \pi t_{n_i}) \in \gamma \circ \gamma \subset \alpha \qquad \text{for all } n_i \geq i_0$$

and we then have a contradiction. The proof of the theorem is now complete.

Theorem 4.9. (X, π) *is positively distal iff* (X_∞, π_∞) *is positively distal.*

Proof. The proof is easy to establish and will be omitted.

Theorem 4.10. (X, π) *is positively almost periodic iff* (X_∞, π_∞) *is positively almost periodic* [9].

Proof. $\Rightarrow$Let $x = (x_t) \in X_\infty$ and $p_s^{-1}(U)$, for some open set U in X, be an open neighborhood of x. Then clearly U is an open neighborhood of x_s in X. Hence there exists a relatively dense set A in R^+ such that $x_s \pi A \subset U$. Let $t \in R^+$. If $t < s$, then

$$x_t \pi A = (x_s \pi(s-t)) \pi A = (x_s \pi A) \pi(s-t) \subset U \pi(s-t)$$

and if $t > s$, then

$$(x_t \pi A) \pi(t-s) = (x_t \pi(t-s)) \pi A = x_s \pi A \subset U.$$

Hence $x \pi_\infty A \subset p_s^{-1}(U)$ and consequently x is positively almost periodic. Thus (X_∞, π_∞) is positively almost periodic. $\Leftarrow$The converse is straightforward and will be omitted.

Theorem 4.11. (X, π) *is* (*positively*) *recurrent iff* (X_∞, π_∞) *is positively recurrent* [9].

Proof. The proof is similar to the one given in the preceding Theorem.

Definition 4.12. [3] A subset M of X is said to be (positively) *Liapunov stable* if for every open neighborhood U of M, there exists an open neighborhood V of M such that $V\pi R^+ \subset U$. The system (X, π) is *positively Liapunov stable* if $\overline{a\pi R^+}$ is positively Liapunov stable for each $a \in X$. The system (X, π) is said to be (*positively*) *Lagrange stable* [4] if $\overline{a\pi R^+}$ is compact for each $a \in X$.

Theorem 4.13. *Let (X, π) be a (positively) Lagrange stable semidynamical system. Then (X, π) is (positively) Liapunov stable iff (X_∞, π_∞) is positively Liapunov stable.*

Proof. $\Rightarrow$Assume that (X, π) is (positively) Liapunov stable. Let $x \in X_\infty$ and let U_∞ be an open neighborhood of $\overline{x\pi_\infty R^+}$. Without loss of generality, we may assume that $U_\infty = \bigcup_{i=1}^n p_0^{-1}(U_i)$ for some open sets U_i in X, $i=1, 2, \cdots, n$. Since $\overline{x_0\pi R^+}$ is compact, it follows that $p_0^{-1}(\overline{x_0\pi R^+})$ is compact and consequently, $\overline{x\pi_\infty R^+}$ is compact. This implies that $p_0(\overline{x\pi_\infty R^+}) = \overline{p_0(x\pi_\infty R^+)} = \overline{x_0\pi R^+}$. Hence there exists an open neighborhood V of $\overline{x_0\pi R^+}$ such that $V\pi R^+ \subset \bigcup_{i=1}^n U_i$. Then

$$\overline{x\pi_\infty R^+} \subset p_0^{-1}(\overline{x_0\pi R^+}) \subset p_0^{-1}(V) \subset \bigcup_{i=1}^n p_0^{-1}(U_i) .$$

Furthermore,

$$p_0^{-1}(V)\pi_\infty R^+ \subset p_0^{-1}(V\pi R^+) \subset p_0^{-1}\left(\bigcup_{i=1}^n U_i\right) = U_\infty .$$

Hence $\overline{x\pi_\infty R^+}$ is positively Liapunov stable. The proof of the converse is easy and will be omitted.

Conclusion. At the risk of being obvious we pose the following problems:

(1) It can be shown easily that (X, π) is (a) topologically transitive [9] (b) of characteristic 0^+ [1] if (X_∞, π_∞) has, respectively, the same property. It is not known to the author whether the converse of the above two statements is true or false.

(2) Does Theorem 4.13 remain true if the assumption of Lagrange stability is omitted?

(3) It is desirable if one could develop section (4) without assuming that X is first countable.

(4) There is a lot of work to be done concerning the negative versions of dynamical properties such as attraction, stability, characteristic 0^+, recurrence, $\cdots$ etc.. Some work in this direction can be found in [9]. In a forthcoming paper the author is going to investigate such notions.

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