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On Global Simplification of a Singularly Perturbed System of Linear Ordinary Differential Equations

By

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§ 1. Periodic and non-periodic systems.

Let $J=[a, b]$, $J_\infty=(-\infty, \infty)$, $\mathscr{S}_\varepsilon(c)=\{\varepsilon \mid 0<|\varepsilon|<c, |\arg \varepsilon|\leq \delta\pi/(2h)\}$, where h is a positive integer, a, b, c and δ are constants with $c>0$, $0<\delta<1$. Consider a system of linear ordinary equations:

$$\varepsilon^h \frac{dy}{dt}=\{A(t)+q(\varepsilon)Q(t, \varepsilon)\}y. \tag{E$_1$}$$

Here $A(t)$ and $Q(t, \varepsilon)$ are n by n matrices, $q(\varepsilon)$ is a differentiable scalar function satisfying

$$\lim q(\varepsilon)=0, \qquad \text{as } \varepsilon \text{ tends to 0 in } \mathscr{S}_\varepsilon(c). \tag{1.1}$$

Let $\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)$ be eigenvalues of $A(t)$. We shall study the global simplification of (E_1) under two different sets of assumptions. The first set is:

(A1) *$A(t)$ is in $C^2(J)$;*
(A2) *$Q(t, \varepsilon)$ is in $C^1(J\times\mathscr{S}_\varepsilon(c))$;*
(A3) *There is a positive integer k such that the set of indices $\{1, 2, \cdots, n\}$ is divided into $I_1=\{1, 2, \cdots, k\}$, $I_2=\{k+1, k+2, \cdots, n\}$ and*

$$\operatorname{Re}\{\lambda_i(t)-\lambda_j(t)\}\neq 0, \quad i\in I_1, \quad j\in I_2, \tag{1.2}$$

for all $t\in J$;

The second set of assumptions is:

(B1) *$A(t)$ is in $C^2(J_\infty)$ and periodic with period $b-a$;*
(B2) *$Q(t, \varepsilon)$ is in $C^1(J_\infty\times\mathscr{S}_\varepsilon(c))$ and periodic in t with period $b-a$;*
(B3) (1.2) *holds for all $t\in J_\infty$, $i\in I_1$, $j\in I_2$.*

For the non-periodic case, on a finite interval J, we shall prove:

Theorem A. *Under the assumptions* (A1), (A2) *and* (A3), *there exists an n by n matrix $T(t, \varepsilon)$, nonsingular and in class C^1 for $t\in J$, $\varepsilon\in\mathscr{S}_\varepsilon(c_1)$, $(c_1<c)$, such that the transformation*

(1.3) $$y=T(t,\varepsilon)w$$

reduces (E_1) *to*

(E_2) $$\varepsilon^h\frac{dw}{dt}=\begin{pmatrix}B_1(t,\varepsilon) & 0\\ 0 & B_2(t,\varepsilon)\end{pmatrix}w,$$

where $B_1(t,\varepsilon)$ *and* $B_2(t,\varepsilon)$ *are* k *by* k *and* $(n-k)$ *by* $(n-k)$ *matrices, respectively, in* $C^1(J\times\mathscr{S}_\varepsilon(c_1))$ *such that* $\lim B_1(t,\varepsilon)$ *and* $\lim B_2(t,\varepsilon)$, *as* ε *tends to* 0 *in* $\mathscr{S}_\varepsilon(c_1)$, *have eigenvalues* $\{\lambda_1(t),\lambda_2(t),\cdots,\lambda_k(t)\}$ *and* $\{\lambda_{k+1}(t),\lambda_{k+2}(t),\cdots,\lambda_n(t)\}$ *respectively.*

For the periodic case, we shall prove

Theorem B. *Under the assumptions* (B1), (B2) *and* (B3), *there exists an* n *by* n *matrix* $\hat{T}(t,\varepsilon)$, *nonsingular and in class* C^1 *for* $t\in J_\infty$, $\varepsilon\in\mathscr{S}_\varepsilon(c_1)$, $(c_1<c)$, *periodic in* t *with period* $b-a$, *such that the transformation*

(1.4) $$y=\hat{T}(t,\varepsilon)w$$

reduces (E_1) *to* (E_2) *where* $B_1(t,\varepsilon)$ *and* $B_2(t,\varepsilon)$ *are* k *by* k *and* $(n-k)$ *by* $(n-k)$ *matrices, respectively, in* $C^1(J_\infty\times\mathscr{S}_\varepsilon(c_1))$, *periodic in* t *with period* $b-a$, *such that* $\lim B_1(t,\varepsilon)$ *and* $\lim B_2(t,\varepsilon)$, *as* ε *tends to* 0 *in* $\mathscr{S}_\varepsilon(c_1)$, *have eigenvalues* $\{\lambda_1(t),\lambda_2(t),\cdots,\lambda_k(t)\}$ *and* $\{\lambda_{k+1}(t),\lambda_{k+2}(t),\cdots,\lambda_n(t)\}$ *respectively.*

The proof of Theorem A is given in § 2 and § 3, and that of Theorem B is in § 4.

The block-diagonalization process, as stated in these theorems, is an important method in the study of ordinary differential equations, not only for asymptotic expansions of the solutions, also for the boundary value problems, for example, see W. Wasow [11]. In order to obtain such simplification transformation, one usually has to shrink the domain of the independent variable. Under a rather restrictive condition $H(\delta)$ or $H^*(\delta)$, and applying a result of Y. Sibuya [10], B. L. J. Braaksma [2] successfully overcomes such deficiency. However, his result is not applicable to a systems with the relaxed assumptions (A1), (A2) and to a periodic system as stated in Theorem B.

As we can see in the proofs of these two theorems, not much difference is needed in handling periodic and non-periodic cases. We shall employ a recent result of H. Gingold [8] to prove these theorems. Furthermore, Theorem A and our proof can be extended to the equations of a complex independent variable, and will be studied in § 5 under a condition (K), similar to $H(\delta)$ imposed by B. L. J. Braaksma [2]. Also, the condition $H(\delta)$ of B. L. J. Braaksma [2] and our condition (K) are similar to a condition imposed by H. Gingold [7] in the study of simplification of a linear homogeneous system with moving singularities.

§ 2. Initial reductions and a non-linear differential equations.

First, we shall use the following result proved by H. Gingold [8]. This result also provides a *constructive mean* for an essential transformation to be used.

Theorem 2.1. (i) *Under the assumptions* (A1) *and* (A3), *there exists a non-singular matrix* $T_1(t)$ *in* $C^2(J)$ *such that*

$$T_1^{-1}(t)A(t)T_1(t)=A_1(t)\oplus A_2(t), \tag{2.1}$$

with $\lambda_1(t), \cdots, \lambda_k(t)$ *the eigenvalues of* $A_1(t)$ *and* $\lambda_{k+1}(t), \cdots, \lambda_n(t)$ *the eigenvalues of* $A_2(t)$.

(ii) *Under the assumptions* (B1) *and* (B3), $T_1(t)$ *in* (2.1) *is in* $C^2(J_\infty)$ *and periodic with period* $b-a$.

We shall prove Theorem A first. Applying Theorem 2.1, put

$$y=T_1(t)z. \tag{2.2}$$

Then z satisfies the differential equation

$$\varepsilon^h z'=(A_1(t)\oplus A_2(t))z+\tilde{q}(\varepsilon)\tilde{Q}(t, \varepsilon)z, \qquad \left('=\frac{d}{dt}\right) \tag{2.3}$$

where $\tilde{q}(\varepsilon)$ and $\tilde{Q}(t, \varepsilon)$ satisfy the analytic properties of $q(\varepsilon)$ and $Q(t, \varepsilon)$, and

$$\lim \tilde{q}(\varepsilon)=0, \qquad \text{as } \varepsilon \text{ tends to 0 in } \mathscr{S}_\varepsilon(c). \tag{2.4}$$

Let

$$\tilde{Q}(t, \varepsilon)=\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \tag{2.5}$$

where Q_{11}, Q_{12}, Q_{21} and Q_{22} are k by k, k by $n-k$, $n-k$ by k and $n-k$ by $n-k$ matrices, respectively. Put

$$z=(I_n+P(t, \varepsilon))w, \tag{2.6}$$

where I_n is the n by n identity matrix, $P(t, \varepsilon)$ has the form

$$P(t, \varepsilon)=\begin{pmatrix} 0 & P_{12} \\ P_{21} & 0 \end{pmatrix} \tag{2.7}$$

with P_{12} and P_{21} being k by $n-k$ and $n-k$ by k matrices. If $B_1(t, \varepsilon)$, $B_2(t, \varepsilon)$, $P_{12}(t, \varepsilon)$ and $P_{21}(t, \varepsilon)$ satisfy the following equations:

$$\begin{cases} B_1(t, \varepsilon)=A_1(t)+\tilde{q}(\varepsilon)Q_{11}(t, \varepsilon)+\tilde{q}(\varepsilon)Q_{12}(t, \varepsilon)P_{21}(t, \varepsilon), \\ B_2(t, \varepsilon)=A_2(t)+\tilde{q}(\varepsilon)Q_{22}(t, \varepsilon)+\tilde{q}(\varepsilon)Q_{21}(t, \varepsilon)P_{12}(t, \varepsilon), \end{cases} \tag{2.8}$$

$$(2.9)\qquad \begin{cases} \varepsilon^h P_{12}' = \tilde{q}(\varepsilon)Q_{12} + (A_1 + \tilde{q}(\varepsilon)Q_{11})P_{12} - P_{12}(A_2 + \tilde{q}(\varepsilon)Q_{22}) - \tilde{q}(\varepsilon)P_{12}Q_{21}P_{12}, \\ \varepsilon^h P_{21}' = \tilde{q}(\varepsilon)Q_{21} + (A_2 + \tilde{q}(\varepsilon)Q_{22})P_{21} - P_{21}(A_1 + \tilde{q}(\varepsilon)Q_{11}) - \tilde{q}(\varepsilon)P_{21}Q_{12}P_{21}, \end{cases}$$

then the transformation (2.6) takes the differential equation (2.3) into (E_3). Furthermore if $P_{12}(t, \varepsilon)$ and $P_{21}(t, \varepsilon)$ are obtained by (2.9), then $B_1(t, \varepsilon)$ and $B_2(t, \varepsilon)$ are given by (2.8).

By "stacking" up the entries of P_{12} and P_{21}, (e.g. see R. Bellman [1], p. 125), we obtain an N-column vector, $N=2k(n-k)$, that must satisfy the quadratic differential equation for V:

$$(2.10)\qquad \varepsilon^h V' = C(t)V + \tilde{q}(\varepsilon)H(t, \varepsilon)V + \tilde{q}(\varepsilon)\begin{pmatrix} V^T L_1 V \\ \vdots \\ V^T L_N V \end{pmatrix} + \tilde{q}(\varepsilon)G(t, \varepsilon),$$

where $C(t)$ is an N by N matrix having the property (A1), $H(t, \varepsilon)$, $L_l(t, \varepsilon)$, $(l=1, 2, \cdots, N)$ and $G(t, \varepsilon)$ are N by N matrices having the property (A2). Furthermore, $C(t)$ has the eigenvalues $\mu_1(t), \cdots, \mu_N(t)$ consisting of $\lambda_i(t)-\lambda_j(t)$ with $i \in I_1$, $j \in I_2$ or vice versa. Thus, there is a positive integer r, such that

$$(2.11)\qquad \operatorname{Re}\mu_1(t), \cdots, \operatorname{Re}\mu_r(t)<0;\ \operatorname{Re}\mu_{r+1}(t), \cdots, \operatorname{Re}\mu_N(t)>0 \qquad \text{for all } t \in J.$$

Applying Theorem 2.1 again, there is an N by N nonsingular matrix $T_2(t)$ in $C^2(J)$, such that

$$(2.12)\qquad T_2^{-1}(t)C(t)T_2(t) = \tilde{C}_1(t) \oplus \tilde{C}_2(t)$$

where $\tilde{C}_1(t)$ is an r by r matrix in $C^2(J)$ having eigenvalues $\mu_1(t), \cdots, \mu_r(t)$, and $\tilde{C}_2(t)$ is an $N-r$ by $N-r$ matrix in $C^2(J)$ having eigenvalues $\mu_{r+1}(t), \cdots, \mu_N(t)$. Put

$$(2.13)\qquad V = T_2(t)S.$$

Then, S satisfies a differential equation

$$(2.14)\qquad \varepsilon^h S' = (\tilde{C}_1(t) \oplus \tilde{C}_2(t))S + \tilde{q}_1(\varepsilon)\tilde{H}(t, \varepsilon)S + \tilde{q}(\varepsilon)\begin{pmatrix} S^T \tilde{L}_1 S \\ \vdots \\ S^T \tilde{L}_N S \end{pmatrix} + \tilde{q}(\varepsilon)\tilde{G}(t, \varepsilon),$$

where $\tilde{H}(t, \varepsilon)$, $\tilde{L}_l(t, \varepsilon)$, $(l=1, 2, \cdots, N)$, and $\tilde{G}(t, \varepsilon)$ are N by N matrices having the property (A2), and $\tilde{q}_1(\varepsilon)$ has the same analytic property as that of $\tilde{q}(\varepsilon)$ and satisfies (2.4).

Now let $S_{1H}(t, a, \varepsilon)$ and $S_{2H}(t, b, \varepsilon)$ be the fundamental matrices of

$$(2.15)\qquad \varepsilon^h S_{iH}' = \tilde{C}_i(t)S_{iH}, \qquad i=1, 2,$$

satisfying

(2.16) $$S_{1H}(a, a, \varepsilon)=I_r, \qquad S_{2H}(b, b, \varepsilon)=I_{N-r}.$$

Let

(2.17) $$\begin{cases} S_{1H}(t, s, \varepsilon)=S_{1H}(t, a, \varepsilon)S_{1H}^{-1}(s, a, \varepsilon), \\ S_{2H}(t, s, \varepsilon)=S_{2H}(t, b, \varepsilon)S_{2H}^{-1}(s, b, \varepsilon). \end{cases}$$

For an n by n matrix A, let $\|A\|_1$ denote some proper norm of A. Let $\varepsilon=|\varepsilon|e^{i\theta}$. By applying results of W. A. Coppel [5], K. W. Chang and W. A. Coppel [4], L. Flatto and N. Levinson [6], and J. J. Levin [9], K. W. Chang [3] proved the exponential dichotomy:

(2.18) $$\begin{cases} \|S_{1H}(t, s, \varepsilon)\|_1 \leq K_1 \exp\left\{-\dfrac{k_1}{|\varepsilon|^h}(t-s)\cos h\theta\right\}, & \text{for } b\geq t\geq s\geq a, \\ \|S_{2H}(t, s, \varepsilon)\|_1 \leq K_2 \exp\left\{-\dfrac{k_2}{|\varepsilon|^h}(s-t)\cos h\theta\right\}, & \text{for } b\geq s\geq t\geq a, \end{cases}$$

where k_1, k_2, K_1 and K_2 are suitable positive constants independent of ε. Denote

(2.19) $$S_H(t, s, \varepsilon)=S_{1H}(t, s, \varepsilon)\oplus S_{2H}(t, s, \varepsilon).$$

Then, (2.14) is equivalent to the following integral equation

(2.20) $$\begin{aligned} S(t, \varepsilon)=&(S_{1H}(t, a, \varepsilon)\oplus S_{2H}(t, b, \varepsilon))\begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix}+\frac{\tilde{q}(\varepsilon)}{\varepsilon^h}\int S_H(t, s, \varepsilon)\tilde{G}(s, \varepsilon)ds \\ &+\frac{\tilde{q}_1(\varepsilon)}{\varepsilon^h}\int S_H(t, s, \varepsilon)\tilde{H}(s, \varepsilon)S(s, \varepsilon)ds+\frac{\tilde{q}(\varepsilon)}{\varepsilon^h}\int S_H(t, s, \varepsilon)\begin{pmatrix} S^T\tilde{L}_1 S \\ \vdots \\ S^T\tilde{L}_N S \end{pmatrix}ds \end{aligned}$$

where y_1^0 and y_2^0 are suitable r-column and $N-r$ column vectors and the integrals are from a to t for the first r components and from t to b for the last $N-r$ components.

§ 3. Completion of proof for Theorem A.

Note first, that by (2.18), we have

(3.1) $$\begin{cases} \displaystyle\int_a^t \|S_{1H}(t, s, \varepsilon)\|_1\, ds \leq K_1 \int_a^t \exp\left\{-\frac{k_1}{|\varepsilon^h|}(t-s)\cos h\theta\right\}ds, \\ \displaystyle\int_t^b \|S_{2H}(t, s, \varepsilon)\|_1\, ds \leq K_2 \int_t^b \exp\left\{-\frac{k_2}{|\varepsilon|^h}(s-t)\cos h\theta\right\}ds \end{cases}$$

where the right hand sides are both $0(|\varepsilon|^h)$, since $|h\theta|\leq\delta\pi/2$. Denote $\|S\|_2=\max\|S\|_1$, for $a\leq t\leq b$. Set the right hand side of (2.20) by $\mathscr{F}S$, and choose $y_1^0=0$, $y_2^0=0$. Then, there exist positive constants m, f, l and a non-negative function $u(\varepsilon)$ with

(3.2) $$\lim_{\varepsilon\to 0} u(\varepsilon)=0, \qquad \varepsilon \in \mathscr{S}_\varepsilon(c),$$

such that

(3.3) $$\|\mathscr{F}S\|_2 \leq u(\varepsilon)[m+f\|S\|_2+l\|S\|_2^2].$$

Define $p(\lambda;\varepsilon)$ by

(3.4) $$p(\lambda;\varepsilon)=u(\varepsilon)[m+f\lambda+l\lambda^2].$$

Then, it is easy to see that $p(\lambda;\varepsilon)$ is monotonic increasing for $\lambda\geq 0$, and that the equation

(3.5) $$p(\lambda;\varepsilon)-\lambda=0$$

has a positive root $\lambda_1(\varepsilon)=0(u(\varepsilon))$ for sufficiently small ε, which satisfies $\lim \lambda_1(\varepsilon)=0$ as ε tend to 0 in $\mathscr{S}_\varepsilon(c)$.

Now, we can see that

(3.6) $$\|\mathscr{F}S\|_2 \leq \lambda_1(\varepsilon) \qquad \text{if } \|S\|_2\leq \lambda_1(\varepsilon),$$

and furthermore, $\mathscr{F}S$ is a contraction mapping. As a matter of fact, by (3.3) and (3.4)

(3.7) $$\|\mathscr{F}S\|_2 \leq p(\|S\|_2;\varepsilon)\leq p(\lambda_1;\varepsilon)=\lambda_1(\varepsilon).$$

Furthermore, for $\|S\|_2\leq\lambda_1(\varepsilon)$, $\|W\|_2\leq\lambda_1(\varepsilon)$,

(3.8) $$\begin{aligned}\mathscr{F}S-\mathscr{F}W=&\frac{\tilde{q}_1(\varepsilon)}{\varepsilon^h}\int S_H(t,s,\varepsilon)\tilde{H}(s,\varepsilon)[S-W]ds\\ &+\frac{\tilde{q}(\varepsilon)}{\varepsilon^h}\int S_H(t,s,\varepsilon)\left[\begin{pmatrix}(S^T-W^T)\tilde{L}_1S\\ \vdots\\ (S^T-W^T)\tilde{L}_NS\end{pmatrix}+\begin{pmatrix}W^T\tilde{L}_1(S-W)\\ \vdots\\ W^T\tilde{L}_N(S-W)\end{pmatrix}\right]ds,\end{aligned}$$

where the integrals are taken in the same way as for (2.20). By (3.1), there exist positive numbers M and L such that

(3.9) $$\|\mathscr{F}S-\mathscr{F}W\|_2\leq u(\varepsilon)[M+NL\lambda_1(\varepsilon)]\,\|S-W\|_2.$$

By (3.2), we can choose c_1 sufficiently small that $\mathscr{F}S$ is a contraction mapping for t in J and ε in $\mathscr{S}_\varepsilon(c_1)$.

Thus, we have the solution S of (2.20) satisfying

(3.10) $$\|S(t,\varepsilon)\|_2=0(u(\varepsilon))$$

and consequently, there exists a solution V of (2.10) satisfying

(3.11) $$\|V(t, \varepsilon)\|_2 = 0(u(\varepsilon)).$$

Hence, the matrices P_{12} and P_{21} in (2.7) satisfy

(3.12) $$\|P_{12}\|_2 = 0(u(\varepsilon)), \qquad \|P_{21}\|_2 = 0(u(\varepsilon)).$$

Therefore, by choosing c_1 small enough, we have

(3.13) $$\|P(t, \varepsilon)\|_2 < \|I\|_1,$$

for (t, ε) in $J \times \mathscr{S}_\varepsilon(c_1)$, and thus $T(t, \varepsilon)$ in (1.3) is nonsingular in $J \times \mathscr{S}_\varepsilon(c_1)$.

§ 4. Proof of Theorem B.

In order to prove Theorem B, applying the second part of Theorem 2.1, we get an equation similar to (2.14) with $\tilde{C}_1(t)$, $\tilde{C}_2(t)$, $\tilde{H}$, $\tilde{L}_i$ and $\tilde{G}$ being periodic in t of period $b-a$, or equivalently, an integral equation similar to (2.20). Let

(4.1) $$\mathscr{G}(t, \varepsilon; S) = \frac{\tilde{q}(\varepsilon)}{\varepsilon^h} \int S_H(t, s, \varepsilon)\tilde{G}(s, \varepsilon)ds + \frac{\tilde{q}_1(\varepsilon)}{\varepsilon^h} \int S_H(t, s, \varepsilon)\tilde{H}(s, \varepsilon)S(s, \varepsilon)ds + \frac{\tilde{q}(\varepsilon)}{\varepsilon^h} \int S_H(t, s, \varepsilon) \begin{pmatrix} S^T \tilde{L}_1 S \\ \vdots \\ S_H^T \tilde{L}_N S \end{pmatrix} ds.$$

In order that the solution of (2.20) is periodic in t of period $b-a$, y_1^0, y_2^0 and S should satisfy

(4.2) $$(S_{1H}(t, a, \varepsilon) \oplus S_{2H}(t, b, \varepsilon)) \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix} + \mathscr{G}(t, \varepsilon; S) = (S_{1H}(t-a+b, a, \varepsilon) \oplus S_{2H}(t-a+b, b, \varepsilon)) \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix} + \mathscr{G}(t-a+b, \varepsilon; S).$$

In particular, for $t=a$ and using (2.16),

(4.3) $$[(I_r - S_{1H}(b, a, \varepsilon)) \oplus (S_{2H}(a, b, \varepsilon) - I_{N-r})] \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix} = \mathscr{G}(b, \varepsilon; S) - \mathscr{G}(a, \varepsilon; S).$$

Now, put

(4.4) $$Z(t, a, b, \varepsilon) = (S_{1H}(t, a, \varepsilon) \oplus S_{2H}(t, b, \varepsilon))[(I_r - S_{1H}(b, a, \varepsilon)) \oplus (S_{2H}(a, b, \varepsilon) - I_{N-r})]^{-1},$$

and let

(4.5) $$S_0(t, \varepsilon) = Z(t, a, b, \varepsilon)[\mathscr{G}(b, \varepsilon; 0) - \mathscr{G}(a, \varepsilon; 0)] + \mathscr{G}(t, \varepsilon; 0),$$

and

(4.6) $$S_{\nu+1}(t,\varepsilon)=Z(t,a,b,\varepsilon)[\mathscr{G}(b,\varepsilon;S_\nu)-\mathscr{G}(a,\varepsilon;S_\nu)]+\mathscr{G}(t,\varepsilon;S_\nu),\qquad \nu=0,1,\cdots.$$

Similar to (3.3), there exist positive numbers m_1, f_1 and l_1 such that

(4.7) $$\|\mathscr{G}(t,\varepsilon;S)\|_2\leq u(\varepsilon)[m_1+f_1\|S\|_2+l_1\|S\|_2^2]$$

and, similar to (3.6), consequently,

(4.8) $$\|\mathscr{G}(t,\varepsilon;S)\|_2\leq\hat{\lambda}(\varepsilon)\qquad \text{if } \|S\|_2\leq\hat{\lambda}(\varepsilon)$$

where $\hat{\lambda}(\varepsilon)$ is the positive root of

(4.9) $$u(\varepsilon)[m_1+f_1\hat{\lambda}+l_1\hat{\lambda}^2]-\hat{\lambda}=0$$

satisfying $\hat{\lambda}(\varepsilon)=0(u(\varepsilon))$ for sufficiently small ε. Moreover, similar to (3.9), there exist positive numbers M_1 and L_1 such that

(4.10) $$\|\mathscr{G}(t,\varepsilon;S)-\mathscr{G}(t,\varepsilon;W)\|_2\leq u(\varepsilon)\{M_1+NL_1\hat{\lambda}(\varepsilon)\}\|S-W\|_2.$$

By (2.18), if $\hat{c}$ is chosen sufficiently small,

(4.11) $$\|Z(t,a,b,\varepsilon)\|_2\leq 1\qquad \text{for } t\in J_\infty,\ \varepsilon\in\mathscr{S}_\varepsilon(\hat{c}).$$

Now, let

(4.12) $$S(t,\varepsilon)=S_0(t,\varepsilon)+\sum_{\nu=0}^{\infty}[S_{\nu+1}(t,\varepsilon)-S_\nu(t,\varepsilon)].$$

By (4.10) and (4.11), if $\tilde{c}$ is chosen sufficiently small, there exist positive numbers M_2 and L_2 such that

(4.13) $$\|S_{\nu+1}(t,\varepsilon)-S_\nu(t,\varepsilon)\|_2\leq u(\varepsilon)\{M_2+NL_2\hat{\lambda}(\varepsilon)\}\|S_\nu(t,\varepsilon)-S_{\nu-1}(t,\varepsilon)\|_2$$

for $t\in J_\infty$, $\varepsilon\in\mathscr{S}_\varepsilon(\tilde{c})$, $\nu=0,1,2,\cdots$. Furthermore, by (3.2), we can choose c_1 so small that

(4.14) $$u(\varepsilon)\{M_2+NL_2\hat{\lambda}(\varepsilon)\}<1\qquad \text{for } \varepsilon\in\mathscr{S}_\varepsilon(c_1).$$

Thus, the series (4.12) converges uniformly for $t\in J_\infty$ and $\varepsilon\in\mathscr{S}_\varepsilon(c_1)$ and satisfies

(4.15) $$S(t,\varepsilon)=Z(t,a,b,\varepsilon)[\mathscr{G}(b,\varepsilon;S(t,\varepsilon))-\mathscr{G}(a,\varepsilon;S(t,\varepsilon))]+\mathscr{G}(t,\varepsilon;S(t,\varepsilon)).$$

Moreover, due to (4.2), (4.3) and (4.4), $S(t,\varepsilon)$ is periodic with period $b-a$.

§ 5. Complex equations.

Let t be a complex variable and D be a simply-connected domain with a piecewise smooth boundary C in the t-plane. Assume that

(C1) *$A(t)$ is holomorphic and bounded in D;*

(C2) *$Q(t, \varepsilon)$ is holomorphic and bounded in (t, ε) for t in D and ε in $\mathscr{S}_\varepsilon(c)$;*

(C3) *(1.2) holds for all $t \in D$, $i \in I_1$, $j \in I_2$.*

In addition, we assume

(K) *There exist two points a and b on C and positive constants α_1, α_2 and $\hat{\delta}$ ($\hat{\delta}<\pi/2$), such that every point $\hat{t}$ in D can be connected with a and b by a smooth Jordan curve in D*

(5.1) $$\Gamma_{\hat{t}}: t=t(\tau), \qquad 0\leq\tau\leq 1,$$

satisfying

(5.2) $$t(0)=a, \quad t(1)=b, \quad t(\xi)=\hat{t}, \quad (0<\xi<1),$$

(5.3) $$0<\alpha_1\leq\left|\frac{dt}{d\tau}\right|\leq\alpha_2, \qquad \text{for every } \Gamma_{\hat{t}}.$$

(5.4) $$-\frac{\pi}{2}+\hat{\delta}\leq\arg\left\{[\lambda_i(t(\tau))-\lambda_j(t(\tau))]\frac{dt}{d\tau}\right\} -h\arg\varepsilon\leq\frac{\pi}{2}-\hat{\delta}, \qquad (\text{mod } 2\pi)$$

or

(5.5) $$\frac{\pi}{2}+\hat{\delta}\leq\arg\left\{[\lambda_i(t(\tau))-\lambda_j(t(\tau))]\frac{dt}{d\tau}\right\} -h\arg\varepsilon\leq\frac{3\pi}{2}-\hat{\delta}, \qquad (\text{mod } 2\pi)$$

for every $\Gamma_{\hat{t}}$, $0\leq\tau\leq 1$, $i \in I_1$, $j \in I_2$ or vice versa, $\varepsilon \in \mathscr{S}_\varepsilon(c)$.

Similar to Theorem A, we will have

Theorem C. *Under the assumptions* (C1), (C2), (C3) *and* (K), *there exists an n by n matrix $\tilde{T}(t, \varepsilon)$ nonsingular and holomorphic in (t, ε) for t in D, ε in $\mathscr{S}_\varepsilon(c_1)$, $(0<c_1<c)$ such that the transformation*

(5.6) $$y=\tilde{T}(t, \varepsilon)w$$

reduces (E_1) *to* (E_2), *where $B_1(t, \varepsilon)$ and $B_2(t, \varepsilon)$ are k by k and $(n-k)$ by $(n-k)$ matrices, respectively, holomorphic in (t, ε) for t in D and ε in $\mathscr{S}_\varepsilon(c_1)$ such that* lim $B_1(t, \varepsilon)$ *and* lim $B_2(t, \varepsilon)$ *as ε tends to* 0 *in $\mathscr{S}_\varepsilon(c_1)$, have eigenvalues $\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_k(t)\}$ and $\{\lambda_{k+1}(t), \lambda_{k+2}(t), \cdots, \lambda_n(t)\}$, respectively.*

Remarks. 1. Since for every t in D there exists a curve Γ_t satisfying condition (K) the domain D actually consists of all such curves.

2. Suppose that D is a simply connected domain bounded by conjugate circular arcs C_1 and C_2 connecting the origin $t=0$ and $t=b$, $(b>0)$ such that their tangents at $t=0$ are the straight lines $\arg t=\pm\gamma$, $(0<\gamma<\pi/2)$, respectively. If the condition (1.2) is satisfied for all $t \in D$, then for any $\hat{t} \in D$, $(\operatorname{Im}\hat{t}\neq 0)$, $\Gamma_{\hat{t}}$ can be taken to be the circular arc passing through $t=0$, $t=\hat{t}$ and $t=b$. For $\hat{t}$ on $\overline{0b}$, $\Gamma_{\hat{t}}$ is taken to be the line segment $\overline{0b}$.

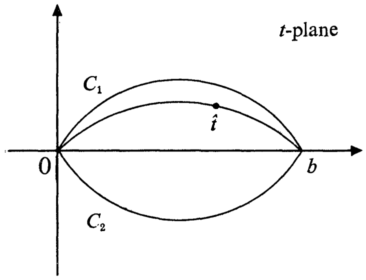

Fig. 1

Let the tangent line of $\Gamma_{\hat{t}}$ at $t=0$ be $\arg t=\alpha$, $(|\alpha|<\gamma)$. A simple geometric argument will show that

$$(5.7) \qquad \Gamma_{\hat{t}}\colon t(\tau)=\frac{b\sin\alpha\tau}{\sin\alpha}e^{i\alpha(1-\tau)}, \qquad 0\leq\tau\leq 1.$$

Then, $t(\tau)$ is analytic for both α and τ, $(-\gamma\leq\alpha\leq\gamma,\ 0\leq\tau\leq 1)$, and

$$(5.8) \qquad \frac{dt}{d\tau}=\frac{b\,\alpha}{\sin\alpha}e^{i\alpha(1-2\tau)}.$$

Since

$$(5.9) \qquad 1\leq\frac{\alpha}{\sin\alpha}\leq\frac{\gamma}{\sin\gamma} \qquad \text{for } |\alpha|\leq\gamma<\frac{\pi}{2},$$

thus, the condition (K) is satisfied for all $\Gamma_{\hat{t}}$, $\hat{t}\in D$ if γ, δ and $\hat{\delta}$ are small enough; namely, $\arg t$ and $\arg\varepsilon$ are small enough for $t\in D$ and $\varepsilon\in\mathscr{S}_{\varepsilon}(c)$.

§ 6. Proof of Theorem C.

Let t be a point in D and $\Gamma_{\hat{t}}$ be the curve given by (5.1) and satisfies the condition (K). Note first that on $\Gamma_{\hat{t}}$, (E_1) can be written as

$$(E_3) \qquad \varepsilon^h\frac{dy}{d\tau}=\left\{A(t(\tau))\frac{dt}{d\tau}-q(\varepsilon)\,Q(t(\tau),\varepsilon)\frac{dt}{d\tau}\right\}y, \qquad 0\leq\tau\leq 1.$$

Since $\lambda_i(t(\tau))dt/d\tau$ $(i=1, 2, \cdots, n)$ are eigenvalues of $A(t(\tau))dt/d\tau$, the conditions (A1)–(A3) are satisfied. Thus, by applying Theorem A, there is a matrix $T_1(\tau, \varepsilon)$ such that

$$y=T_1(\tau, \varepsilon)w \tag{6.1}$$

reduces (E_3) to

$$\varepsilon^h \frac{dw}{d\tau}=\begin{pmatrix} \tilde{B}_1(\tau, \varepsilon) & 0 \\ 0 & \tilde{B}_2(\tau, \varepsilon)\end{pmatrix} w \tag{6.2}$$

with $T_1(\tau, \varepsilon)$, $\tilde{B}_1(\tau, \varepsilon)$, $\tilde{B}_2(\tau, \varepsilon)$ satisfying the properties described in Theorem A. By (5.3), there exists the inverse function, $\tau=\tau(t)$, of $t=t(\tau)$. Replacing τ by $\tau(t)$ in (6.1) and (6.2) and denoting

$$\tilde{T}(t, \varepsilon)=T_1(\tau(t), \varepsilon), \quad B_j(t, \varepsilon)=\tilde{B}_j(\tau(t), \varepsilon)\frac{dt}{d\tau}, \quad (j=1, 2), \tag{6.3}$$

we get the desirable transformation (5.6) on Γ_t.

To show the existence of the transformation (5.6) for all t in D, note that D consists of all curves Γ_t satisfying condition (K). Moreover on each curve, the transformation (6.1) is obtained by solving the integral equation (2.20). Similar to (2.11), $\mu_l(t)=\lambda_i(t)-\lambda_j(t)$ ($i \in I_1$, $j \in I_2$ or vice versa) satisfies (5.4) for $l=1, 2, \cdots, r$, and satisfies (5.5) for $l=r+1, r+2, \cdots, N$. Rewrite (2.2) as

$$S(t, \varepsilon)=S_0(t, \varepsilon)+\mathscr{H}S \tag{6.4}$$

where

$$S_0(t, \tau)=(S_{1H}(t, a, \varepsilon)\oplus S_{2H}(t, b, \varepsilon))\begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix}+\frac{\tilde{q}(\varepsilon)}{\varepsilon^h}\int S_H(t, s, \varepsilon)\tilde{G}(s, \varepsilon)ds \tag{6.5}$$

and

$$\mathscr{H}S=\frac{\tilde{q}(\varepsilon)}{\varepsilon^h}\int S_H(t, s, \varepsilon)\tilde{H}(s, \varepsilon)S(s, \varepsilon)ds+\frac{\tilde{q}(\varepsilon)}{\varepsilon^h}\int S_H(t, s, \varepsilon)\begin{pmatrix} S^T\tilde{L}_1 S \\ \vdots \\ S^T\tilde{L}_N S\end{pmatrix}ds \tag{6.6}$$

with the integrals being taken from a to t along Γ_t for the first r components and from t to b along Γ_t for the last $N-r$ components. Note first that $\mathscr{H}\,0=0$. Furthermore, the constants k_1, k_2, K_1 and K_2 in the inequalities (2.18) and (3.1) depend only on the bounds of $A(t)$ and $\lambda_i(t)-\lambda_j(t)$, ($i \in I_1$, $j \in I_2$), therefore these inequalities hold for all t in D and all curves Γ_t. Moreover, since $S_H(t, s, \varepsilon)$ is holomorphic for t in D, s in D, ε in $\mathscr{S}_\varepsilon(c)$, and $\tilde{H}(t, \varepsilon)$ and $\tilde{L}_j(t, \varepsilon)$, $(j=1, 2, \cdots, N)$, are holomorphic for t in D and ε in $\mathscr{S}_\varepsilon(c)$, $\mathscr{H}S(t, \varepsilon)$ is holomorphic for t in D, ε in $\mathscr{S}_\varepsilon(c)$ if $S(t, \varepsilon)$ is holomorphic for t in D and ε in $\mathcal{S}_\varepsilon(c)$. Similarly, since $\tilde{G}(t, \varepsilon)$ is holomorphic for t in D and ε in $\mathcal{S}_\varepsilon(c)$, $S_0(t, \varepsilon)$ is holomorphic for t in D and ε in $\mathcal{S}_\varepsilon(c)$. Consequently, the paths of integration in (6.5) and (6.6) can be replaced by a rectifiable curve as long as it connects a and t and stays in D for the first r components and it connects t and b and stays in D for the last $N-r$ components.

Now, let

$$S_{\nu+1}(t, \varepsilon)=S_0(t, \varepsilon)+\mathcal{H}S_\nu, \qquad \nu=0, 1, 2, \cdots. \tag{6.7}$$

Since $S_0(t, \varepsilon)$ is holomorphic for t in D and ε in $\mathcal{S}_\varepsilon(c)$, $S_\nu(t, \varepsilon)$ is holomorphic for t in D and ε in $\mathcal{S}_\varepsilon(c)$ ($\nu=1, 2, \cdots$). Define

$$S(t, \varepsilon)=S_0(t, \varepsilon)+\sum_{\nu=0}^{\infty} [S_{\nu+1}(t, \varepsilon)-S_\nu(t, \varepsilon)]. \tag{6.8}$$

By a similar argument as that in § 3 for the operator $\mathcal{F}$, we know that $\mathcal{H}S$ is a contraction mapping for t in D and ε in $\mathcal{S}_\varepsilon(c_1)$ if c_1 is chosen small enough; namely, there is a constant β independent of ε, $0<\beta<1$, such that

$$\|\mathcal{H}S_\nu-\mathcal{H}S_{\nu-1}\|_2\leq\beta\,\|S_\nu-S_{\nu-1}\|_2, \qquad \nu=1, 2, 3, \cdots \tag{6.9}$$

for t in D and ε in $\mathcal{S}_\varepsilon(c_1)$. Here $\|S\|_2=\sup\|S\|_1$, $t\in D$, for some norm $\|S\|_1$ of the vector S. Thus the series in (6.8) converges uniformly for t in D and ε in $\mathcal{S}_\varepsilon(c_1)$, if c_1 is sufficiently small.

Thus, the integral equation (2.20) has a holomorphic solution $S(t, \varepsilon)$ for t in D and $\varepsilon\in\mathcal{S}_\varepsilon(c_1)$.

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