

## Periodic Solutions of Functional Differential Equations with Large Delays

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### § 1. Introduction.

In this paper we are concerned with the existence of periodic solutions for the  $\omega$ -periodic functional differential equation

$$(E_1) \quad \dot{x}(t) = f(t, x_t)$$

with a delay  $r$  which is large comparing with the period  $\omega$ . As far as concerned with the  $\omega$ -periodic solutions of the difference-differential equations of the form  $f(t, x_t) = f(t, x(t), x(t-r))$  with a finite delay  $r$ , it is known that we may assume the condition  $0 \leq r < \omega$  without any loss of generalities. In fact, this can be easily seen by considering the auxiliary equation  $\dot{x}(t) = f(t, x(t), x(t-r_1))$  with  $r_1 = r - k\omega$ , where  $k$  is an integer and  $0 \leq r_1 < \omega$ , refer to Krasnosel'skii [8, p. 278]. The same reduction can be observed even for some integral equations with infinite delay  $\dot{x}(t) = Ax(t) +$

$$\int_{-\infty}^t B(t-s)x(s)ds + p(t),$$

whose  $\omega$ -periodic solution is not different from the one of

$$\dot{x}(t) = Ax(t) + \int_{t-\omega}^t H(t-s)x(s)ds + p(t), \quad H(s) = \sum_{k=0}^{\infty} B(s+k\omega),$$

refer to Grimmer [5],

Leitman and Mizel [9]. But such a reduction is not obvious for Equation (E<sub>1</sub>) in a general form, and in many theorems the condition  $r \leq \omega$  is posed as an important assumption, for example, refer to Theorems 37.1 and 37.2 in Yoshizawa [11], and Theorems 4.20 and 4.21 in Halanay [6], which fails to hold for any  $\omega > 0$  when  $r = \infty$ . In these theories, the condition  $r \leq \omega$  is assumed to ensure the complete continuity of the translation mapping  $T$  particularly in applying fixed point theorems for  $T$  to provide a periodic solution of  $\omega$ -periodic functional differential equations. On the other hand, several efforts have been made to drop the restriction  $r \leq \omega$  for the functional differential equations. Hale and Lopes [7] overcome the deficiency by imposing an additional condition on  $T$ . And further efforts are made for the functional differential equation with infinite delay, see Chow and Hale [1].

In this article, we shall show that the reduction in the above is always possible for a general  $\omega$ -periodic equation (E<sub>1</sub>). More precisely, we shall construct an auxiliary  $\omega$ -periodic equation

$$(E_2) \quad \dot{x}(t) = g(t, x_t)$$

with the delay  $\omega$  such that the  $\omega$ -periodic solutions of  $(E_1)$  are those of the auxiliary equation  $(E_2)$ . To obtain our results, we first consider some retractions from a Banach space of continuous functions onto some closed subsets, and define the auxiliary function  $g(t, \psi)$  for  $f(t, \phi)$  by using a suitable retraction so that the existence problem of  $\omega$ -periodic solutions for  $(E_1)$  can be reduced to that for  $(E_2)$  (Section 2). In Section 3, we prove a Razumikhin type theorem concerning the existence of  $\omega$ -periodic solutions of  $(E_1)$  by using a strongly convex Liapunov function defined by strengthening the conditions for a convex Liapunov function in [5]. Finally, we present an application of the results obtained in this section to an  $\omega$ -periodic functional differential equation.

Let  $R^+$  and  $R$  denote the intervals  $0 \leq t < \infty$ , and  $-\infty < t < \infty$ , respectively. For a given  $r$ ,  $0 < r \leq \infty$ ,  $C_r$  denotes the Banach space of continuous and bounded functions defined by

$$C_r = \{\phi: [-r, 0] \rightarrow R^n, \text{ continuous}\}, \quad 0 < r < \infty,$$

or

$$C_\infty = \{\phi: (-\infty, 0] \rightarrow R^n, \text{ continuous and bounded}\},$$

with the uniform norm,  $|\phi| = \sup \{|\phi(\theta)|: -r < \theta \leq 0\}$ , where  $|\cdot|$  denotes the usual Euclidean norm in  $R^n$ . For a given continuous function  $x(s)$ , the symbol  $x_t$  will denote the element of  $C_r$  such that  $x_t(\theta) = x(t + \theta)$ ,  $-r < \theta \leq 0$ .

Let  $f(t, \phi): R \times C_r \rightarrow R^n$  be a completely continuous function which is  $\omega$ -periodic in  $t$ , that is,  $f(t + \omega, \phi) = f(t, \phi)$  for all  $(t, \phi) \in R \times C_r$  and some positive constant  $\omega$ .

Consider a functional differential equation

$$(1.1) \quad \dot{x}(t) = f(t, x_t),$$

where  $\dot{\phantom{x}}$  denotes the right hand derivative.

## § 2. Retractions of $C_\omega$ .

It is well known that if  $S$  is a nonempty closed convex subset of a Banach space  $X$ , then there exists a retraction from  $X$  onto  $S$  (see Dugundji [3]). In this section, we shall define particular retractions from the Banach space  $C_\omega$  onto some closed subsets of  $C_\omega$ . Define a set  $S$  by

$$(2.1) \quad S = \{\psi \in C_\omega: \psi(-\omega) = \psi(0)\}.$$

Then we have the following lemma.

**Lemma 2.1.** *There exists a retraction  $\rho$  from  $C_\omega$  onto  $S$  such that  $\rho(\psi)(-\omega) = \psi(-\omega)$  and  $|\rho(\psi)| \leq |\psi|$ .*

*Proof.* Since  $S$  is a nonempty closed convex subset of  $C_\omega$ , by Dugundji's theorem, there exists a retraction  $\rho_0$  from  $C_\omega$  onto  $S$  (a simple example is:  $\rho_0(\psi)(\theta) = \psi(\theta) - (\theta/\omega)(\psi(0) - \psi(-\omega))$ ,  $-\omega \leq \theta \leq 0$ ). For any retraction  $\rho_0$  from  $C_\omega$  onto  $S$ , the mapping  $\rho_1: C_\omega \rightarrow S$  defined by

$$\rho_1(\psi)(\theta) = \rho_0(\psi)(\theta) - \rho_0(\psi)(-\omega) + \psi(-\omega), \quad -\omega \leq \theta \leq 0$$

is a retraction from  $C_\omega$  onto  $S$  such that  $\rho_1(\psi)(-\omega) = \psi(-\omega)$ . For such a retraction  $\rho_1$  from  $C_\omega$  onto  $S$ , the mapping  $\rho: C_\omega \rightarrow S$  defined by

$$\rho(\psi)(\theta) = \begin{cases} \rho_1(\psi)(\theta) & \text{if } |\rho_1(\psi)(\theta)| \leq |\psi| \\ \frac{|\psi|}{|\rho_1(\psi)(\theta)|} \rho_1(\psi)(\theta) & \text{if } |\rho_1(\psi)(\theta)| > |\psi| \end{cases}$$

is a retraction from  $C_\omega$  onto  $S$  such that  $\rho(\psi)(-\omega) = \psi(-\omega)$  and  $|\rho(\psi)| \leq |\psi|$ .

Let  $\Sigma$  be a closed set in  $R \times R^n$  satisfying the conditions

(2.2)  $\Sigma$  contains  $(t, 0)$  in its interior for all  $t \in R$ ,

and

(2.3)  $^* \{\lambda \in R^+ : (t, \lambda x) \in \partial \Sigma \cap A(t, x)\} \leq 1$  for each  $t \in R, x \in R^n$ ,

where  $A(t, x) = \{(t, \lambda x) : \lambda \in R^+\}$  and  $^*A$  denotes the number of the set  $A$ . For  $t \in R$  and  $x \in R^n$ , let  $\lambda(t, x): R \times R^n \rightarrow (0, 1]$  be a function defined in the following manner.

$$(2.4) \quad \begin{aligned} \lambda(t, x) &= 1 && \text{if } (t, x) \in \Sigma \\ (t, \lambda(t, x)x) &\in \partial \Sigma && \text{if } (t, x) \notin \Sigma. \end{aligned}$$

From (2.2) and (2.3), it is clear that such a  $\lambda$  is well-defined, and we have  $\lambda(t, x) < 1$  and  $(t, \lambda(t, x)x) \in \partial \Sigma$  whenever  $(t, x) \notin \Sigma$ . Moreover we have the following lemma.

**Lemma 2.2.** *Under the above assumptions on  $\Sigma$ , the function  $\lambda(t, x)$  on  $R \times R^n$  defined by (2.4) is continuous in  $(t, x)$ .*

*Proof.* Suppose that the conclusion is false. Then there exist  $\varepsilon_0 > 0$ ,  $(t, x) \in R \times R^n$ , and a sequence  $\{(t_k, x_k)\}$  in  $R \times R^n$  such that  $\{(t_k, x_k)\}$  converges to  $(t, x)$  and  $|\lambda(t_k, x_k) - \lambda(t, x)| \geq \varepsilon_0$ . Since  $\lambda(t_k, x_k) \in (0, 1]$  for all  $k$ , we may assume that  $\{\lambda(t_k, x_k)\}$  converges to some  $\lambda \in [0, 1] \setminus \{\lambda(t, x)\}$  by taking a subsequence if necessary. Then the sequence  $\{\lambda(t_k, x_k)x_k\}$  converges to  $\lambda x$ , and we have  $|\lambda - \lambda(t, x)| \geq \varepsilon_0$  and  $(t, \lambda x) \in \Sigma$ , since  $\Sigma$  is closed and  $(t_k, \lambda(t_k, x_k)x_k) \in \Sigma$  for all  $k$ . If  $\lambda = 1$ , that is,  $(t, x) \in \Sigma$ , then

we obtain  $\lambda(t, x) = 1$  from (2.4), and this contradicts  $|\lambda - \lambda(t, x)| \geq \varepsilon_0 > 0$ . In the case  $0 \leq \lambda < 1$ , we have  $(t, \lambda x) \in \partial\Sigma$ , since  $\partial\Sigma$  is closed and  $(t_k, \lambda(t_k, x_k)x_k) \in \partial\Sigma$  for all sufficiently large  $k$ . Thus we obtain  $(t, x) \notin \partial\Sigma$  and  $\lambda > 0$  by (2.2) and (2.3).

Now we prove that  $(t, x) \notin \Sigma$ . Suppose that this is false. Then  $N = \Sigma \setminus \partial\Sigma$  is an open neighborhood of  $(t, x)$ , and  $N_t = \{(t, y) : (t, y) \in N\}$  is an open neighborhood of  $(t, \lambda x) \in \partial\Sigma$ . Hence there exists a point  $(s, z) \in N \setminus \{(t, x)\}$  such that  $(s, \lambda z) \in N_t \setminus \Sigma$ . Let  $\kappa_1$  and  $\kappa_2$  be numbers such that  $\kappa_1 = \inf \{\kappa : (s, \mu z) \notin \Sigma \text{ for } \mu \in (\kappa, \lambda]\}$  and  $\kappa_2 = \sup \{\kappa : (s, \mu z) \notin \Sigma \text{ for } \mu \in [\lambda, \kappa)\}$ . Then we have  $\kappa_1 < \lambda < \kappa_2$ ,  $(s, \kappa_1 z) \in \partial\Sigma$ , and  $(s, \kappa_2 z) \in \partial\Sigma$  where we note that  $\kappa_1 > 0$  by (2.2) while  $\kappa_2 < 1$  since  $(s, z) \in N = \Sigma \setminus \partial\Sigma$ . But this contradicts the condition (2.3), and we obtain  $(t, x) \notin \Sigma$ . Therefore  $\partial\Sigma$  must contain the two different points  $(t, \lambda x)$  and  $(t, \lambda(t, x)x)$ . But this contradicts the condition (2.3) again. Thus the proof is completed.

The following is one of the sufficient conditions for  $\Sigma$  to satisfy the conditions (2.2) and (2.3). Let  $\Sigma$  be a set in  $R \times R^n$ , and denote by  $\Sigma_s$  its cross section  $\Sigma_s = \{x \in R^n : (s, x) \in \Sigma\}$ . Then we have the following result as a corollary to Lemma 2.2.

**Corollary 2.1.** *Suppose that  $\Sigma_s$  is closed (in  $R^n$ ) and contains 0 in its interior for all  $s \in R$ ,  $\Sigma_s$  and  $\partial\Sigma_s$  are continuous in the sense of Hausdorff distance, where  $\partial\Sigma_s$  denotes the boundary of  $\Sigma_s$  in the topology of  $R^n$ , and that  $\Sigma_s$  satisfies the condition*

$$(2.5) \quad \#\{\lambda \in R^+ : \lambda x \in \partial\Sigma_s\} \leq 1 \quad \text{for each } s \in R, x \in R^n.$$

*Then the function  $\lambda(t, x)$  in (2.4) is well-defined and continuous on  $R \times R^n$ .*

For the Hausdorff distance, see Dieudonné [2, p. 58].

*Proof.* Under the assumptions on  $\Sigma_s$ , it is not difficult to show that  $\Sigma$  is closed. On the other hand, if  $\partial\Sigma$  is coincident with  $B = \{(s, y) : s \in R, y \in \partial\Sigma_s\}$ , then the condition (2.3) follows from (2.5), and  $\Sigma$  contains  $(t, 0)$  in its interior for all  $t \in R$ , and hence, the conclusion of the corollary is immediate by Lemma 2.2.

Since clearly  $B$  is a subset of  $\partial\Sigma$ , it is sufficient to prove that  $\partial\Sigma$  is contained in  $B$ . Suppose that this is false. Then there exists  $(t, x) \in R \times R^n$  such that  $(t, x) \in \partial\Sigma \setminus B$ , and we can find a sequence  $\{(t_k, x_k)\}$  in  $R \times R^n \setminus \Sigma$  such that  $(t_k, x_k)$  converges to  $(t, x)$ . Since we have  $(t, x) \in \Sigma$  and  $x \notin \partial\Sigma_t$ ,  $\Sigma_t$  contains  $x$  in its interior. On the other hand, since  $x_k \notin \partial\Sigma_{t_k}$ , there exists a sequence  $\{\lambda_k\}$  in  $(0, 1)$  such that  $\lambda_k x_k \in \partial\Sigma_{t_k}$ . By taking a subsequence if necessary, we may assume that  $\lambda_k$  converges to some  $\lambda \in [0, 1]$ . Then, we have  $\lambda x \in \partial\Sigma_t$  and  $\lambda \in (0, 1)$ . Here we note that  $B$  is also a closed set in  $R \times R^n$  under the continuity of  $\partial\Sigma_s$ ,  $\Sigma_t$  contains 0 in its interior, and  $x \notin \partial\Sigma_t$ . By the similar arguments used in the proof of Lemma 2.2 to prove  $(t, x) \notin \Sigma$ , the facts  $\lambda x \in \partial\Sigma_t$  and  $x \in \Sigma_t \setminus \partial\Sigma_t$  yield a contradiction, and the conclusion is true.

*Remark.* We should note that the condition (2.5) suffices to prove that we can

define  $\lambda(t, x)$  by (2.4), but  $\lambda(t, x)$  may not be continuous without the continuity of  $\partial\Sigma_s$ . It is easily seen by the following simple example in  $R \times R^2$ : Define the set  $\Sigma_s$  by  $\Sigma_s = \{(x, y) : x^2 + y^2 \leq 1\}$  if  $s \leq 0$ , and

$$\Sigma_s = \{(x, y) \in \Sigma_0 : |y| \geq s(x - \frac{1}{2}) \text{ for } x > \frac{1}{2}\} \text{ if } s > 0,$$

and let  $\Sigma$  be the set in  $R \times R^2$  whose cross section at  $s$  is  $\Sigma_s$ . Then  $\Sigma$  satisfies all assumptions in Corollary 2.1 except the continuity of  $\partial\Sigma_s$  at  $s=0$ , and we have  $(0, x, 0) \in \partial\Sigma \setminus B$  for  $\frac{1}{2} \leq x < 1$ . Furthermore,  $\lambda(t, 1, 0) = 1$  for  $t \leq 0$  while  $\lambda(t, 1, 0) = \frac{1}{2}$  for  $t > 0$ .

Now we are ready to define retractions from  $C_\omega$  onto the set  $S_t = \{\psi \in C_\omega : (t + \theta, \psi(\theta)) \in \Sigma, -\omega \leq \theta \leq 0\}$  ( $t \in R$ ) or  $S \cap S_t$  by using the function  $\lambda(t, x)$ , where  $\Sigma$  is a set satisfying the assumptions on the set  $\Sigma$  in Lemma 2.2 and  $\Sigma_s$  is  $\omega$ -periodic in  $s$ , and  $S$  is the set given by (2.1). Clearly  $S_t$  is a nonempty closed subset of  $C_\omega$  by the assumptions on  $\Sigma$ . Let  $\pi_t : C_\omega \rightarrow S_t$  be a mapping defined by

$$(\pi_t(\psi))(\theta) = \lambda(t + \theta, \psi(\theta))\psi(\theta), \quad -\omega \leq \theta \leq 0.$$

Then, by Lemma 2.2,  $\pi_t$  is a retraction from  $C_\omega$  onto  $S_t$  such that  $\pi_t$  is  $\omega$ -periodic in  $t$ ,  $|\pi_t(\psi)| \leq |\psi|$ , and  $\pi_t(S) \subset S$ . Let  $\rho : C_\omega \rightarrow S$  be a retraction given by Lemma 2.1, and then the composed mapping  $\pi_t \circ \rho$  (or  $\pi_t \circ \rho \circ \pi_t$ ) is a retraction from  $C_\omega$  onto the nonempty closed subset  $S \cap S_t$  of  $C_\omega$  such that  $\pi_t \circ \rho$  is  $\omega$ -periodic in  $t$ , continuous on  $R \times C_\omega$ , takes bounded sets in  $R \times C_\omega$  into bounded sets, and  $\pi_t(\psi)(-\omega) = (\pi_t \circ \rho)(\psi)(0)$ .

Next, for such a retraction  $\rho_t = \pi_t \circ \rho$  from  $C_\omega$  onto  $S \cap S_t$ , let  $\sigma_t : C_\omega \rightarrow C_r$  be a mapping defined by

$$\sigma_t(\psi)(\theta) = \begin{cases} \pi_t(\psi)(\theta), & -\min\{r, \omega\} \leq \theta \leq 0, \\ \rho_t(\psi)(\theta + k\omega), & -\min\{r, (k+1)\omega\} \leq \theta \leq -k\omega, \quad k=1, 2, \dots, \end{cases}$$

that is,  $\sigma_t(\psi)$  is a function such that  $(t + \theta, \sigma_t(\psi)(\theta)) \in \Sigma$ ,  $\sigma_t(\psi)$  is  $\omega$ -periodic in  $t$ , and the restriction of  $\sigma_t(\psi)(\theta)$  to  $(-r, -\omega]$  is an  $\omega$ -periodic function of  $\theta$  if  $r > 2\omega$ .

Now let  $g(t, \psi) : R \times C_\omega \rightarrow R^n$  be a function defined by

$$(2.6) \quad g(t, \psi) = f(t, \sigma_t(\psi)).$$

Clearly  $g(t, \psi)$  is completely continuous, and  $\omega$ -periodic in  $t$ . For  $g(t, \psi)$  defined by (2.6), consider a functional differential equation

$$(2.7) \quad \dot{x}(t) = g(t, x_t).$$

Then this equation always has a solution for the initial value problem, while Equation (1.1) may fail to have a solution for some initial value problem (see Seifert [10]). However, we have the following theorem.

**Theorem 2.1.** *An  $\omega$ -periodic solution  $x(t)$  of Equation (1.1) is a solution of Equation (2.7) as long as  $(t, x(t)) \in \Sigma$  for all  $t \in R$ , and vice versa.*

Since this theorem can be proved easily from the definition of  $g(t, \psi)$ , we omit the proof.

### § 3. Existence of periodic solutions.

In this section, we shall discuss the existence of periodic solutions of the system (1.1) by applying Theorem 3.2 in [4], where  $\Sigma$  is given by using a strongly convex Liapunov function and a scalar equation

$$(3.1) \quad \dot{u} = h(t, u),$$

where  $h(t, u): R \times R \rightarrow R$  is continuous and locally Lipschitzian with respect to  $u$ .

A function  $V(t, x): R \times R^n \rightarrow R$  is said to be a Liapunov function if  $V(t, x)$  is continuous on  $R \times R^n$  and satisfies  $V(t, x) \geq a(|x|)$  for a continuous function  $a(u)$  such that  $a(u) \rightarrow \infty$  as  $u \rightarrow \infty$ . We shall say a Liapunov function  $V(t, x)$  is convex if for each fixed  $t \in R$  the set  $\{x \in R^n: V(t, x) \leq k\}$  is convex in  $R^n$ . Moreover, a convex Liapunov function  $V(t, x)$  is said to be strongly convex if for each fixed  $t \in R$  the set  $\{x \in R^n: V(t, x) = k\}$  is the boundary of  $\{x \in R^n: V(t, x) \leq k\}$ . We define  $V'_{(1.1)}(t, \phi)$  by

$$V'_{(1.1)}(t, \phi) = \limsup_{\tau \rightarrow 0^+} \frac{1}{\tau} \{V(t + \tau, x(t + \tau, t, \phi)) - V(t, \phi(0))\}.$$

Clearly we have

$$(3.2) \quad V'_{(1.1)}(t, \phi) = \limsup_{\tau \rightarrow 0^+} \frac{1}{\tau} \{V(t + \tau, \phi(0) + \tau f(t, \phi)) - V(t, \phi(0))\}$$

if  $V(t, x)$  is locally Lipschitzian with respect to  $x$ .

The following theorem is given in [4; Theorem 3.2], where the delay  $r$  is finite.

**Theorem 3.1.** *Let  $V: R \times R^n \rightarrow R^+$  be a continuous,  $\omega$ -periodic, convex Liapunov function, and let  $L: R \rightarrow R$  be a continuous, nondecreasing function such that*

$$(3.3) \quad L(u) > u \quad \text{for all } u > 0,$$

or

$$(3.4) \quad L(u) = u \quad \text{for all } u > 0.$$

Suppose that there exists a continuous function  $u(t)$  defined on  $[t_0 - r, t_0 + \omega]$  for some  $t_0$  such that  $u(t)$  is a solution of (3.1) on  $[t_0, t_0 + \omega]$  which satisfies  $u(t_0 + \omega + \theta) \leq u(t_0 + \theta)$

for  $-r \leq \theta \leq 0$ ,  $u(t) \geq V(t, 0)$  for  $t_0 \leq t \leq t_0 + \omega$ , and  $u(t + \theta) \leq L(u(t))$  for  $t_0 \leq t \leq t_0 + \omega$ ,  $-r \leq \theta \leq 0$ , and that we have

$$V'_{(1.1)}(t, \phi) \leq h(t, V(t, \phi(0)))$$

for all functions  $\phi \in C_r$  with the property that

$$(3.5) \quad V(t, \phi(0)) \geq u(t), \quad V(t + \theta, \phi(\theta)) \leq L(V(t, \phi(0))) \quad \text{for } -r \leq \theta \leq 0.$$

Then Equation (1.1) has an  $\omega$ -periodic solution.

We are ready to prove the following theorem concerning the existence of an  $\omega$ -periodic solution of Equation (1.1) with  $r = \infty$ .

**Theorem 3.2.** *Let  $V: R \times R^n \rightarrow R^+$  be a continuous,  $\omega$ -periodic in  $t$ , strongly convex Liapunov function which is locally Lipschitzian with respect to  $x$ , and let  $L: R \rightarrow R$  be a nondecreasing continuous function which satisfies (3.3) or (3.4). Suppose that Equation (3.1) has a solution  $u(t)$  on  $[0, \omega]$  such that  $u(0) \geq u(\omega)$ ,  $u(t) > V(t, 0)$  for  $0 \leq t \leq \omega$ ,  $\bar{u} \leq L(\underline{u})$  where  $\bar{u} = \max_{0 \leq t \leq \omega} u(t)$ ,  $\underline{u} = \min_{0 \leq t \leq \omega} u(t)$ , and that we have*

$$(3.6) \quad V'_{(1.1)}(t, \phi) \leq h(t, V(t, \phi(0)))$$

for all functions  $\phi \in C_\infty$  with the property that

$$(3.7) \quad \phi \in P_\infty, \quad V(t, \phi(0)) \geq u(t), \quad V(t + \theta, \phi(\theta)) \leq L(V(t, \phi(0))) \quad \text{for } -\infty < \theta \leq 0,$$

where  $P_\infty = \{\phi \in C_\infty : \phi(\theta) \text{ is } \omega\text{-periodic on } (-\infty, -\omega]\}$ . Then Equation (1.1) has an  $\omega$ -periodic solution  $x(t)$  such that  $V(t, x(t)) \leq u(t)$  for  $0 \leq t \leq \omega$ .

*Proof.* We note this theorem can be proved in the following way. Choose a suitable transformation  $\bar{\sigma}_t: C_\omega \rightarrow C_\infty$  so that (1.1) is reduced to an equation which is similar to (2.7), and apply Theorem 3.1 to the system with the finite delay  $\omega$ . The rest of the proof follows from Theorem 2.1. Let  $u^*(t)$  be any continuous  $\omega$ -periodic function on  $R$  such that  $u^*(t) > V(t, 0)$  for all  $t \in R$ . Define  $\Sigma_s^*$  by

$$\Sigma_s^* = \{x \in R^n : V(s, x) \leq u^*(s)\}, \quad s \in R,$$

and let  $\Sigma^*$  be the set in  $R \times R^n$  whose cross section at  $s \in R$  is given by  $\Sigma_s^*$ . From the assumptions on  $V(t, x)$  and  $u^*(t)$ , the set  $\Sigma_s^*$  is  $\omega$ -periodic in  $s$  and  $\Sigma^*$  satisfies the conditions assumed on  $\Sigma$  in Corollary 2.1 to Lemma 2.2. The proof of this statement is obvious except for the continuity in the sense of Hausdorff distance of  $\Sigma_s^*$  and  $\partial \Sigma_s^*$  but it can be proved by the standard arguments using the continuity and the strong convexity of  $V(t, x)$ . Thus we may use the results obtained in Section 2.

Now we define a suitable transformation  $\bar{\sigma}_t: C_\omega \rightarrow C_\infty$  in the following manner.

For any fixed  $v \geq L(\underline{u})$ , let  $\Sigma_s^*(v) = \{x \in R^n : V(s, x) \leq v\}$ , and let  $\sigma_t^*(v) : C_\omega \rightarrow C_\infty$  be the transformation defined by taking  $\Sigma_s = \Sigma_s^*(v)$  and  $r = \infty$  in the procedures in Section 2. Let  $\bar{\sigma}_t : C_\omega \rightarrow C_\infty$  be a mapping defined by

$$\bar{\sigma}_t(\psi) = \sigma_t^*(v^*)(\psi), \quad v^* = \max \{v, L(\underline{u})\},$$

where  $v = \max_{-\omega \leq \theta \leq 0} V(t + \theta, \psi(\theta))$ . Then, from the assumptions on  $V(t, x)$ , clearly  $\bar{\sigma}_t$  is  $\omega$ -periodic in  $t$ , continuous on  $R \times C_\omega$ , and takes bounded sets in  $R \times C_\omega$  into bounded sets. By using  $\bar{\sigma}_t$ , we can define an auxiliary function  $g^*(t, \psi)$  for  $f(t, \phi)$  by

$$(3.8) \quad g^*(t, \psi) = f(t, \bar{\sigma}_t(\psi)).$$

Consider the following auxiliary equation

$$(3.9) \quad \dot{x}(t) = g^*(t, x_t).$$

In order to apply Theorem 3.1, it is sufficient to show that

$$(3.10) \quad V'_{(3.9)}(t, \psi) \leq h(t, V(t, \psi(0)))$$

under the condition

$$(3.11) \quad V(t, \psi(0)) \geq u(t), \quad V(t + \theta, \psi(\theta)) \leq L(V(t, \psi(0))) \quad \text{for } -\omega \leq \theta \leq 0.$$

Let  $(t, \psi)$  satisfy (3.11), and let  $v = \max_{-\omega \leq \theta \leq 0} V(t + \theta, \psi(\theta))$  and  $v^* = \max \{v, L(\underline{u})\}$ . If we take  $\phi = \bar{\sigma}_t(\psi)$  for  $(t, \psi)$ , then by the definition of  $\bar{\sigma}_t$ , we have  $\phi(\theta) = \psi(\theta)$  for  $-\omega \leq \theta \leq 0$  and  $V(t + \theta, \phi(\theta)) \leq v^*$  for  $-\infty < \theta \leq 0$ . Hence, we obtain

$$V(t + \theta, \phi(\theta)) \leq v^* \leq L(V(t, \phi(0))) \quad \text{for } -\infty < \theta \leq 0.$$

Thus (3.11) implies (3.7), and consequently we have (3.6), which is not different from (3.10) by (3.2) and (3.8), since  $V(t, x)$  is locally Lipschitzian with respect to  $x$ .

Finally, let  $u(t)$  extend continuously on  $[-\omega, \omega]$  by defining  $u(t) = \max \{u(0), u(t + \omega)\}$  for  $-\omega \leq t \leq 0$ . Then all assumptions of Theorem 3.1 are satisfied with this  $u(t)$  and  $t_0 = 0$ . Then Equation (3.9) has an  $\omega$ -periodic solution  $x(t)$  such that  $V(t, x(t)) \leq u(t)$  for  $0 \leq t \leq \omega$  by Theorem 3.1. Now let  $\Sigma$  be a set whose cross section is  $\Sigma_s = \Sigma_s^*(L(\underline{u}))$ , and let  $\sigma_t : C_\omega \rightarrow C_\infty$  be the corresponding transformation. Then we obtain  $(t, x(t)) \in \Sigma$  and  $\bar{\sigma}_t(x_t) = \sigma_t(x_t)$  for all  $t \in R$ . Thus  $x(t)$  is also an  $\omega$ -periodic solution of Equation (2.7) with  $g(t, \psi)$  defined by (2.6) by using this  $\sigma_t$ , and consequently, we can conclude that  $x(t)$  is an  $\omega$ -periodic solution of Equation (1.1) by Theorem 2.1.

*Remark 1.* For Equation (1.1) with a finite delay  $r$ , the restriction (3.5) is posed in Theorem 3.1 instead of (3.7). When  $r$  is finite, we can obtain a modified version

of Theorem 3.2 by replacing (3.7) by the following restriction, which is more useful than (3.5) if  $r \geq 2\omega$ .

$$(3.12) \quad \phi \in P_r, \quad V(t, \phi(0)) \geq u(t), \quad V(t+\theta, \phi(\theta)) \leq L(V(t, \phi(0))) \quad \text{for } -r \leq \theta \leq 0,$$

where  $P_r = \{\phi \in C_r : \phi(\theta) \text{ is } \omega\text{-periodic on } [-r, -\omega] \text{ if } r \geq 2\omega\}$ .

*Remark 2.* It is easily seen from the proof of Theorem 3.2 in [4] that the condition  $\max_{-\omega \leq \theta \leq 0} V(t+\theta, \psi(\theta)) < \bar{u} + \varepsilon$  can be added to the condition (3.11), where  $\varepsilon$  is an arbitrary positive constant. Thus we can choose a simple transformation  $\sigma_t^*(L(\underline{u}))$  as a suitable transformation  $\bar{\sigma}_t$  if the function  $u(t)$  in this theorem satisfies  $\bar{u} < L(\underline{u})$ .

Now we present an application of Theorem 3.2. Let  $F(t, x) : R \times R \rightarrow R$  be a continuous function which is  $\omega$ -periodic in  $t$  for some positive constant  $\omega$ , and satisfies

$$(3.13) \quad |F(t, x)| \leq |x| \quad \text{whenever } t \in R, \quad |x| > K^*$$

for some positive constant  $K^*$ . Let  $\{r_k(t)\}$  be a sequence of continuous nonnegative  $\omega$ -periodic functions on  $R$ , and let  $\{c_k\}$  be a sequence such that  $\sum_{k=1}^{\infty} |c_k| \leq 1$ . Define a function  $f(t, \phi) : R \times C_{\infty} \rightarrow R$  by

$$f(t, \phi) = \sum_{k=1}^{\infty} c_k F(t, \phi(-r_k(t))).$$

From the assumptions on  $F(t, x)$ ,  $\{r_k(t)\}$  and  $\{c_k\}$ , clearly  $f(t, \phi)$  is completely continuous, and  $\omega$ -periodic in  $t$ . Moreover, we have from (3.13)

$$(3.14) \quad |f(t, \phi)| \leq |\phi| \quad \text{whenever } t \in R, \quad |\phi(0)| > K$$

for some positive constant  $K$ . Consider a scalar  $\omega$ -periodic functional differential equation

$$(3.15) \quad \dot{x}(t) = -x(t) + 4x(t-\omega) - 4x(t-2\omega) + f(t, x_t).$$

If we take  $L(u) = u$  and  $V(x) = x^2/2$ , then  $V$  is a strongly convex Liapunov function which is locally Lipschitzian, and we have

$$V'_{(3.15)}(t, \phi) = -\phi^2(0) + \phi(0)f(t, \phi) \leq -\phi^2(0) + \phi^2(0) \frac{|\phi|}{|\phi(0)|} \frac{|f(t, \phi)|}{|\phi|} \leq 0$$

under the condition (3.7) or (3.12) if the solution  $u(t)$  of (3.1) with  $h(t, u) \equiv 0$  satisfies  $u(t) > K^2/2$ , where  $K$  is the one in (3.14). Therefore, by Theorem 3.2, there exists an  $\omega$ -periodic solution of (3.15).

If  $r = \max \{2\omega, \sup_{k \geq 1} \max_{0 \leq t \leq \omega} r_k(t)\}$  is finite, then (3.15) is an equation with the finite delay  $r$ . But we cannot apply Theorem 37.1 in [11] to (3.15) to conclude

that Equation (3.15) has an  $\omega$ -periodic solution, since  $r$  is greater than the period  $\omega$ . Moreover,  $V'_{(3.15)}(t, \phi)$  cannot be compared with  $h(t, V(t, \phi(0))) \equiv 0$  under the condition (3.5). Thus we have no idea how to apply Theorem 3.1.

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