Funkcialaj Ekvacioj, **26** (1983) 1–9

On a Nonlinear Volterra Integral-Functional Equation

By

B. G. PACHPATTE

(Marathwada University, India)

§ 1. Introduction.

In the present paper we consider the nonlinear Volterra integral-functional equation of the form

$$x(t)=F\left(t, \int_{0}^{\alpha_1(t)} f_1(t, s, x(s))ds, \cdots, \int_{0}^{\alpha_r(t)} f_r(t, s, x(s))ds, x(\beta_1(t)), \cdots, x(\beta_m(t)), u\right), \tag{1}$$

where x is an unknown function. The special cases of equation (1) were considered by many authors, for example, see [1–5, 9, 10, 14] and the references given therein. One of the simplest problem which can be reduced to a particular case of equation (1) is the initial value problem for the functional differential equation of neutral type of the form

$$x^{(n)}(t)=F(t, x(\alpha_1(t)), x'(\alpha_2(t)), \cdots, x^{(n-1)}(\alpha_r(t)), x^{(n)}(\beta_1(t)), \cdots, x^{(n)}(\beta_m(t))), \tag{2}$$

$$x(0)=x'(0)=\cdots=x^{(n-1)}(0)=0. \tag{3}$$

Equations of the type (2)–(3) arise in many areas of applied mathematics as mathematical modles of physical processes (see, [6–8]). Recently, in [11, 13] equation (1) when $u=0$ were discussed by using comparative method developed by Ważewski [15]. In this paper, we study the existence, uniqueness and continuous dependence of solutions of equation (1) by using the well known Banach contraction mapping principle [12]. The results on the existence of maximal and minimal solutions of equation (1) (when $u=0$) are also established by using the monotone iterative method and the notion of upper and lower solutions.

§ 2. Existence and uniqueness.

In this section we shall establish our results on the existence of a unique solution and continuous dependence on parameter of solutions of equation (1). For convenience we first list the following hypotheses.

Hypothesis 1. (i) Let $(B, \|\cdot\|)$ be a Banach space. The functions $\alpha_j, \beta_i \in C[I, I]$, $f_j \in C[I^2 \times B, B]$, $j=1, \cdots, r$, $i=1, \cdots, m$; $F \in C[I \times (B)^{r+m} \times R, B]$, where $I=[0, \infty)$, $R=(-\infty, \infty)$.

(ii) For $u \in R$, $(t, z_1, \cdots, z_r, u_1, \cdots, u_m, u)$, $(t, \bar{z}_1, \cdots, \bar{z}_r, \bar{u}_1, \cdots, \bar{u}_m, u) \in I \times (B)^{r+m} \times R$,

$$\|F(t, z_1, \cdots, z_r, u_1, \cdots, u_m, u) - F(t, \bar{z}_1, \cdots, \bar{z}_r, \bar{u}_1, \cdots, \bar{u}_m, u)\| \leq \sum_{j=1}^{r} L_j \|z_j - \bar{z}_j\| + \sum_{i=1}^{m} M_i \|u_i - \bar{u}_i\|;$$

and for each $j=1, \cdots, r$,

$$\|f_j(t, s, z) - f_j(t, s, \bar{z})\| \leq N_j \|z - \bar{z}\|,$$

where L_j, M_i and N_j are nonnegative constants.

(iii) Let L be a positive number. For every fixed $u \in R$ there exist a constant $P \geq 0$ and $(t, z_1, \cdots, z_r, u_1, \cdots, u_m, u) \in I \times (B)^{r+m} \times R$ such that

$$\left\| F\left(t, \int_0^{\alpha_1(t)} f_1(t, s, 0) ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, 0) ds, 0, \cdots, 0, u\right) \right\| \leq P \exp(Lt), \qquad t \in I.$$

(iv) There exist constants a_1, a_2 such that $0 \leq a_1 + a_2 < 1$ and

$$\sum_{j=1}^{r} L^{-1} L_j N_j \exp(L\alpha_j(t)) \leq a_1 \exp(Lt), \qquad t \in I,$$

$$\sum_{i=1}^{m} M_i \exp(L\beta_i(t)) \leq a_2 \exp(Lt), \qquad t \in I.$$

Hypothesis 2. There exist a constant $K \geq 0$ and a function $A: I \to I$ such that for every $t \in I$, $u, v \in R$, $(z_1, \cdots, z_r, u_1, \cdots, u_m) \in (B)^{r+m}$,

$$\|F(t, z_1, \cdots, z_r, u_1, \cdots, u_m, u) - F(t, z_1, \cdots, z_r, u_1, \cdots, u_m, v)\| \leq A(t) |u - v\|,$$

and

$$\sup_{t \in I} [A(t) \exp(-Lt)] \leq K,$$

where $|w\|$ denotes the absolute value of w.

Let G be a space of those functions $\phi: I \to B$ which are continuous in I and fulfil the condition

$$\|\phi(t)\| = O(\exp(Lt)). \tag{4}$$

In the space G we define the norm (see, [2])

$$(5) \qquad |\phi| = \sup_{t \in I} [\|\phi(t)\| \exp(-Lt)].$$

It is easily seen that G with norm defined in (5) is a Banach space (see, [3]).

We note that the condition (4) implies that there exists a constant $M \geq 0$ such that $\|\phi(t)\| \leq M \exp(Lt)$, $t \in I$. Using this fact in (5) we observe that

$$(6) \qquad |\phi| \leq M.$$

Now we shall prove the following

Theorem 1. *Let Hypothesis* 1 *be fulfilled. Then for every $u \in R$ there exists exactly one solution $x \in G$ of equation* (1), *given as the limit of successive approximations.*

Proof. Let $u \in R$ be fixed. For $x \in G$ we define the transformation $\Phi = T(x)$, where $T(x)$ is defined by the right-hand side of the equation (1). Now we shall prove that Φ maps G into itself. Evidently by (i) Φ is continuous in I and $\Phi(t) \in B$ for $t \in I$. We verify that (4) is fulfilled. Using (ii) and (iii) we obtain

$$\begin{aligned}
&\|\Phi(t)\| \\
&\leq \Big\| F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, x(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, x(s))ds, x(\beta_1(t)), \cdots, x(\beta_m(t)), u\Big) \\
&\qquad - F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, 0)ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, 0)ds, 0, \cdots, 0, u\Big)\Big\| \\
&\qquad + \Big\| F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, 0)ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, 0)ds, 0, \cdots, 0, u\Big)\Big\| \\
&\leq \sum_{j=1}^{r} L_j N_j \int_0^{\alpha_j(t)} \|x(s)\| \exp(-Ls) \exp(Ls) ds \\
&\qquad + \sum_{i=1}^{m} M_i \|x(\beta_i(t))\| \exp(-L\beta_i(t)) \exp(L\beta_i(t)) + P \exp(Lt) \\
&\leq |x| \sum_{j=1}^{r} L^{-1} L_j N_j \exp(L\alpha_j(t)) + |x| \sum_{i=1}^{m} M_i \exp(L\beta_i(t)) + P \exp(Lt).
\end{aligned}$$

Using (iv) and (6) we obtain

$$\|\Phi(t)\| \leq [Ma_1 + Ma_2 + P] \exp(Lt).$$

From which it follows that $\Phi \in G$.

Now we verify that the transformation Φ is a contraction map. We assume that $x, y \in G$ and $\Phi = T(x)$, $\Psi = T(y)$. From Hypothesis 1, we have

$$\begin{aligned}
&\|\Phi(t) - \Psi(t)\| \\
&= \Big\| F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, x(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, x(s))ds, x(\beta_1(t)), \cdots, x(\beta_m(t)), u\Big)
\end{aligned}$$

$$-F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, y(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, y(s))ds,$$
$$y(\beta_1(t)), \cdots, y(\beta_m(t)), u\Big)\Big\|$$

$$\leq \sum_{j=1}^{r} L_j N_j \int_0^{\alpha_j(t)} \|x(s)-y(s)\| \exp(-Ls) \exp(Ls) ds$$
$$+\sum_{i=1}^{m} M_i \|x(\beta_i(t))-y(\beta_i(t))\| \exp(-L\beta_i(t)) \exp(L\beta_i(t))$$
$$\leq |x-y| \sum_{j=1}^{r} L^{-1} L_j N_j \exp(L\alpha_j(t)) + |x-y| \sum_{i=1}^{m} M_i \exp(L\beta_i(t))$$
$$\leq (a_1+a_2)\,|x-y| \exp(Lt).$$

Consequently, we have

$$|\Phi-\Psi| \leq (a_1+a_2)\,|x-y|. \tag{7}$$

By using the known Banach's fixed point principle for contraction maps [12], for every fixed $u \in R$ there exists a unique fixed point of transformation T, i.e. there is a unique solution $x \in G$ of equation (1), and it is given as the limit of successive approximations. This completes the proof of the theorem.

We next establish the following theorem which deals with the problem of continuous dependence of solutions of equation (1) on a parameter u.

Theorem 2. *If Hypotheses* 1 *and* 2 *are fulfilled, then the solution* $x(t, u)$ *of equation* (1) *belonging to* G *is continuous with respect to the variables* (t, u) *in* $I \times R$.

Proof. For $x \in G$ we define the transformation $T_u(x)$ by the right side of the equation (1). From (7) we have

$$|T_u(x)-T_u(y)| \leq (a_1+a_2)\,|x-y|. \tag{8}$$

Next, from Hypothesis 2, we obtain

$$|T_u(x)-T_{u_0}(x)| \leq \sup_{t\in I}\,[A(t)\,|u-u_0\| \exp(-Lt)] \leq K|u-u_0\|. \tag{9}$$

From Theorem 1 there exists a unique function $x(t, u)$ such that $T_u(x(t, u))=x(t, u)$ and $T_{u_0}(x(t, u_0))=x(t, u_0)$ for $t \in I$. Therefore, from (8) and (9) we have

$$|x(t, u)-x(t, u_0)| \leq |T_u(x(t, u))-T_u(x(t, u_0))| + |T_u(x(t, u_0))-T_{u_0}(x(t, u_0))|$$
$$\leq (a_1+a_2)\,|x(t, u)-x(t, u_0)| + K|u-u_0\|.$$

Hence

$$|x(t, u)-x(t, u_0)| \leq (1-a_1-a_2)^{-1} K |u-u_0\|.$$

This shows that the function $x(t, u)$ is continuous with respect to the variable u in R uniformly with respect to the variable t in I, and consequently $x(t, u)$ is also continuous with respect to the two variables (t, u) in $I\times R$, which completes the proof.

We note that the results obtained in Theorems 1 and 2 can be very easily extended to the Volterra integrodifferential-functional equation of the form

$$
\begin{aligned}
x'(t)=F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, x(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, x(s))ds, \\
x(t), x(\beta_1(t)), \cdots, x(\beta_m(t)), u\Big), \\
x(0)=x_0
\end{aligned} \tag{10}
$$

with suitable modifications. Equation (10) is of more general type and contains as a special case the functional differential equation recently studied by Czerwik [5].

§ 3. Maximal and minimal solutions.

In this section we employ the notion of upper and lower solutions to study the existence of maximal and minimal solutions of equation (1) when $u=0$ and discribe how these functions become upper and lower bounds of solutions of the equation

$$
\begin{aligned}
x(t)=F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, x(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, x(s))ds, \\
x(\beta_1(t)), \cdots, x(\beta_m(t))\Big),
\end{aligned} \tag{11}
$$

where x is an unknown function. Hereafter, we assume that $I=[0, a]$, where a is a fixed positive real number, $\alpha_j, \beta_i \in C[I, I], f_j \in C[I^2\times R^n, R^n]$, for $j=1, \cdots, r$, $i=1, \cdots, m$ and $F\in C[I\times (R^n)^{r+m}, R^n]$. Without further mention, we also assume that all the inequalities between vectors are componentwise.

Definition 1. A function $w \in C[I, R^n]$ is said to be an upper solution of (11) if

$$w(t)\geq F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, w(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, w(s))ds, w(\beta_1(t)), \cdots, w(\beta_m(t))\Big),$$

and similarly, a function $v \in C[I, R^n]$ is said to be a lower solution of (11) if

$$v(t)\leq F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, v(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v(s))ds, v(\beta_1(t)), \cdots, v(\beta_m(t))\Big).$$

Definition 2. The functions $\bar{x}, \underline{x} \in C[I, R^n]$ are called maximal and minimal solutions of (11) respectively, if every other solution $x \in C[I, R^n]$ of (11) satisfies the relation $\underline{x}(t)\leq x(t)\leq \bar{x}(t)$ for all $t \in I$.

In our subsequent discussion we need the following hypotheses.

Hypothesis 3. (i) The functions $v, w \in C[I, R^n]$ with $v(t) \leq w(t)$ for $t \in I$ are lower and upper solutions of equation (11).

(ii) For each i, $f_{ji}(t, s, \xi)$ is monotone increasing in ξ for fixed $t, s \in I$, whenever $v(t) \leq \xi \leq w(t)$ for $t \in I$ and $j = 1, \cdots, r$.

(iii) For each i, $F_i(t, z_1, \cdots, z_r, u_1, \cdots, u_m)$ is increasing with respect to the variables $z_1, \cdots, z_r, u_1, \cdots, u_m$.

Hypothesis 4. For $(t, z_1, \cdots, z_r, u_1, \cdots, u_m)$, $(t, \bar{z}_1, \cdots, \bar{z}_r, \bar{u}_1, \cdots, \bar{u}_m) \in I \times (R^n)^{r+m}$,

$$\|F(t, z_1, \cdots, z_r, u_1, \cdots, u_m) - F(t, \bar{z}_1, \cdots, \bar{z}_r, \bar{u}_1, \cdots, \bar{u}_m)\|$$
$$\leq \sum_{j=1}^{r} M_j \|z_j - \bar{z}_j\| + \sum_{i=1}^{m} N_i \|u_i - \bar{u}_i\|,$$

and for each $j = 1, \cdots, r$,

$$\|f_j(t, s, u) - f_j(t, s, \bar{u})\| \leq L_j \|u - \bar{u}\|, \qquad t, s \in I,$$

where M_i, L_i and N_j are nonnegative constants such that

$$a \sum_{j=1}^{r} M_j L_j + \sum_{i=1}^{m} N_i < 1, \tag{12}$$

and $\|\cdot\|$ denote a convenient norm in R^n.

In order to prove the existence of maximal and minimal solutions of equation (11) we define the sequences $\{w_n\}$ and $\{v_n\}$ by the relations

$$\begin{aligned} & w_0(t) = w(t), \\ & w_n(t) = F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, w_{n-1}(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, w_{n-1}(s))ds, \\ & \qquad\qquad w_{n-1}(\beta_1(t)), \cdots, w_{n-1}(\beta_m(t))\Big), \end{aligned} \tag{13}$$

$$\begin{aligned} & v_0(t) = v(t), \\ & v_n(t) = F\Big(t, \int_0^{\alpha_1(t)} f_1(t, s, v_{n-1}(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v_{n-1}(s))ds, \\ & \qquad\qquad v_{n-1}(\beta_1(t)), \cdots, v_{n-1}(\beta_m(t))\Big), \end{aligned} \tag{14}$$

for $n = 1, 2, \cdots$.

In the following theorem we establish the existence of maximal and minimal solutions of equation (11).

Theorem 3. *Let Hypothesis* 3 *be fulfilled. Then the sequence* $\{w_n\}$ *defined by* (13) *converges uniformly from above to a maximal soluiton* $\bar{x}$ *of* (11) *while the sequence*

$\{v_n\}$ *defined by* (14) *converges uniformly from below to a minimal solution* $\underline{x}$ *of* (11). *Furthermore, if* $x(t)$ *is any solution of* (11) *such that* $v(t)\leq x(t)\leq w(t)$ *on* I, *then*

$$v\leq v_1\leq v_2\leq\cdots\leq v_n\leq\cdots\leq\underline{x}\leq x\leq\bar{x}\leq\cdots\leq w_n\leq\cdots\leq w_2\leq w_1\leq w, \tag{15}$$

on I.

Proof. Define $p_i(t)=v_{1i}(t)-v_i(t)$ for $t\in I$, then

$$\begin{aligned}p_i(t)\geq &F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v(s))ds, v(\beta_1(t)), \cdots, v(\beta_m(t))\right)\\ &-F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v(s))ds, v(\beta_1(t)), \cdots, v(\beta_m(t))\right)\\ &=0,\end{aligned}$$

which implies $v(t)\leq v_1(t)$ on I. Let us assume that for some integer $k>0$, $v_{k-1}(t)\leq v_k(t)$ on I. Then setting $p_i(t)=v_{(k+1)i}(t)-v_{ki}(t)$, we get

$$\begin{aligned}p_i(t)=&F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v_k(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v_k(s))ds, v_k(\beta_1(t)), \cdots, v_k(\beta_m(t))\right)\\ &-F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v_{k-1}(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v_{k-1}(s))ds,\right.\\ &\qquad\qquad\left. v_{k-1}(\beta_1(t)), \cdots, v_{k-1}(\beta_m(t))\right)\\ \geq &F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v_{k-1}(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v_{k-1}(s))ds,\right.\\ &\qquad\qquad\left. v_{k-1}(\beta_1(t)), \cdots, v_{k-1}(\beta_m(t))\right)\\ &-F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v_{k-1}(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v_{k-1}(s))ds,\right.\\ &\qquad\qquad\left. v_{k-1}(\beta_1(t)), \cdots, v_{k-1}(\beta_m(t))\right)\\ =&0,\end{aligned}$$

which implies $v_k(t)\leq v_{k+1}(t)$ on I. Hence, it follows by induction $v_{n-1}(t)\leq v_n(t)$ on I. By following the above argument we can show that $w_n(t)\leq w_{n-1}(t)$ on I.

Now define $p_i(t)=w_{1i}(t)-v_{1i}(t)$, then using the fact that $v(t)\leq w(t)$ on I, we have

$$\begin{aligned}p_i(t)=&F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, w(s)), \cdots, \int_0^{\alpha_r(t)} f_r(t, s, w(s))ds, w(\beta_1(t)), \cdots, w(\beta_m(t))\right)\\ &-F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v(s))ds, v(\beta_1(t)), \cdots, v(\beta_m(t))\right)\\ \geq &F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v(s))ds, v(\beta_1(t)), \cdots, v(\beta_m(t))\right)\end{aligned}$$

$$-F_i\left(t, \int_0^{\alpha_1(t)} f_1(t, s, v(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, v(s))ds, v(\beta_1(t)), \cdots, v(\beta_m(t))\right)$$
$$=0,$$

which implies $v_1(t) \leq w_1(t)$ on I. By following an induction argument we have $v_n(t) \leq w_n(t)$ on I. Thus, the sequences $\{v_n\}$, $\{w_n\}$ are monotone nondecreasing, nonincreasing respectively, and $v(t) \leq v_n(t) \leq w_n(t) \leq w(t)$ on I. Furthermore, using the standard arguments, it follows that these sequences converge uniformly and monotonically to the solutions $\underline{x}(t)$ and $\bar{x}(t)$ of (11).

Let $x(t)$ be any solution of (11) on I such that $v(t) \leq x(t) \leq w(t)$ on I. Then, by the induction argument, it is easily seen that $x(t) \leq w_n(t)$ and $x(t) \geq v_n(t)$ on I for every $n=0, 1, 2, \cdots$. Hence, we have $\underline{x}(t) \leq x(t) \leq \bar{x}(t)$ on I. This shows that $\bar{x}(t)$ is a maximal solution and $\underline{x}(t)$ is a minimal solution of (11). This completes the proof of the theorem.

Our next theorem shows that under some additional conditions on the functions involved in (11) the maximal and minimal solutions obtained in Theorem 3 coincide on I.

Theorem 4. *Let the Hypotheses* 3 *and* 4 *hold. Then the maximal solution* $\bar{x}(t)$ *and the minimal solution* $\underline{x}(t)$ *obtained in Theorem* 3 *coincide on* I.

Proof. Since $\bar{x}(t)$ and $\underline{x}(t)$ are solutions of (1), we have

$$\begin{aligned}\|\bar{x}(t)-\underline{x}(t)\| = \Big\| & F\left(t, \int_0^{\alpha_1(t)} f_1(t, s, \bar{x}(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, \bar{x}(s))ds, \bar{x}(\beta_1(t)), \cdots, \bar{x}(\beta_m(t))\right) \\ & -F\left(t, \int_0^{\alpha_1(t)} f_1(t, s, \underline{x}(s))ds, \cdots, \int_0^{\alpha_r(t)} f_r(t, s, \underline{x}(s))ds, \underline{x}(\beta_1(t)), \cdots, \underline{x}(\beta_m(t))\right)\Big\| \\ \leq & \sum_{j=1}^{r} M_j L_j \int_0^{\alpha_j(t)} \|\bar{x}(s)-\underline{x}(s)\| ds + \sum_{i=1}^{m} N_i \|\bar{x}(\beta_i(t))-\underline{x}(\beta_i(t))\| \\ \leq & \left[a \sum_{j=1}^{r} M_j L_j + \sum_{i=1}^{m} N_i\right] \sup_I \|\bar{x}(t)-\underline{x}(t)\|.\end{aligned}$$

Then setting $s=\sup_I \|\bar{x}(t)-\underline{x}(t)\|$, we get

$$s \leq \left[a \sum_{j=1}^{r} M_j L_j + \sum_{i=1}^{m} N_i\right] s,$$

which implies, in virtue of the condition (12) that $s=0$, i.e. $\bar{x}(t)=\underline{x}(t)$ on I. This completes the proof of the theorem.

We note that in general the maximal and minimal solutions obtained in Theorem 3 are different. However, when (11) has a unique solution, x, they coincide. In this case, the iteration scheme (15) has the advantage that $\{v_n\}$ and $\{w_n\}$ converge to x on I and $v_n \leq x \leq w_n$. That is, an approximate solution can be constructed to any degree of accuracy by making $\|v_n - w_n\|$ sufficiently small.

References

[1] Baron, K., Note on the existence of continuous solutions of a functional equation of n-th order, Ann. Polon. Math., **30** (1974), 77–80.

[2] Bielecki, A., Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaries, Bull. Acad. Polon. Sci., **4** (1956), 261–264.

[3] Czerwik, S., Special solutions of a functional equation, Ann. Polon. Math., **31** (1975), 141–144.

[4] ——, On the global existence of solutions of a functional-differential equation, Period. Math. Hungar., **6** (1975), 347–351.

[5] ——, On a differential equation with deviating argument, Comm. Math., **19** (1977), 183–187.

[6] Driver, R. D., *A functional-differential system of neutral type arising in a two-body problem of classical electro-dynamics*, International Symposium Nonlinear Differential Equations and Nonlinear Mechanics, p. 474, Academic Press, New York 1963.

[7] El'sgol'ts, L. E., *Introduction to the theory of differential equations with deviating arguments*, New York, Holden-Day, 1966.

[8] Hale, J. K., *Theory of functional differential equations*, Springer-Verlag, New York, 1976.

[9] Kuczma, M., *Functional equations in a single variable*, Monografie Mat. 46, Warszawa, 1968.

[10] Kwapisz, M., On the existence and uniqueness of solutions of a certain integral-functional equation, Ann. Polon. Math., **31** (1975), 23–41.

[11] Kwapisz, M. and Turo, J., On the existence and convergence of successive approximations for some functional equations in a Banach space, J. Differential Equations, **16** (1974), 298–318.

[12] Liusternik, L. A. and Sobolev, V. J., *Elements of functional analysis*, Frederick Ungar Publishing Co., New York, 1961.

[13] Nowicka, K., On the existence of solutions of a certain integral-functional equation, Demonstratio Math., **11** (1978), 435–451.

[14] Pelczar, A., Remarks on some functional equations and inequalities, Zeszyty Nauk. Uniw. Jagiello., Prace Matematyczne, **12** (1968), 47–52.

[15] Ważewski, T., Sur une procédé de prouver la convergence des approximations successive sans utilisation des séries de comparison, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., **8** (1960), 45–52.

nuna adreso:
Department of Mathematics and Statistics
Marathwada University
Aurangabad—431 004
(Maharashtra)
India

(Ricevita la 15-an de marto, 1982)