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On a Class of Nonlinear Functional Pseudoparabolic Problems

By

A. G. KARTSATOS and M. E. PARROTT

(University of South Florida, U.S.A.)

§ 1. Introduction-preliminaries.

Let X, Y be two real Banach spaces. The pseudoparabolic problem

$$\text{(E)}\qquad \begin{aligned}&(Bu)'+Au=G(t, u), \qquad t\in[0, T),\\ &u(0)=u_0\end{aligned}$$

was recently studied by Brill [2]. Brill assumed that A, B are linear, closed and densely defined operators with domains $D(A)$, $D(B)\subset D(A)$ in X and ranges in Y. His method for the existence of solutions of (E) was actually based on the reduction of (E) to the problem

$$\text{(E)}_b\qquad \begin{aligned}&v'+AB^{-1}v=G(t, B^{-1}v), \qquad t\in[0, T),\\ &v(0)=Bu_0.\end{aligned}$$

Naturally, it is assumed here that $u_0\in D(B)$. Since AB^{-1}: $Y\rightarrow Y$ is a bounded linear operator in (E)_b, solutions of the problem (E)_b can be obtained from the variation of constants formula

$$v(t)=X(t)Bu_0+\int_0^t X(t)X^{-1}(s)G(s, B^{-1}v(s))ds,$$

where $X(t)$, $t\in[0, T)$, is the unique operator-valued solution of the problem

$$X'(t)+AB^{-1}X(t)=0, \qquad X(0)=I.$$

The symbol I denotes the identity operator on Y.

Our purpose in this paper is the study of functional pseudoparabolic problems of the form

$$\text{(FDE)}\qquad \begin{aligned}&(B(t)u(t))'+A(t, u_t)u(t)=G(t, u_t), \qquad t\in[0, T),\\ &B(t)u(t)=\phi(t), \qquad t\in[-r, 0],\end{aligned}$$

where r is a fixed positive constant. Let C denote the space of all continuous functions $\psi: [-r, 0]\rightarrow Y$. The function ϕ in (FDE) is a fixed function in C and, given

u: $[-r, T) \to Y$, continuous, u_t denotes the function in C defined by: $u_t(s) = u(t+s)$, $s \in [-r, 0]$. We also assume that for each $(t, \psi) \in [0, T) \times C$, $B(t)$ and $A(t, \psi)$ are linear, closed and densely defined operators while G: $[0, T) \times C \to Y$ is continuous.

In order to obtain local strong solutions of (FDE), we apply the Schauder-Tychonov theorem to an operator associated with (FDE) under additional conditions on A, B, G, which include the compactness of the operators $B^{-1}(t)$: $Y \to X$. This is one of the conditions imposed by Brill [2] in the case of the problem (E). An extendability result is also given for the local solutions thus obtained. To illustrate the results of the paper we include an application involving partial elliptic differential operators of even order.

For a variety of results concerning pseudoparabolic problems the reader is referred to the book of Showalter [13] as well as the paper of Brill [2], and the references therein. For finite dimensional differential equations of the type (FDE) ($B \equiv I$, $r=0$), the reader is referred to the papers [4–6] of the first author and their references. For problems (FDE) with $B(t) \equiv I$, $r=0$, in Banach spaces, we cite the papers of Becker [1], Kartsatos [7], Kartsatos and Parrott [8], Kato [9], Murphy [11], Ward [15], and the book of Friedman [3, Section 16]. For functional problems (FDE) with A and B constant operators the paper of Lightbourne and Rankin [10] is our only reference.

We denote by $L(X, Y)$ the Banach space of all bounded linear operators T: $X \to Y$. The symbol $\|\cdot\|$ denotes the norm of all the spaces and bounded linear operators considered in this paper. It also denotes the sup-norm of any bounded continuous function.

Unless otherwise specified, $L(X, Y)$ will be assumed to be associated with the uniform operator topology. Let $M \subset X$ and A: $M \to Y$ be given. Then A is said to be "compact" if A is continuous on M and maps bounded subsets of M onto relatively compact subsets of Y. Let $J \subset R$ $(=(-\infty, \infty))$ be a bounded interval and let the operator A: $J \times X \to Y$ be given. We say that $A(t, u)$ is continuous in t X-uniformly in u, if for every bounded subset M of X we have

$$\lim_{\substack{t \to t_0 \\ t \in J}} \sup_{u \in M} \|A(t, u) - A(t_0, u)\| = 0$$

for every $t_0 \in J$. We denote by $C(J, Y)$ the space of all continuous functions f: $J \to Y$ with the sup-norm. If $J=[-r, 0]$, then this space has already been denoted by C. The symbol "$\to$" ("$\rightharpoonup$") denotes strong (weak) convergence. The space Y^2 will be assumed to have the norm $\|(x, y)\| = \|x\| + \|y\|$ with respect to which it is a Banach space.

§ 2. Nonlinear delay equations. Local existence.

In this section we establish a local existence result for the delay problem

$$\text{(DE)} \quad \begin{aligned} &u'(t)+A(t, u(t), u(t-r))u(t)=G(t, u(t), u(t-r)), \quad t \in [0, T), \\ &u(t)=\phi(t), \qquad t \in [-r, 0], \end{aligned}$$

where the operator $A(t, u, v)w$ is linear and bounded in w. This result and its method of proof are then used in order to obtain a corresponding local existence theorem (Theorem 2) for the pseudoparabolic problem

$$\text{(FDE)}_1 \quad \begin{aligned} &(B(t)u(t))'+A(t, u(t), u(t-r))u(t)=G(t, u(t), u(t-r)), \qquad t \in [0, T), \\ &B(t)u(t)=\phi(t), \qquad t \in [-r, 0]. \end{aligned}$$

We are going to need the following conditions:

(S_1) $A(t, u, v) \in L(Y, Y)$ for every $(t, u, v) \in [0, T) \times Y^2$. Moreover, $A(t, u, v)$ is compact in (u, v) and continuous in t Y^2-uniformly in (u, v).

(S_2) $G: [0, T) \times Y^2 \to Y$ and $G(t, u, v)$ is compact in (u, v) and continuous in t Y^2-uniformly in (u, v).

(S_3) ϕ is a given Lipschitzian in C.

Theorem 1. *Let the assumptions* (S_1)–(S_3) *be satisfied. Then there exists a number* $T_1 \in (0, T)$ *and a continuous function* $u: [-r, T_1] \to Y$ *such that*: $u(t)=\phi(t)$, $t \in [-r, 0]$, *and* $u(t)$ *is strongly continuously differentiable and satisfies the differential equation* (DE) *on* $[0, T_1]$.

Proof. Given $f \in C([-r, T_1], Y)$, for some $T_1 \in (0, T)$, we let $X_f(t)$, $t \in [0, T_1]$, $X_f(0)=I$, denote the fundamental operator of the equation

$$(1) \qquad x'+A(t, f(t), f(t-r))x=0, \qquad x(0)=\phi(0).$$

Then $X_f \in C([0, T_1], L(Y, Y))$ and X_f is the unique continuously differentiable solution of the problem

$$(2) \qquad X'+A(t, f(t), f(t-r))X=0, \quad X(0)=I, \quad t \in [0, T_1].$$

Moreover, $X_f^{-1} \in C([0, T_1], L(Y, Y))$ and X_f^{-1} is the unique continuously differentiable solution of the problem

$$(3) \qquad X'-XA(t, f(t), f(t-r))=0, \quad X(0)=I, \quad t \in [0, T_1].$$

Now, let $L>\|\phi(0)\|+\|\phi\|$ be given. Assume that T_0 is a fixed constant in $[0, T)$ and let $T_1 \in (0, T_0)$, $f: [-r, T_1] \to Y$ be such that $\|f\| \le L$. Then f can be extended to the interval $[-r, T_0]$ by setting $\bar{f}(t)=f(t)$, $t \in [-r, T_1]$, $\bar{f}(t)=f(T_1)$, $t \in (T_1, T_0]$. We have

$$\begin{aligned} K(T_1) &= \sup_{\substack{t\in[0,T_1]\\ f\in C([-r,T_1],Y),\|f\|\le L}} \|A(t,f(t),f(t-r))\| \\ &\le \sup_{\substack{t\in[0,T_0]\\ f\in C([-r,T_1],Y),\|f\|\le L}} \|A(t,\bar f(t),\bar f(t-r))\| \\ &\le \sup_{\substack{t\in[0,T_0]\\ u\in Y,\|u\|\le L\\ v\in Y,\|v\|\le L}} \|A(t,u,v)\| < +\infty. \end{aligned}$$

The boundedness of $\|A(t, u, v)\|$ above follows from the fact that the operator $(t, u, v) \to A(t, u, v)$ is compact on the set $[0, T_0]\times\{u \in Y; \|u\|\le L\}^2$. Thus, since $K(T_1)T_1\to 0$ as $T_1\to 0^+$, there exists $T_1 \in (0, T_0]$ such that

$$e^{K(T_1)T_1}\|\phi(0)\|+\|\phi\|<L.$$

Since we also have

$$K_2(T_1)=T_1 \sup_{\substack{t\in[0,T_1]\\ \|u\|\le L\\ \|v\|\le L}} \|G(t, u, v)\|\to 0 \qquad \text{as } t\to 0^+,$$

we may (and do) choose T_1 so that

$$e^{K(T_1)T_1}\|\phi(0)\|+\|\phi\|+e^{2K(T_1)T_1}K_2(T_1)\le L. \tag{4}$$

We also set $K=K(T_1)$, $K_1=e^{K(T_1)T_1}$, $K_2=K_2(T_1)$. Integrating the equations (2) and (3) from 0 to $t\le T_1$ and then applying Gronwall's inequality, we obtain

$$\|X_f(t)\|,\quad \|X_f^{-1}(t)\|\le K_1,\quad t \in [0, T_1]$$

for every $f \in C([-r, T_1], Y)$ with $\|f\|\le L$. Given such an f, consider the problem

$$\begin{aligned} &u'(t)+A(t,f(t),f(t-r))u(t)=G(t,f(t),f(t-r)), \qquad t \in [0, T_1],\\ &u(t)=\phi(t), \qquad t \in [-r, 0]. \end{aligned} \tag{5}$$

Since the operators $A(t, u, v)$, $G(t, u, v)$ are continuous on $[0, T_1]\times Y^2$, this problem has a unique solution $u_f(t)$, $t \in [-r, T_1]$, such that

$$u_f(t)=X_f(t)\phi(0)+\int_0^t X_f(t)X_f^{-1}(s)G(s,f(s),f(s-r))ds \tag{6}$$

for every $t \in [0, T_1]$. Let $M=\{f \in C([-r, T_1], Y); f(t)=\phi(t),\ t \in [-r, 0],\ \|f\|\le L$ and $\|f(t)-f(t')\|\le N|t-t'|,\ t, t' \in [0, T_1]\}$, where $N=[(K+(1/T_1))K_1K_2+K\|\phi(0)\|]K_1$. $M\ne\emptyset$ because the function $f\colon [-r, T_1]\to Y$ with $f(t)=\phi(t)$, $t \in [-r, 0]$ and $f(t)=\phi(0)$, $t \in [0, T_1]$, belongs to M. Let $V\colon M\to C([-r, T_1], Y)$ be the operator that maps $f \in M$ into u_f. In order to apply the Schauder-Tychonov theorem on M, we first show that $VM\subset M$. In fact, given $f \in M$, equation (6) implies

$$\|u_f(t)\| \le \|X_f(t)\| \, \|\phi(0)\| + \int_0^t \|X_f(t)\| \, \|X_f^{-1}(s)\| \, \|G(s, f(s), f(s-r))\| \, ds$$

$$\le K_1 \|\phi(0)\| + K_1^2 \int_0^{T_1} \sup_{\substack{s \in [0,T_1] \\ u \in Y, \|u\| \le L \\ v \in Y, \|v\| \le L \\ t \in [0,T_1]}} \|G(s, u, v)\| \, ds$$

$$= K_1 \|\phi(0)\| + K_1^2 K_2,$$

while $\|u_f(t)\| = \|\phi(t)\| \le \|\phi\|$, $t \in [-r, 0]$. It follows that $\|u_f\| \le L$. Now, given t, t' $\in [0, T_1]$, we have

$$\|u_f(t) - u_f(t')\| \le \|X_f(t) - X_f(t')\| \, \|\phi(0)\|$$

$$+ \|X_f(t) - X_f(t')\| \left| \int_0^t \|X_f^{-1}(s)\| \, \|G(s, f(s), f(s-r))\| \, ds \right|$$

$$+ \|X_f(t')\| \left| \int_{t'}^t \|X_f^{-1}(s)\| \, \|G(s, f(s), f(s-r))\| \, ds \right|$$

$$\le N |t - t'|.$$

Here we have used the appraisal

$$\|X_f(t) - X_f(t')\| \le \left| \int_t^{t'} \|A(s, f(s), f(s-r))\| \, \|X_f(s)\| \, ds \right|$$

$$\le K K_1 |t - t'|.$$

It follows that $VM \subset M$. To show that V is continuous, let $f_n, f \in M$ be given with $\|f_n - f\| \to 0$ as $n \to \infty$. Then from

$$\|X_{f_n}(t) - X_f(t)\| \le \int_0^t \|A(s, f_n(s), f_n(s-r)) X_{f_n}(s) - A(s, f(s), f(s-r)) X_f(s)\| \, ds$$

$$\le \int_0^t \|A(s, f_n(s), f_n(s-r)) - A(s, f(s), f(s-r))\| \, \|X_{f_n}(s)\| \, ds$$

$$+ \int_0^t \|A(s, f(s), f(s-r))\| \, \|X_{f_n}(s) - X_f(s)\| \, ds$$

and Gronwall's inequality, we get

$$\|X_{f_n}(t) - X_f(t)\| \le K_1^2 \int_0^{T_1} \|A(s, f_n(s), f_n(s-r)) - A(s, f(s), f(s-r))\| \, ds$$

for every $t \in [0, T_1]$, which shows that $\|X_{f_n} - X_f\| \to 0$ as $n \to \infty$. Similarly, using (3), we have that $\|X_{f_n}^{-1} - X_f^{-1}\| \to 0$ as $n \to \infty$. From the continuity of G we also get that $G(t, f_n(t), f_n(t-r))$ converges uniformly to $G(t, f(t), f(t-r))$ on $[0, T_1]$. Using these facts, we can now obtain from (6), and the corresponding equation with f replaced by f_n, that $\|u_{f_n} - u_f\| \to 0$ as $n \to \infty$. Consequently, V is continuous on M.

Before we show that VM is a relatively compact set, we first prove that the operators $A_1: M \to C([0, T_1], L(Y, Y))$, $G_1: M \to C([0, T_1], Y)$ with

$$(A_1 f)(t) = A(t, f(t), f(t-r)), \qquad (G_1 f)(t) = G(t, f(t), f(t-r)),$$

are compact. To this end, let $\{f_n\}$ be a sequence in M. We first observe that $\|A(t, f_n(t), f_n(t-r))\| \leq K$, $n = 1, 2, \cdots$, $t \in [0, T_1]$. Given $t, t_0 \in [0, T_1]$, we find

$$\begin{aligned}
&\|A(t, f_n(t), f_n(t-r)) - A(t_0, f_n(t_0), f_n(t_0-r))\| \\
&\quad \leq \|A(t, f_n(t), f_n(t-r)) - A(t_0, f_n(t), f_n(t-r))\| \\
&\qquad + \|A(t_0, f_n(t), f_n(t-r)) - A(t_0, f_n(t_0), f_n(t_0-r))\| \\
&\quad \leq \sup_{\substack{u \in Y, \|u\| \leq L \\ v \in Y, \|v\| \leq L}} \|A(t, u, v) - A(t_0, u, v)\| \\
&\qquad + \|A(t_0, f_n(t), f_n(t-r)) - A(t_0, f_n(t_0), f_n(t_0-r))\|,
\end{aligned}$$

which, by the Y^2-uniform continuity of $A(t, u, v)$ and the uniform Lipschitz continuity of the functions f_n on $[-r, T_1]$, implies the equicontinuity of the set of all functions $F_n(t) \equiv A(t, f_n(t), f_n(t-r))$, $t \in [0, T_1]$, $n = 1, 2, \cdots$. Now, let $t_0 \in [0, T_1]$ be given. Then since $\{f_n(t_0)\}$ is a bounded sequence, the compactness of $A(t_0, u, v)$ in (u, v) implies the relative compactness of the set $\{A(t_0, f_n(t_0), f_n(t_0-r))\}$. Consequently, the operator A_1 is compact. A similar argument proves the compactness of G_1. Thus, given a sequence $\{f_n\} \subset M$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $B \in C([0, T_1], L(Y, Y))$, $H \in C([0, T_1], Y)$ such that $A(t, f_{n_k}(t), f_{n_k}(t-r)) \to B(t)$, $G(t, f_{n_k}(t), f_{n_k}(t-r)) \to H(t)$ uniformly on $[0, T_1]$ as $k \to \infty$. Let $X(t)$ denote the fundamental operator for the problem

$$x' + B(t)x = 0, \qquad x(0) = \phi(0).$$

Then

$$u(t) = X(t)\phi(0) + \int_0^t X(t)X^{-1}(s)H(s)ds, \qquad t \in [0, T_1],$$

is the unique solution of the problem

$$x' + B(t)x = H(t), \quad x(0) = \phi(0), \quad t \in [0, T_1].$$

It is easy to see now that $X_{f_{n_k}}(t) \to X(t)$ and $X_{f_{n_k}}^{-1}(t) \to X^{-1}(t)$ uniformly on $[0, T_1]$. It follows that $u_{f_{n_k}}(t) \to u(t)$ uniformly on $[0, T_1]$. Since $u_{f_{n_k}}(t) = \phi(t)$, $t \in [-r, 0]$, we have actually shown the compactness of VM. Any fixed point of the operator V in M is a solution to our problem.

We now consider the pseudoparabolic delay problem $(\text{FDE})_1$. We show that this problem can be reduced to the problem (DE), whose local solution has been shown to exist in Theorem 1, under the following assumptions.

(S_4) For each $(t, u, v) \in [0, T)\times X^2$, $A(t, u, v)$ is linear, closed and densely defined with domain $D(A)$ (independent of (t, u, v)) in X and range in Y.

(S_5) For each $t \in [-r, T)$, $B(t)$ is linear, closed and densely defined with domain $D(B)$ (independent of t) in $D(A)$ and range Y. Moreover, for each $t \in [-r, T)$, $B^{-1}(t): Y \to X$ exists and is compact while $B^{-1}(t)u$ is continuous in t for each $u \in Y$.

(S_6) For each $(t, u, v) \in [0, T)\times Y^2$, $A(t, B^{-1}(t)u, B^{-1}(t-r)v)B^{-1}(t) \in L(Y, Y)$ (boundedness follows from assumptions (S_4), (S_5)) and is continuous in (t, u, v) with its continuity in t Y^2-uniform in (u, v).

(S_7) For each $(t, u, v) \in [0, T)\times Y^2$, $G(t, B^{-1}(t)u, B^{-1}(t-r)v) \in Y$ and is continuous in (t, u, v) with its continuity in t Y^2-uniform in (u, v). $\phi: [-r, 0]\to Y$ is a Lipschitz continuous function.

Theorem 2. *Let the assumptions* (S_4)–(S_7) *be satisfied. Then the problem* $(\mathrm{FDE})_1$ *has a local solution in the following sense: there exists a number* $T_1 \in (0, T)$ *and a continuous function* $u(t)$, $t \in [-r, T_1]$, *such that* $u(t) \in D(B)$, $t \in [-r, T_1]$, $B(t)u(t)=\phi(t)$, $t \in [-r, 0]$, $B(t)u(t)$ *is strongly continuously differentiable on* $[0, T_1]$ *and the delay equation in* $(\mathrm{FDE})_1$ *is satisfied on* $[0, T_1]$.

Proof. We consider the problem

$$(7)\qquad \begin{aligned} &v'(t)+A(t, B^{-1}(t)v(t), B^{-1}(t-r)v(t-r))B^{-1}(t)v(t)\\ &\qquad =G(t, B^{-1}(t)v(t), B^{-1}(t-r)v(t-r)), \qquad t \in [0, T),\\ &v(t)=\phi(t), \qquad t \in [-r, 0]. \end{aligned}$$

If $v(t)$, $t \in [-r, T_1]$, is a solution of this problem in the sense of Theorem 1, then $u(t)=B^{-1}(t)v(t)$, $t \in [-r, T_1]$, satisfies the conclusion of the theorem. In order to solve (7) locally, it suffices to show that the operators $A(t, B^{-1}(t)u, B^{-1}(t-r)v)B^{-1}(t)$, $G(t, B^{-1}(t)u, B^{-1}(t-r)v)$ are compact in (u, v). We prove this property only for the first operator. A similar proof covers the operator $G(t, B^{-1}(t)u, B^{-1}(t-r)v)$. Let $\{u_n\}, \{v_n\}\subset Y$ be two bounded sequences. Since the sets $\{B^{-1}(t_0)u_n\}$, $\{B^{-1}(t_0-r)v_n\}$ are relatively compact, there exist subsequences $\{u_{n_k}\}$, $\{v_{n_k}\}$ of $\{u_n\}$, $\{v_n\}$, respectively, such that $B^{-1}(t_0)u_{n_k}\to u_0 \in X$, $B^{-1}(t_0-r)v_{n_k}\to v_0 \in X$. Since

$$A(t_0, B^{-1}(t_0)u, B^{-1}(t_0-r)v)B^{-1}(t_0)$$

is continuous in (u, v), we have that

$$A(t_0, B^{-1}(t_0)u_{n_k}, B^{-1}(t_0-r)v_{n_k})B^{-1}(t_0)\to A(t_0, u_0, v_0)B^{-1}(t_0).$$

Consequently, $A(t_0, B^{-1}(t_0)u, B^{-1}(t_0-r)v)B^{-1}(t_0)$ is compact in (u, v).

§ 3. Functional pseudoparabolic problems. Reflexive spaces Y.

In this section we establish the existence of local solutions of the problem (FDE) in the introduction for spaces Y which are reflexive. We state the following conditions:

(S_8) Y is reflexive and for each $(t, \psi) \in [0, T)\times C_0$ ($C_0 = C([-r, 0], X)$), $A(t, \psi)$ is linear, closed and densely defined with domain $D(A)\subset X$ (independent of (t, ψ)) and range in Y.

(S_9) For each $t \in [-r, T)$, $B(t)$ is linear, closed and densely defined on $D(B)\subset D(A)$ (independent of t) and range Y. Moreover, $B^{-1}(t)$: $Y\to X$ exists and is compact for each $t \in [-r, T)$ while $B^{-1}(\cdot)$: $[-r, T)\to L(Y, X)$ is continuous.

(S_{10}) The operator $A(\cdot, \cdot)B^{-1}(\cdot)$: $[0, T)\times C_0\to L(Y, Y)$ (boundedness follows from Assumptions (S_8) and (S_9)) is continuous in $(t, \psi) \in [0, T)\times C_0$ with its continuity in t C_0-uniform in ψ. Moreover, it maps bounded subsets of $[0, T)\times C_0$ onto bounded subsets of $L(Y, Y)$.

(S_{11}) G: $[0, T)\times C_0\to Y$ is continuous in $(t, \psi) \in [0, T)\times C_0$ with its continuity in t C_0-uniform in ψ. $\phi \in C$ is a fixed function satisfying a Lipschitz condition on $[-r, 0]$.

In what follows, $B^{-1}u$ denotes the function h with $h(t)\equiv B^{-1}(t)u(t)$.

Theorem 3. *Let the conditions* (S_8)–(S_{11}) *be satisfied. Then the problem* (FDE) *has at least one solution* $u(t)$, $t \in [0, T_1]$, *for some* $T_1 \in (0, T)$, *with the following properties*: $u(t)$ *is continuous and* $B(t)u(t)$ *is strongly continuously differentiable on* $[0, T_1]$. *Moreover, the functional equation in* (FDE) *is satisfied on* $[0, T_1]$.

Proof. We consider the equation

$$v'(t)+A(t, (B^{-1}f)_t)v(t)=G(t, (B^{-1}f)_t), \qquad v_0=\phi \tag{8}$$

for a function f: $[-r, T_0]\to Y$, where T_0 is a fixed number in $(0, T)$. Let $T_1 \in (0, T_0]$ be given. Let $C(t)=C([-r, t], Y)$, $t>0$. Every function $\psi \in C(T_1)$ can be extended to a function $\bar{\psi}$ on the interval $[-r, T_0]$ by letting $\bar{\psi}(t)=\psi(t)$, $t \in [-r, T_1]$, and $\bar{\psi}(t)=\psi(T_1)$ for $t \in (T_1, T_0]$. Let $L>\|\phi(0)\|+\|\phi\|$ and $A_1(t, \psi)=A(t, (B^{-1}\psi)_t)B^{-1}(t)$. We have

$$\begin{aligned} K(T_1)&= \sup_{\substack{t\in[0,T_1]\\ \psi\in C(T_1),\|\psi\|\le L}} \|A_1(t, \psi)\| \\ &\le \sup_{\substack{t\in[0,T_0]\\ \psi\in C(T_1),\|\psi\|\le L}} \|A_1(t, \bar{\psi})\| \\ &\le \sup_{\substack{t\in[0,T_0]\\ f\in C(T_0),\|f\|\le L}} \|A_1(t,f)\|<+\infty. \end{aligned} \tag{9}$$

The last inequality holds because $A_1(t, \psi)$ maps bounded subsets of $[0, T)\times C_0$ onto bounded subsets of $L(Y, Y)$. Similarly, we obtain

$$(10)\qquad \begin{aligned} K_2(T_1) &= T_1 \sup_{\substack{t\in[0,T_1]\\ \psi\in C(T_1),\|\psi\|\le L}} \|G(t, (B^{-1}\psi)_t)\| \\ &\le T_1 \sup_{\substack{t\in[0,T_0]\\ f\in C(T_0),\|f\|\le L}} \|G(t, (B^{-1}f)_t)\| < +\infty. \end{aligned}$$

The proof now follows as in Theorem 1 by letting $T_1 \in (0, T_0)$ be such that

$$e^{K(T_1)T_1}\|\phi(0)\|+\|\phi\|+e^{2K(T_1)T_1}K_2(T_1)\le L \quad \text{and} \quad X_f(t),$$

$t \in [0, T_1]$, be the fundamental operator corresponding to (8) with $X_f(0)=I$. Letting M be the set of the proof of Theorem 1, we only show here that the operator U that maps $f \in M$ into the function $A(t, (B^{-1}f)_t)B^{-1}(t) \in L(Y, Y)$, $t \in [0, T_1]$, is compact. A similar statement is true for the function G.

Let $\{f_n\}\subset M$ be given. Then the uniform Lipschitz continuity of $\{f_n\}$ implies that the functions $F_n(t)\equiv A(t, (B^{-1}f_n)_t)B^{-1}(t)$ are equicontinuous. Since they are also uniformly bounded, it remains to show that, given $t_0 \in [0, T_1]$, the set $S(t_0)=\{A(t_0, (B^{-1}f_n)_{t_0})B^{-1}(t_0); n=1, 2, \cdots\}$ is relatively compact in $L(Y, Y)$. To this end, fix $t_0 \in [0, T_1]$ and consider the functions $\bar{f}_n(s)=f_n(t_0+s)$, $s \in [-r, 0]$, $n=1, 2, \cdots$. The sequence $\{\bar{f}_n\}$ is uniformly bounded and equicontinuous on $[-r, 0]$. As such, and because Y is reflexive, there exists a subsequence of $\{f_n\}$, denoted again by $\{f_n\}$, and a weakly continuous function f: $[-r, T_1]\to Y$ such that $\bar{f}_n(s)\to\bar{f}(s)$ uniformly on $[-r, 0]$ (cf. Szep [14]). This means that for every $y^* \in Y^*$ the sequence $y^*(\bar{f}_n(s)-\bar{f}(s))$ converges to zero uniformly on $[-r, 0]$. In order to show that we also have $B^{-1}(t_0+s)\bar{f}_n(s)\to B^{-1}(t_0+s)\bar{f}(s)$ uniformly on $[-r, 0]$, we let $T(s)=B^{-1}(t_0+s)$, $g_n(s)=\bar{f}_n(s)-\bar{f}(s)$, and we assume that the contrary is true. Then

$$\max_{s\in[-r,0]} \|T(s)g_n(s)\|=\|T(s_n)g_n(s_n)\|\not\to 0$$

as $n\to\infty$. The maximum above is actually attained at some point $s_n \in [-r, 0]$ because the function $T(s)g_n(s)$ is continuous on the compact interval $[-r, 0]$ for every $n=1, 2, \cdots$. The sequence $\{s_n\}$ has a subsequence, denoted again by $\{s_n\}$, such that $s_n\to s_0 \in [-r, 0]$. We have

$$(11)\qquad \begin{aligned} \|T(s_n)g_n(s_n)\| &\le \|(T(s_n)-T(s_0))g_n(s_n)\|+\|T(s_0)g_n(s_n)\| \\ &\le \|T(s_n)-T(s_0)\|\,\|g_n(s_n)\|+\|T(s_0)g_n(s_n)\|. \end{aligned}$$

Since the sequence $\{g_n(s_n)\}$ is bounded and $T(s_n)\to T(s_0)$, the first term in the last member of the above inequality converges to zero as $n\to\infty$. Now, we observe that if $y^* \in Y^*$ is given, then $y^*(g_n(s))\to y^*(0)=0$ uniformly as $n\to\infty$. This, however, implies that for every $s_0 \in [-r, 0]$ and every sequence $\{s_n\}\subset[-r, 0]$ with $s_n\to s_0$

we have that $y^*(g_n(s_n))\to 0$ as $n\to\infty$. Consequently, for the chosen sequence $\{s_n\}$, $g_n(s_n)\to 0$ as $n\to\infty$. Since $T(s_0)$ is compact, we have that $T(s_0)g_n(s_n)\to 0$ as $n\to\infty$. Combining this with (11), we obtain that $T(s_n)g_n(s_n)\to 0$ as $n\to\infty$. Since we could have started with any subsequence of $\{s_n\}$ instead of $\{s_n\}$ itself, we have actually shown the following: every subsequence $\{s_{n_k}\}$ of $\{s_n\}$ contains a further subsequence $\{s_{n_{k'}}\}$ such that $T(s_{n_{k'}})g_{n_{k'}}(s_{n_{k'}})\to 0$ as $k'\to\infty$. This implies that $T(s_n)g_n(s_n)\to 0$, i.e., a contradiction.

Thus, $B^{-1}(t_0+s)f_n(t_0+s)\to B^{-1}(t_0+s)f(t_0+s)$ uniformly on $[-r, 0]$. This is equivalent to saying that $(B^{-1}f_n)_{t_0}\to(B^{-1}f)_{t_0}$ in the norm of C_0. Consequently,

$$A(t_0, (B^{-1}f_n)_{t_0})B^{-1}(t_0)\to A(t_0, (B^{-1}f)_{t_0})B^{-1}(t_0),$$

and the proof of the theorem is complete.

It should be noted at this point that if $B(t)\equiv I$ and $u_t(s)=u(t)$, $s\in[-r, 0]$, then Theorem 3 is an extension of the main result of Ward in [15]. Naturally, in this case $A(t, u)v$ is a bounded linear operator in v mapping X into $Y=X$. Ward considered weakly continuous operators $A(t, u)v$, $G(t, u)$ in separable reflexive Banach spaces X with A m-accretive in v. For several basic results concerning first order equations involving weakly continuous mappings, the reader is referred to Szep [14] and Zigler [16].

§ 4. Extendability.

Let $S(\cdot)\in C([-r, T], L(Y, X))$, $(Su)(t)=S(t)u(t)$ and consider

$$(\mathrm{E}_1)\qquad u'(t)+F(t, u(t), (Su)_t)=0, \qquad u_0=\phi.$$

Here, $F\colon R_+\times Y\times C_0\to Y$ is continuous and ϕ is a given function in C. Let $T\in(0, \infty]$ ($T\in(0, \infty)$) be fixed. By a solution of (E_1) on $[-r, T)$ ($[-r, T]$), we mean a function $u\colon[-r, T)\to Y$ ($u\colon[-r, T]\to Y$) which is strongly continuously differentiable and satisfies the functional equation in (E_1) on $[0, T)$ ($[0, T]$) while $u_0=\phi$. A solution $u(t)$, $t\in[-r, T)$, $0<T<\infty$, of (E_1) is called "extendable" to $T_1\in[T, \infty)$ if there exists another solution $y(t)$ of (E_1) on $[-r, T_1]$ such that $y(t)\equiv u(t)$, $t\in[-r, T)$. If $T_1=T$, then we obviously have $y(T)=\lim_{t\to T^-}u(t)$. A solution $u(t)$, $t\in[-r, T)$, $0<T\le+\infty$, is said to be "nonextendable" if either $T=+\infty$ or (in case $T<+\infty$) $u(t)$ is not extendable to T.

The following lemma will be used in order to show the extendability to arbitrary $T>0$ of the solution obtained in Theorem 3.

Lemma 1. *Let F map bounded subsets of $R_+\times Y\times C_0$ onto bounded subsets of Y. Let $u(t)$, $t\in[-r, T)$, $0<T<+\infty$, be a nonextendable solution of the problem* (E_1). *Then* $\limsup_{t\to T^-}\|u(t)\|=+\infty$.

Proof. Let $u(t)$, $t \in [-r, T)$, $0<T<+\infty$ be a nonextendable solution of (E_1) such that $\limsup_{t\to T^-} \|u(t)\|<+\infty$. Then since $u(t)$ it continuous, we have that $\|u(t)\|\leq K$, $t \in [-r, T)$. The boundedness of F implies that $\|u'(t)\|=\|F(t, u(t), (Su)_t)\|\leq K_1$, $t \in [0, T)$. Let $u(t)\to L \in Y$ as $t\to T^-$. This limit exists by the Lipschitz continuity of $u(t)$. It can be easily seen that $F(t, u(t), (Su)_t)\to F(T, \bar{u}(T), (S\bar{u})_T)$ as $t\to T^-$, where $\bar{u}(t)=u(t)$, $t \in [-r, T)$, $\bar{u}(t)=L$ at $t=T$. This implies easily that

$$\bar{u}(t)=\phi(0)-\int_0^t F(s, \bar{u}(s), (S\bar{u})_s)ds, \qquad t \in [0, T],$$

which contradicts the fact that $u(t)$ is nonextendable.

Theorem 4. *Let the conditions* (S_8)–(S_{11}) *be satisfied with* $T=+\infty$. *Assume, further, that there exist three locally* L^1 *functions* p_i: $R_+\to R_+$, $i=1, 2, 3$, *such that*

$$\|A(t, \psi)B^{-1}(t)\|\leq p_1(t), \qquad \|G(t, \psi)\|\leq p_2(t)\,\|\psi\|+p_3(t)$$

for every $(t, \psi) \in R_+\times C_0$. *Then every solution* $u(t)$, $t \in [-r, T_1]$, $0<T_1<+\infty$, *of the problem* (FDE) *can actually be defined on* $[-r, \infty)$.

Proof. Before we prove the theorem, we should show first that every solution $u(t)$ as above can be extended to a solution $\bar{u}(t)$, $t \in [-r, T)$, such that $T \in (T_1, \infty]$ and $\bar{u}(t)$ is nonextendable. To this end, consider the equations

$$(E_2) \qquad \begin{aligned} &v'(t)+A(t, (B^{-1}v)_t)B^{-1}(t)v(t)=G(t, (B^{-1}v)_t), \qquad t \in [0, T_1],\\ &v(t)=\phi(t), \qquad t \in [-r, 0],\end{aligned}$$

where $v(t)=B(t)u(t)$, $t \in [-r, 0]$. To prove our assertion, it suffices to obtain a non-extendable extension of $v(t)$ on $[-r, T)$, with T as above. This in turn will have been shown if we prove that $v(t)$ is extendable to a point T_2 to the right of T_1. Thus, we consider the problem

$$(E_3) \qquad \begin{aligned} &x'(t)+A(t, (B^{-1}x)_t)B^{-1}(t)x(t)=G(t, (B^{-1}x)_t), \qquad t \in [T_1, T_1+T)\\ &x(t)=v(t), \qquad t \in [T_1-r, T_1].\end{aligned}$$

In order to solve (E_3), we let $\tau=t-T_1$ and $x(t)=x(\tau+T_1)\equiv w(\tau)$, $\bar{B}(\tau)=B(\tau+T_1)$. Then ($E_3$) reduces to

$$(E_4) \qquad \begin{aligned} &w'(\tau)+A(\tau+T_1, (\bar{B}^{-1}w)_\tau)B^{-1}(\tau+T_1)w(\tau)=G(\tau+T_1, (\bar{B}^{-1}w)_\tau), \qquad \tau \in [0, T),\\ &w(\tau)=v(\tau+T_1), \qquad \tau \in [-r, 0].\end{aligned}$$

This problem has a solution $w(\tau)$ on $[0, T_2]$, $0<T_2<T$, by Theorem 3. It is easy to see now that the solution $v(t)$ is extendable to the point T_1+T_2. Let $[-r, T)$, $T_1<T<+\infty$, be the maximal interval of existence of the solution $u(t)$ as in the statement of the theorem. Then, integrating (E_2) (with $v(t)=B(t)u(t)$, $t \in [-r, T)$) from 0 to $t \in [0, T)$ and then applying Gronwall's inequality, we find that $v(t)$ is bounded on the interval $[-r, T)$. Since $v(t)$ is nonextendable, this violates the conclusion of Lemma 1 with $S(t)\equiv B^{-1}(t)$ and $F(t, u, (B^{-1}u)_t)\equiv A(t, (B^{-1}u)_t)B^{-1}(t)u-G(t, (B^{-1}u)_t)$ in (E_1).

§ 5. Application.

Let $\Omega\subset R^n$, $n\geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. Given an integer $m\geq 0$ and a real number $p \in (1, \infty)$ we denote by $L^p=(L^p(\Omega), \|\cdot\|_p)$, $W^{m,p}=(W^{m,p}(\Omega), \|\cdot\|_{m,p})$ the usual Sobolev spaces. We consider the following two elliptic operators

$$\mathscr{A}(x, t, u)v=\sum_{|\alpha|\leq 2l} b_\alpha(x, t, \xi(u))D^\alpha v,$$

$$\mathscr{B}(x)u=\sum_{|\alpha|\leq 2m} c_\alpha(x)D^\alpha u,$$

where $\xi(u)=\{D^\alpha u;\ |\alpha|\leq q\}$ (cf. notation in [12, p. 272] with $R^{N_m}=R^{d_0}$ or R^{d_1} below), q is defined below, and m, l are two positive integers with $l\leq m$. We assume that $(2m-1)p>n$ and we let $q=(2m-1)-(n/p)$. We also let

$$d_0=\sum_{|\alpha|\leq q} 1, \qquad d_1=\sum_{|\alpha|\leq 2m-1} 1.$$

The functions $b_\alpha\colon \bar{\Omega}\times R_+\times R^{d_0}\to R$, $|\alpha|\leq 2l$, $c_\alpha\colon \bar{\Omega}\to R$, $|\alpha|\leq 2m$, are continuous, uniformly bounded and such that

$$\left.\begin{aligned}&\sum_{|\alpha|=2l} b_\alpha(x, t, \xi)\eta_1^{\alpha_1}\cdots\eta_n^{\alpha_n}\neq 0\\ &\sum_{|\alpha|=2m} c_\alpha(x)\eta_1^{\alpha_1}\cdots\eta_n^{\alpha_n}\neq 0\end{aligned}\right\}\qquad \begin{aligned}&(x, t, \xi) \in \Omega\times R_+\times R^{d_0},\\ &0\neq(\eta_1, \cdots, \eta_n) \in R^n.\end{aligned}$$

Moreover, there exist constants $k_1, k_2\geq 0$ such that

$$|b_\alpha(x, t, \xi)-b_\alpha(x, t', \xi')|\leq k_1|t-t'|+k_2|\xi-\xi'|$$

for every $x \in \Omega$, $t, t' \in R_+$, $\xi, \xi' \in R^{d_0}$, where $|\xi|=\sum_{|\alpha|\leq q}|\xi_\alpha|$. The boundary operators $\{B_i\}_{i=1}^m$, $\{B_i^1\}_{i=1}^k$ and the spaces $W^{2m,p}(\Omega; \{B_i\}_{i=1}^m)$, $W^{2l,p}(\Omega; \{B_i\}_{i=1}^k)$ are defined in Section 4 of Brill [2] (see also Friedman [3, p. 74]). We are planning to solve the problem

$$\text{(PDE)}\qquad \begin{aligned}&(\partial/\partial t)(\mathscr{B}(x)u(x, t))+\mathscr{A}(x, t, u(x, t-r))u(x, t)\\ &\qquad =g(x, (D^\alpha u(x, t-r))_{|\alpha|\leq 2m-1}, u(x, t)), \qquad (x, t) \in \Omega\times(0, \infty),\\ &B_i u(x, t)=0, \quad (x, t) \in \partial\Omega\times(0, \infty), \quad 1\leq i\leq m,\\ &\mathscr{B}(x)u(x, t)=\phi(x, t), \qquad (x, t) \in \Omega\times[-r, 0].\end{aligned}$$

For the Nemytskii operator g we assume the following:

$g: \Omega \times R^{d_1} \times R \to R$ is continuous and there exist $h \in L^p$ and a constant $k_3 > 0$ such that

$$|g(x, \xi, u)| \leq h(x) + k_3 \sum_{|\alpha| \leq 2m-1} |\xi_\alpha|, \qquad (x, \xi, u) \in \Omega \times R^{d_1} \times R.$$

For the function ϕ in (PDE) we assume that $\phi(\cdot, t) \in L^p$, $t \in [-r, 0]$, and is Lipschitz continuous in t uniformly with respect to $x \in \Omega$.

Following Brill [2], we set $X = W^{2m-1,p}$, $Y = L^p$ and we define

$$(Bu)(x) = \sum_{|\alpha| \leq 2m} c_\alpha(x)(D^\alpha u)(x), \quad u \in D(B) = W^{2m,p}(\Omega; \{B_i\}_{i=1}^m), \quad x \in \Omega,$$

$$(A(t, u)v)(x) = \sum_{|\alpha| \leq 2l} b_\alpha(x, t, ((D^\alpha u)(x))_{|\alpha| \leq q})(D^\alpha v)(x), \qquad u \in W^{2m-1,p},$$

$$v \in D(A) = W^{2l,p}(\Omega; \{B_i^1\}_{i=1}^k) \cap X \supset D(B), \qquad (x, t) \in \Omega \times [0, \infty),$$

$$(G(u, v))(x) = g(x, ((D^\alpha u)(x))_{|\alpha| \leq 2m-1}, v(x)), \quad u, v \in W^{2m-1,p}, \quad x \in \Omega.$$

If we assume, in addition to the above, that the operator B is bijective, then the abstract problem $(\text{FDE})_1$, corresponding to the problem (PDE), will have at least one solution $u(t)$, $t \in [-r, \infty)$, by Theorems 2 and 4, provided that condition (S_6) is satisfied. In fact, all the other poperties of A, B, G required by Theorems 2 and 4 hold true in the present setting (cf. also Brill [2]).

In order to show that (S_6) holds, we fix $u_1, u_2 \in X$, $v \in D(A)$ and $t, t' \in [0, \infty)$. Then we have

$$\|[A(t, u_1) - A(t', u_2)]v\|_p$$
$$\leq \sum_{|\alpha| \leq 2l} \left[\int_\Omega |[b_\alpha(x, t, \xi(u_1(x))) - b_\alpha(x, t', \xi(u_2(x)))](D^\alpha v)(x)|^p \, dx\right]^{1/p}.$$

Fixing α, with $|\alpha| \leq 2l$, we have

$$\left[\int_\Omega [|b_\alpha(x, t, \xi(u_1(x))) - b_\alpha(x, t', \xi(u_2(x)))| \, |(D^\alpha v)(x)|]^p \, dx\right]^{1/p}$$
$$\leq \left[\int_\Omega ([k_1 |t - t'| + k_2 \sum_{|\beta| \leq q} |(D^\beta(u_1 - u_2))(x)|] \, |(D^\alpha v)(x)|)^p dx\right]^{1/p}$$
$$\leq k_1 |t - t'| \, \|D^\alpha v\|_p + k_2' d_0 \|u_1 - u_2\|_{2m-1,p} \|D^\alpha v\|_p$$
$$\leq k_4(|t - t'| + \|u_1 - u_2\|_{2m-1,p}) \|D^\alpha v\|_p,$$

where $k_4 = \max\{k_1, k_2' d_0\}$. Here we have used the fact that there exists a constant $k' = k'(\Omega)$ such that

$$\|D^\alpha(u_1 - u_2)\|_p \leq k' \|u_1 - u_2\|_{2m-1,p}, \qquad |\alpha| \leq q$$

(cf. [12, p. 58]). We have put $k_2' = k_2 k'$.

Summing up above, we obtain

$$\|[A(t, u_1)-A(t', u_2)]v\|_p \leq k_4(|t-t'|+\|u_1-u_2\|_{2m-1,p})\|v\|_{2l,p}$$
$$\leq k_4(|t-t'|+\|u_1-u_2\|_{2m-1,p})\|v\|_{2m-1,p}.$$

Thus, for $\bar{u}_1, \bar{u}_2, \bar{v} \in Y=L^p$, we get

$$\|[A(t, B^{-1}\bar{u}_1)-A(t', B^{-1}\bar{u}_2)]B^{-1}\bar{v}\|_p$$
$$\leq k_4(|t-t'|+\|B^{-1}\bar{u}_1-B^{-1}\bar{u}_2\|_{2m-1,p})\|B^{-1}\bar{v}\|_{2m-1,p}$$
$$\leq k_4(|t-t'|+\|B^{-1}\|\,\|\bar{u}_1-\bar{u}_2\|_p)\|B^{-1}\|\,\|\bar{v}\|_p,$$

which implies

$$\|A(t, B^{-1}\bar{u}_1)B^{-1}-A(t', B^{-1}\bar{u}_2)B^{-1}\| \leq k_4\|B^{-1}\|(|t-t'|+\|B^{-1}\|\,\|\bar{u}_1-\bar{u}_2\|_p).$$

This proves that (S_6) is satisfied.

It should be noted that the above existence result could be considerably strengthened with certain modifications in the hypotheses, e.g., Caratheodory conditions instead of continuity conditions, time varying B, G, local Lipschitzian b_α's instead of global ones, etc..

§ 6. Discussion.

The lack of compactness of the inverse $B^{-1}(t)$ in Theorems 2 and 3 can be compensated for by compactness assumptions involving the proper arguments of the operators A and G.

It would be interesting to see extensions of the above results to problems involving infinite delays as well as other boundary value problems on finite and infinite intervals.

If $B(t)\equiv I$ and u_t is replaced by $u(t-r)$ in (FDE), then Theorem 1 can be proved by assuming that $A(t, u)v$, $G(t, u)$ are continuous on their proper domains. In fact, using the method of steps, we can define $u(t)$ on $[0, r]$ as the unique solution $u_1(t)$ of

$$(E_\phi) \qquad u'(t)+A(t, \phi(t-r))u(t)=G(t, \phi(t-r))$$

with $u(0)=\phi(0)$. Then $u(t)$ can be defined on $[r, 2r]$ as the unique solution $u_2(t)$ of (E_{u_1}) with $u(r)=u_1(r)$, etc.. A similar remark covers Theorems 2–4.

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nuna adreso:
Department of Mathematics
University of South Florida
Tampa, Florida 33620
U.S.A.

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