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An Optimal Control Problem of the Prey-Predator System

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Introduction.

In his research of the prey-predator system, V. Volterra [3] introduced the differential equations

$$(0.1)\qquad \begin{aligned} x_1' &= x_1(\lambda_1+\mu_1 x_2) \\ x_2' &= x_2(\lambda_2+\mu_2 x_1) \end{aligned} \qquad \lambda_1>0,\quad \lambda_2<0,\quad \mu_1<0,\quad \mu_2>0$$

in which x_1 and x_2 are respectively the numbers of preys and predators.

If the preys and predators are separated from each other completely or incompletely, then (0.1) should be replaced with the following system

$$(0.2)\qquad \begin{aligned} x_1' &= x_1(\lambda_1+u\mu_1 x_2) \\ x_2' &= x_2(\lambda_2+u\mu_2 x_1) \end{aligned} \qquad 0\leqq u\leqq 1$$

where $1-u$ means the rate of separation. We shall study an optimal control problem regarding u as a control function. The independent variable t in (0.2) is commented as the time and varies from zero to a given terminal time T, x_1 and x_2 should satisfy the initial condition

$$(0.3)\qquad \begin{aligned} x_1(0) &= a_1 \\ x_2(0) &= a_2 \end{aligned} \qquad a_1>0,\quad a_2>0.$$

The cost functional is given as

$$(0.4)\qquad J[u]=-(x_1(T)+x_2(T))$$

of which minimum is to be looked for. The cost functional of the form $-(\rho_1 x_1(T)+\rho_2 x_2(T))$ with positive weights ρ_1, ρ_2, representing the individual economical values of preys and predators, can be reduced to (0.4) by an obvious transformation $\tilde{x}_1=\rho_1 x_1$, $\tilde{x}_2=\rho_2 x_2$.

In § 1, after some preliminary comments in which the existence of the optimal control in the class of bounded measurable functions is pointed out, Pontrjagin's maximum principle and its attendant transversality condition are invoked. We specify three cases according to the signum of $\mu_1+\mu_2$.

In § 2, cases $\mu_1+\mu_2<0$ and $\mu_1+\mu_2=0$ are treated and we see that the optimal controls are $u=0$.

In § 3, case $\mu_1+\mu_2>0$ is treated and we prove that every optimal u takes only the values 0 or 1 and its number of switching times is at most one.

In § 4, we give the switching time $\bar{\tau}_1$ of the critical optimal recsponse, having the terminal value equal to the critical value of the control system.

In § 5, optimal responses with terminal values near to the critical one are treated, and as for the switching time τ_1 of them we give sufficient conditions for $\tau_1<\bar{\tau}_1$ and for $\tau_1>\bar{\tau}_1$ respectively.

§ 1. Preliminaries.

If (x_1, x_2) is the solution of (0.2) and (0.3) then it is positive: $x_1(t)>0$, $x_2(t)>0$, because the solution is unique and $x_1=0$, $x_2=0$ is a solution of (0.2). Further, it is bounded. Indeed from $\mu_1<0$ and (0.2) we have

$$(0<)x_1(t)\leqq a_1e^{\lambda_1 t}\leqq a_1e^{\lambda_1 T}$$

which, together with $\lambda_2<0$, gives

$$(0<)x_2(t)\leqq a_2 \exp(\mu_2 a_1 e^{\lambda_1 T}t)\leqq a_2 \exp(\mu_2 a_1 e^{\lambda_1 T}T).$$

The target set at $t=T$ is originally the whole (x_1, x_2)-plane, but the above result allows us to take it as a compact set in that plane, and we can take use of the theorem 5.1 in chapter III of Berkovitz [1] to see the existence of an optimal control in the class of bounded measurable functions.

Since optimal control and corresponding adjoint variables ψ_0, ψ_1, ψ_2 satisfy Pontrjagin's maximum principle, we have

$$H(\psi(t), x(t), u(t))=\max_{0\leqq z\leqq 1} H(\psi(t), x(t), z) \qquad \text{a.e. } t\in[0, T] \tag{1.1}$$

where

$$x=\begin{pmatrix}x_1\\x_2\end{pmatrix}, \qquad \psi=\begin{pmatrix}\psi_0\\\psi_1\\\psi_2\end{pmatrix},$$

$$\begin{aligned}H(\psi, x, u)&=\psi_1x_1(\lambda_1+u\mu_1x_2)+\psi_2x_2(\lambda_2+u\mu_2x_1)\\&=\lambda_1\psi_1x_1+\lambda_2\psi_2x_2+ux_1x_2(\mu_1\psi_1+\mu_2\psi_2).\end{aligned}$$

The same principle gives also the transversality condition at the terminal time $t=T$;

$$\begin{aligned}&\psi_1(T)+\psi_0=0\\&\psi_2(T)+\psi_0=0.\end{aligned} \tag{1.2}$$

It is $\psi_0=\text{const}\leqq 0$ and ψ_1, ψ_2 satisfy

$$\text{(1.3)}\qquad \begin{aligned} \psi_1' &= -\frac{\partial H}{\partial x_1} = -\lambda_1\psi_1 - ux_2(\mu_1\psi_1+\mu_2\psi_2) \\ \psi_2' &= -\frac{\partial H}{\partial x_2} = -\lambda_2\psi_2 - ux_1(\mu_1\psi_1+\mu_2\psi_2). \end{aligned}$$

In general theory, the transversality condition in its general form requires the continuity of the optimal control u at the terminal time (Berkovitz [1], chapter V, theorem 3.1 and exercise 3.5), but in our present problem, due to the simplicity of the boundary condition at $t=T$, it is easily verified by rescrutinizing the proof of that theorem given in ibid. Chapter VI section 7, that (1.2) is valid without the continuity of the optimal control at $t=T$ (see Note (1)).

On account of nontrivialness of ψ and (1.2), we may assume without any loss of generality that

$$\text{(1.4)}\qquad \psi_0=-1, \qquad \psi_1(T)=\psi_2(T)=1.$$

Since $x_1(t)$ and $x_2(t)$ are positive, we have from (1.1) that optimal control u is determined by

$$\text{(1.5)}\qquad u(t)=\begin{cases} 0 & \text{for a.e. } t \text{ satisfying } (\mu_1\psi_1(t)+\mu_2\psi_2(t))<0 \\ 1 & \text{for a.e. } t \text{ satisfying } (\mu_1\psi_1(t)+\mu_2\psi_2(t))>0. \end{cases}$$

We specify three cases according to the signum of $\mu_1+\mu_2$;
Case I: $\mu_1+\mu_2<0$,
Case II: $\mu_1+\mu_2=0$,
Case III: $\mu_1+\mu_2>0$.

§ 2. Cases I and II.

First we take up the Case I and investigate the behavior of the optimal control u. Since $\mu_1+\mu_2=\mu_1\psi_1(T)+\mu_2\psi_2(T)<0$ in this case, it follows from the continuity of ψ_1 and ψ_2 that there is an $\varepsilon>0$ such that

$$\text{(2.1)}\qquad \mu_1\psi_1(t)+\mu_2\psi_2(t)<0 \qquad \text{for } t\in(T-\varepsilon, T]$$

which means for a.e. $t\in(T-\varepsilon, T]$

$$\text{(2.2)}\qquad u(t)=0$$

and since the values of u on a null set exert no effect upon the behavior of the response and adjoint variables, we have

$$\text{(2.3)}\qquad \begin{aligned} \psi_1' &= -\lambda_1\psi_1 \\ \psi_2' &= -\lambda_2\psi_2 \end{aligned}$$

hence

$$\psi_1(t)=e^{\lambda_1(T-t)}, \qquad \psi_2(t)=e^{\lambda_2(T-t)}.$$

Since $\lambda_1>0$, $\lambda_2<0$, $\mu_1<0$ and $\mu_2>0$, $\mu_1\psi_1(t)+\mu_2\psi_2(t)$ is monotone increasing and we see that (2.1) and (2.2) are valid for $t\in[0, T]$. Neglecting the values of control on null set, we have the following

Proposition 1. *In the Case I, there is no switching time of the optimal control u,* $u(t)=0$ *for* $t\in[0, T]$ *and*

$$x_1(t)=a_1e^{\lambda_1 t}, \quad x_2(t)=a_2e^{\lambda_2 t}, \quad J[u]=-(a_1e^{\lambda_1 T}+a_2e^{\lambda_2 T}).$$

Now we investigate the Case II, in which $\mu_1+\mu_2=0$. If we put $-\mu_1=\mu_2=\mu>0$, them (1.3) assumes the form

$$\begin{aligned}\psi_1'&=-\lambda_1\psi_1-ux_2\mu(-\psi_1+\psi_2)\\ \psi_2'&=-\lambda_2\psi_2-ux_1\mu(-\psi_1+\psi_2).\end{aligned} \tag{2.4}$$

Since $-\psi_1(T)+\psi_2(T)=0$, we have

$$\begin{aligned}\mu_1\psi_1(t)+\mu_2\psi_2(t)&=\mu(-\psi_1(t)+\psi_2(t))\\ &=-\int_t^T[\mu(\lambda_1\psi_1(s)-\lambda_2\psi_2(s))+u(s)\mu^2(-x_1(s)+x_2(s))\\ &\qquad\qquad\times(-\psi_1(s)+\psi_2(s))]ds.\end{aligned}$$

Since $\lambda_1\psi_1(s)-\lambda_2\psi_2(s)\to\lambda_1-\lambda_2>0$ and $-\psi_1(s)+\psi_2(s)\to0$ as $s\to T-0$, there is an $\varepsilon>0$ such that for $t\in(T-\varepsilon, T)$

$$\mu_1\psi_1(t)+\mu_2\psi_2(t)=\mu(-\psi_1(t)+\psi_2(t))<0. \tag{2.5}$$

By the same argument as in the Case I, we know that (2.5) and $u(t)=0$ are valid for $t\in[0, T]$, and can state

Proposition 2. *In the Case II, there is no switching time of the optimal control u.* $u(t)=0$ *for* $t\in[0, T]$ *and*

$$x_1(t)=a_1e^{\lambda_1 t}, \quad x_2(t)=a_2e^{\lambda_2 t}, \quad J[u]=-(a_1e^{\lambda_1 T}+a_2e^{\lambda_2 T}).$$

§ 3. Case III; the switching time τ_1.

As $\mu_1+\mu_2>0$ in this case, there is an $\varepsilon>0$ such that

$$\mu_1\psi_1(t)+\mu_2\psi_2(t)>0 \qquad \text{for } t\in(T-\varepsilon, T]. \tag{3.1}$$

Let τ be any positive number such that $0\leqq\tau\leqq T$ and the inequality (3.1) be satisfied for $t\in(\tau, T]$, and let τ_1 be the infimum of such τ's, then the optimal control $u(t)=1$ for a.e. $t\in(\tau_1, T]$ and $\mu_1\psi_1(\tau_1)+\mu_2\psi_2(\tau_1)=0$.

If $\tau_1>0$ then for $t \in (\tau_1, T]$ we have

$$\begin{aligned}\psi_1' &= -\lambda_1\psi_1 - x_2(\mu_1\psi_1+\mu_2\psi_2)\\ \psi_2' &= -\lambda_2\psi_2 - x_1(\mu_1\psi_1+\mu_2\psi_2).\end{aligned} \tag{3.2}$$

Since $x_2(\mu_1\psi_1+\mu_2\psi_2)>0$, we have $\psi_1(t)\geqq\varphi(t)=\exp\lambda_1(T-t)$, φ being the solution of $\varphi'=-\lambda_1\varphi$ and $\varphi(T)=1$, hence $\psi_1(t)\geqq 1$ for $t \in [\tau_1, T]$. Since $\mu_1\psi_1(\tau_1)+\mu_2\psi_2(\tau_1)=0$, it must be $\psi_2(\tau_1)\geqq-\mu_1/\mu_2>0$, and

$$\mu_1\psi_1(t)+\mu_2\psi_2(t)=\int_t^{\tau_1}[\lambda_1\mu_1\psi_1(s)+\lambda_2\mu_2\psi_2(s) \\ +u(s)(\mu_1x_2(s)+\mu_2x_1(s))(\mu_1\psi_1(s)+\mu_2\psi_2(s))]ds<0$$

for $t \in (\tau_1-\varepsilon, \tau_1)$ with an $\varepsilon>0$, because $\lambda_1\mu_1\psi_1(\tau_1)+\lambda_2\mu_2\psi_2(\tau_1)<0$ and $\mu_1\psi_1(s)+\mu_2\psi_2(s)\to 0$ as $s\to\tau_1-0$. Neglecting the values on null set and agreeing that u is left continuous at τ_1, we have $u(t)=0$ for $t \in (\tau_1-\varepsilon, \tau_1]$, hence τ_1 is a switching time of the optimal control u.

By the same argument as given in § 2, we know $u(t)=0$ for a.e. $t \in [0, \tau_1]$ and have the following proposition, in which we neglect the values of u on null set.

Proposition 3. *In the Case III, number of the switching times of optimal control is at most one. If there is actually a switching time* τ_1, *then*

$$u(t)=0, \quad x_1=a_1e^{\lambda_1 t}, \quad x_2=a_2e^{\lambda_2 t} \qquad \textit{for } t \in [0, \tau_1]$$

and

$$u(t)=1, \qquad \textit{for } t \in (\tau_1, T].$$

If there is no switching time, then $u(t)=1$, $t \in [0, T]$ *is the optimal control.*

For $u(t)=1$, (0.2) coincides with the Volterra's system (0.1) and can be solved easily (Volterra [3] and Yamaguti [4]), giving

$$x_1^{\lambda_2}e^{\mu_2 x_1}x_2^{-\lambda_1}e^{-\mu_1 x_1}=\text{const.}$$

The orbit is a closed curve (or a part of it) around the critical point $(-\lambda_2/\mu_2, -\lambda_1/\mu_1)$ in the phase plane.

§ 4. Value of $\bar{\tau}_1$ for the optimal critical response.

As we have seen in § 3, every optimal response x in the Case III satisfies the control system (0.2) with the corresponding optimal control u equal to 1 for $t \in (\tau_1, T]$ with some $\tau_1 \in [0, T)$. Since τ_1 is the largest zero point of $\mu_1\psi_1+\mu_2\psi_2$ (or equal to 0 in case of no switching time) and depends continuously on the terminal value $x(T)$

of the optimal response x, the state of the orbit of optimal response in a bounded domain in the phase plane is the same as that of the orbit of (0.1), the Volterra's system, for a t interval $\tau \leqq t \leqq T$.

Let $\bar{x}=(\bar{x}_1, \bar{x}_2)$ be the optimal response having the terminal value $\bar{x}(T)=(\bar{x}_1(T), \bar{x}_2(T))$ equal to the critical value $(-\lambda_2/\mu_2, -\lambda_1/\mu_1)$ of (0.2) and $\bar{u}$ be the corresponding optimal control, $\bar{u}(t)=1$ for $t \in (\bar{\tau}_1, T]$, $\bar{\tau}_1$ being the switching time of $\bar{u}$ (or equal to 0). The initial value of this optimal critical response is obtained very easily in terms of $\bar{\tau}_1$ by solving (0.2) backwardly with $u=\bar{u}(t)=0$ for $t \in [0, \tau_1]$ and with $\bar{x}_1(\bar{\tau}_1)=-\lambda_2/\mu_2$, $\bar{x}_2(\bar{\tau}_1)=-\lambda_1/\mu_1$, giving

$$\bar{x}_1(0)=-\frac{\lambda_2}{\mu_2}e^{-\lambda_1\bar{\tau}_1}, \qquad \bar{x}_2(0)=-\frac{\lambda_1}{\mu_1}e^{-\lambda_2\bar{\tau}_1},$$

while the value of $\bar{\tau}_1$ is obtained by the following elementary calculations.

For $t \in (\bar{\tau}_1, T]$, (1.3) assumes the form

$$(4.1)\qquad \begin{aligned}\psi_1' &= \lambda_1\frac{\mu_2}{\mu_1}\psi_2\\ \psi_2' &= \lambda_2\frac{\mu_1}{\mu_2}\psi_1\end{aligned}$$

hence it follows

$$(4.2)\qquad \begin{aligned}&\frac{d}{dt}(\mu_1\psi_1+\mu_2\psi_2)=(\lambda_2\mu_1\psi_1+\lambda_1\mu_2\psi_2)\\ &\frac{d}{dt}(\lambda_2\mu_1\psi_1+\lambda_1\mu_2\psi_2)=\lambda_1\lambda_2(\mu_1\psi_1+\mu_2\psi_2).\end{aligned}$$

Let

$$A=\begin{pmatrix}0 & 1\\ \lambda_1\lambda_2 & 0\end{pmatrix},$$

then a fundamental matrix of (4.2) is given by

$$(4.3)\qquad e^{At}=\begin{pmatrix}\cos(\sqrt{\lambda_1|\lambda_2|}\,t) & \dfrac{1}{\sqrt{\lambda_1|\lambda_2|}}\sin(\sqrt{\lambda_1|\lambda_2|}\,t)\\ -\sqrt{\lambda_1|\lambda_2|}\sin(\sqrt{\lambda_1|\lambda_2|}\,t) & \cos(\sqrt{\lambda_1|\lambda_2|}\,t)\end{pmatrix}$$

hence

$$\begin{pmatrix}\mu_1\psi_1+\mu_2\psi_2\\ \lambda_2\mu_1\psi_1+\lambda_1\mu_2\psi_2\end{pmatrix}=e^{A(t-T)}\begin{pmatrix}\mu_1+\mu_2\\ \lambda_2\mu_1+\lambda_1\mu_2\end{pmatrix}$$

from which we have

$$\mu_1\psi_1+\mu_2\psi_2=(\mu_1+\mu_2)\cos(\sqrt{\lambda_1|\lambda_2|}(T-t)) \\ -\frac{(\lambda_2\mu_1+\lambda_1\mu_2)}{\sqrt{\lambda_1|\lambda_2|}}\sin(\sqrt{\lambda_1|\lambda_2|}(T-t))$$

and

$$T-\bar{\tau}_1=\frac{1}{\sqrt{\lambda_1|\lambda_2|}}\tan^{-1}\frac{\sqrt{\lambda_1|\lambda_2|}(\mu_1+\mu_2)}{\lambda_2\mu_1+\lambda_1\mu_2}.$$

Recollecting the results, we have

Proposition 4. *In the Case III, the switching time* $\bar{\tau}_1$ *of the optimal control* $\bar{u}$ *corresponding to the response having the critical terminal value, appears if and only if* $T>\theta/\sqrt{\lambda_1|\lambda_2|}$, *and its value is*

$$\bar{\tau}_1=T-\frac{1}{\sqrt{\lambda_1|\lambda_2|}}\theta \quad \textit{with} \quad \theta=\tan^{-1}\frac{\sqrt{\lambda_1|\lambda_2|}(\mu_1+\mu_2)}{\lambda_2\mu_1+\lambda_1\mu_2}.$$

§ 5. Responses near to the optimal critical response.

In this section we investgate optimal responses of which terminal values are sufficiently near to the critical one. As we have seen in the preceeding section, orbit of such a response is a closed curve (or a part of it) around the critical value $(-\lambda_2/\mu_2, -\lambda_1/\mu_1)$ in the phase plane for $t \in (\tau_1, T]$, where τ_1 is the switching time of its optimal control or is equal to 0 in case of no switching time. Let $(\bar{x}_1, \bar{x}_2)$ be the optimal critical response, considered in § 4, of which terminal value is $(-\lambda_2/\mu_2, -\lambda_1/\mu_1)$, (consequently $\bar{x}_1=-\lambda_2/\mu_2$, $\bar{x}_2=-\lambda_1/\mu_1$ on $[\bar{\tau}_1, T]$), and let (x_1, x_2) be an optimal response of which control u is $u(t)=1$ on $(\tau_1, T]$ with the switching time τ_1, and $(x_1(T), x_2(T))$ is sufficiently near to $(-\lambda_2/\mu_2, -\lambda_1/\mu_1)$. We can state sufficient conditions for $\bar{\tau}_1>\tau_1$ and for $\bar{\tau}_1<\tau_1$ in terms of the location of the terminal value $(x_1(T), x_2(T))$. Let $y_1=x_1-\bar{x}_1$, $y_2=x_2-\bar{x}_2$, then $|y_1|, |y_2|$ are small on $(\tau_0, T]$ with $\tau_0=\max\{\tau_1, \bar{\tau}_1\}$ and (y_1, y_2) approximately satisfies (cf. [3], [4])

$$y_1'=-\lambda_2\frac{\mu_1}{\mu_2}y_2 \qquad\qquad t\in(\tau_0, T]. \tag{5.1}$$
$$y_2'=-\lambda_1\frac{\mu_2}{\mu_1}y_1$$

Let (ψ_1, ψ_2) be the adjoint variable corresponding to (x_1, x_2) then we have for $t \in (\tau_0, T]$

$$\psi_1'=-\lambda_1\psi_1-x_2(\mu_1\psi_1+\mu_2\psi_2)=\lambda_1\frac{\mu_2}{\mu_1}\psi_2-y_2(\mu_1\psi_1+\mu_2\psi_2)$$

$$\psi_2' = -\lambda_2\psi_2 - x_1(\mu_1\psi_1+\mu_2\psi_2) = \lambda_2\frac{\mu_1}{\mu_2}\psi_1 - y_1(\mu_1\psi_1+\mu_2\psi_2).$$

Putting $\varphi_1=\mu_1\psi_1+\mu_2\psi_2$, $\varphi_2=\lambda_2\mu_1\psi_1+\lambda_1\mu_2\psi_2$, we have $\varphi_1(T)=\mu_1+\mu_2$, $\varphi_2(T)=\lambda_2\mu_1+\lambda_1\mu_2$ and

$$\begin{aligned}\varphi_1' &= \varphi_2-(\mu_2y_1+\mu_1y_2)\varphi_1=\varphi_2-h_1\varphi_1\\ \varphi_2' &= \lambda_1\lambda_2\varphi_1-(\lambda_1\mu_2y_1+\lambda_2\mu_1y_2)\varphi_1=\lambda_1\lambda_2\varphi_1-h_2\varphi_1\end{aligned}$$

in which we used the notations $h_1=\mu_2y_1+\mu_1y_2$, $h_2=\lambda_1\mu_2y_1+\lambda_2\mu_1y_2$.

Let $(\bar\psi_1, \bar\psi_2)$ and $(\bar\varphi_1, \bar\varphi_2)$ be the variables corresponding to $(\bar x_1, \bar x_2)$ and derived from it in the same fashion as above, then $\bar\varphi_1(T)=\varphi_1(T)$, $\bar\varphi_2(T)=\varphi_2(T)$, and

$$\begin{aligned}\bar\varphi_1' &= \bar\varphi_2\\ \bar\varphi_2' &= \lambda_1\lambda_2\bar\varphi_1\end{aligned}$$

since $\bar y_1=\bar y_2=0$.

Let $\alpha_1=\varphi_1-\bar\varphi_1$, $\alpha_2=\varphi_2-\bar\varphi_2$ then $\alpha_1(T)=\alpha_2(T)=0$ and

$$\begin{aligned}\alpha_1' &= \alpha_2-h_1\varphi_1\\ \alpha_2' &= \lambda_1\lambda_2\alpha_1-h_2\varphi_1\end{aligned}\tag{5.2}$$

while (h_1, h_2) satisfies approximately

$$\begin{aligned}h_1' &= -h_2\\ h_2' &= -\lambda_1\lambda_2h_1.\end{aligned}\tag{5.3}$$

Let

$$A=\begin{pmatrix}0 & 1\\ \lambda_1\lambda_2 & 0\end{pmatrix}$$

as in § 4, then from (5.3) we have

$$\begin{pmatrix}h_1(t)\\ h_2(t)\end{pmatrix}=e^{A(T-t)}\begin{pmatrix}h_1(T)\\ h_2(T)\end{pmatrix}\tag{5.4}$$

and from (5.2)

$$\begin{aligned}\begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix} &= -e^{At}\int_T^t e^{-As}\varphi_1(s)\begin{pmatrix}h_1(s)\\ h_2(s)\end{pmatrix}ds\\ &= \int_t^T \varphi_1(s)e^{A(t-s)}e^{A(T-s)}\begin{pmatrix}h_1(T)\\ h_2(T)\end{pmatrix}ds\\ &= \int_t^T \varphi_1(s)e^{A(T+t-2s)}\begin{pmatrix}h_1(T)\\ h_2(T)\end{pmatrix}ds\end{aligned}$$

where by (4.3)

$$e^{A(T+t-2s)}=\begin{pmatrix} \cos(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s)) & \frac{1}{\sqrt{\lambda_1|\lambda_2|}}\sin(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s)) \\ -\sqrt{\lambda_1|\lambda_2|}\sin(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s)) & \cos(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s)) \end{pmatrix}$$

hence

$$\alpha_1(t)=\int_t^T \varphi_1(s)\Big[\cos(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s))h_1(T) + \frac{1}{\sqrt{\lambda_1|\lambda_2|}}\sin(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s))h_2(T)\Big]ds. \tag{5.5}$$

Now assume

$$h_1(T)>0 \tag{5.6}$$

and

$$\cos(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s))h_1(T)+\frac{1}{\sqrt{\lambda_1|\lambda_2|}}\sin(\sqrt{\lambda_1|\lambda_2|}\,(T+t-2s))h_2(T)>0 \quad \text{for } t<s<T,\ t\in(\bar\tau_1, T) \tag{5.7}$$

then we can show that $\tau_1<\bar\tau_1$.

Indeed, if $\tau_1\geqq\bar\tau_1$ then $\tau_0=\tau_1$ and from (5.5) and (5.7) we have $\alpha_1(t)>0$ for $t\in[\tau_1, T)$. Hence

$$\varphi_1(t)=\bar\varphi_1(t)+\alpha_1(t)>\bar\varphi_1(t)\geqq 0 \qquad \text{for } t\in[\tau_1, T).$$

Since $\varphi_1(t)$ is continuous and $\varphi_1(T)>0$, $\varphi_1(t)=\mu_1\psi_1(t)+\mu_2\psi_2(t)>0$ for $t\in[\tau_1-\varepsilon, T]$ with an $\varepsilon>0$. Hence τ_1 cannot be the switching time of the optimal control u of the (x_1, x_2), which is contradiction.

Let

$$\theta_1=\frac{1}{\sqrt{\lambda_1|\lambda_2|}}\tan^{-1}\left(\frac{h_1(T)\sqrt{\lambda_1|\lambda_2|}}{|h_2(T)|}\right). \tag{5.8}$$

Since $t<s<T$ means $T+t-2s\in(t-T, T-t)$, it is easily observed that if

$$\theta_1\geqq T-\bar\tau_1 \tag{5.9}$$

then (5.7) follows from (5.6).

Recollecting the results, we can state

Proposition 5. *In the Case III, if* (5.6) *and* (5.9) *hold for an optimal response of which terminal value is sufficiently near to the critical value* $(-\lambda_2/\mu_2, -\lambda_1/\mu_1)$, *then the switching time* τ_1 *of the corresponding control satisfies* $\tau_1<\bar\tau_1$.

By the same argument as above, we can prove the following proposition provided that (5.6) and (5.7) are replaced with

$$h_1(T)<0 \tag{5.10}$$

and

$$\cos(\sqrt{\lambda_1|\lambda_2|}(T+t-2s))h_1(T)+\frac{1}{\sqrt{\lambda_1|\lambda_2|}}\sin(\sqrt{\lambda_1|\lambda_2|}(T+t-2s))h_2(T)<0 \quad \text{for } t<s<T,\ t\in(\tau_1, T) \tag{5.11}$$

respectively:

Proposition 6. *Let* (5.6) *in the assumptions of the Proposition* 5 *be replaced with* (5.10) *then the switching time* τ_1 *satisfies* $\tau_1>\bar{\tau}_1$.

Note

(1) Indeed, in the cited proof, it is proved that a vector of the form $c=(c_0^0, c_0, c_1^0, c_1)$ (c_0 and c_1 are 2-dimensional vectors) is orthogonal to every column vector of the matrix

$$\begin{pmatrix} T_{0\sigma} \\ X_{0\sigma} \\ T_{1\sigma} \\ X_{1\sigma} \end{pmatrix},$$

which is in our case equal to

$$\begin{pmatrix} & O & \\ 1 & & 0 \\ 0 & & 1 \end{pmatrix},$$

O being 4×2 null matrix. It is $c_1=(\psi_1(T)-\psi_0, \psi_2(T)-\psi_0)$ in our case. c_0^0 and c_1^0 are given originally in form of integrals on a small interval. Continuity of $u(t)$ is used to express c_0^0 and c_1^0 in form of differentials, but it is needless for us as observed immediately from the form of the matrix

$$\begin{pmatrix} T_{0\sigma} \\ X_{0\sigma} \\ T_{1\sigma} \\ X_{1\sigma} \end{pmatrix}.$$

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