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Generic Properties of Functional-Differential Equations of Neutral Type in Separable Banach Spaces

By

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§ 1. Introduction.

In recent years much work has been done on functional-differential equations of neutral type ([2], [8], [9], [15], [16], [19]). These are differential equations in which the present dynamics of the systems are influenced by its past behavior.

The main result of the present paper states that all functional-differential equations of neutral type in a separable Banach space with existence, uniqueness and continuous dependence on the initial date and the right-hand side of solutions constitute a residual set in any complete metric space. It is proved furthermore, that nonconvergence of successive approximations of such type equations is in any sense a rare case. This type property of differential equations is said to be generic.

The study of generic properties of differential equations was started by W. Orlicz ([17]), who showed that the subset all f for which the Cauchy problem for the equation $\dot{x}=f(t, x)$ has not uniqueness of solutions is of the first Baire's category in the space of all continuous and bounded f with values in the n-dimensional Euclidean space R^n, equipped with a natural metric. Latter on, this result has been generalized by A. Alexiewicz and Orlicz [1], A. Lasota and J. Yorke [14], M. Kisielewicz [11]–[13], T. Castello [3], J. Piórek [20], F. De Blasi and J. Myjak [4]–[6], G. Vidossich [21], [22] and W. Orlicz and S. Szufla [18]. Generic properties of functional equations have been considered in [7].

The results of this paper extend, among others, the main results of Lasota and Yorke ([14], Th. 1) and De Blasi and Myjak ([6], Th. 31, 4.8) on the case of functional-differential equations of neutral type in separable Banach spaces.

Let E be a separable Banach space with a norm $|\cdot|$ and let C_α and L_α denote the Banach spaces of all strongly continuous and Bochner integrable functions, respectively from $[-r, \alpha]$ into E with the usual norms $\|\cdot\|$ and $\mathbf{|}\cdot\mathbf{|}$, respectively, where $\alpha>0$ and $r>0$. Denote by $\mathscr{L}_{ab}$ $(a<b)$, the Banach space all Bochner integrable functions of $[a, b]$ into E with the usual norm $|\cdot|_{ab}$. If $a=0$ we will write $\mathscr{L}_b$ and $|\cdot|_b$ instead of $\mathscr{L}_{0b}$ and $|\cdot|_{0b}$, respectively.

Let us denote by $\mathscr{A}_\alpha$ the space of all absolutely continuous functions $x\colon [-r, \alpha] \to E$ with a norm $\|\mathbf{|}x\mathbf{|}\|_\alpha=\|x\|_\alpha+\mathbf{|}\dot{x}\mathbf{|}_\alpha$, where $\dot{x}$ denotes the strong derivative of x.

Similarly to the proof as in [19] it can be proved that $(\mathscr{A}_\alpha, \|\cdot\|)$ is a complete metric space. The results of this paper are concerned with functional-differential equations of the form

$$(1) \qquad \dot{x}(t)=f(t, x_t, \dot{x}_t) \qquad \text{for a.e. } t \in [0, T]$$

with $f\colon [0, T]\times C_0\times L_0\to E$ satisfying the following conditions:

(i) $\quad f(\cdot, x, y) \in \mathscr{L}_T \quad$ for fixed $(x, y) \in C_0\times L_0$

and

(ii) mapping $g\colon C_0\times L_0\to\mathscr{L}_T$ defined by $g(x, y)=f(\cdot, x, y)$ for $(x, y) \in C_0\times L_0$ is continuous.

Here, for given $x \in \mathscr{A}_T$ and $t \in [0, T]$, by x_t, $\dot{x}_t$ we denote elements of C_0 and L_0, respectively defined by $x_t(s)=x(t+s)$ and $\dot{x}_t(s)=\dot{x}(t+s)$ for $s \in [-r, 0]$.

For given $\varphi \in \mathscr{A}_0$ by solution of (1) with an initial condition

$$(2) \qquad x(t)=\varphi(t) \qquad \text{for } t \in [-r, 0]$$

we mean an absolutely continuous function $x\colon [-r, T]\to E$ satisfying (1) and (2).

It is useful to associate with the initial value problem (1) and (2) the following equivalent functional integral equation

$$(3) \qquad x(t)=\begin{cases}\varphi(t) & \text{for } t \in [-r, 0]\\ \varphi(0)+\int_0^t f(s, x_s, \dot{x}_s)ds & \text{for } t \in [0, T].\end{cases}$$

§ 2. The metric space $(\mathscr{F}, \rho)$ and an approximation theorem.

Let us introduce in the space F of all functions $f\colon [0, T]\times C_0\times L_0\to E$ satisfying conditions (i), (ii) an equivalence relation "$\sim$" defined by $f_1\sim f_2$ iff

$$\sup\{|f_1(\cdot, x, y)-f_2(\cdot, x, y)|_T\colon (x, y) \in C_T\times L_T\}=0.$$

Let $\mathscr{F}$ denote the space of all equivalence classes of F defined by $\sim$. Notationally, we shall not distinguish between elements of $\mathscr{F}$ and F.

For given $0\leqslant\alpha<\beta\leqslant T$ and $f_1, f_2 \in \mathscr{F}$ let

$$\rho_{\alpha\beta}(f_1, f_2)=\sup\left\{\frac{|f_1(\cdot, x, y)-f_2(\cdot, x, y)|_{\alpha\beta}}{1+|f_1(\cdot, x, y)-f_2(\cdot, x, y)|_{\alpha\beta}}\colon (x, y) \in C_0\times L_0\right\}.$$

We say that f satisfying (i), is locally Lipschitzean if for every $(x, y) \in C_0\times L_0$ there are a neighborhood U_{xy} of (x, y) and a continuous increasing function $K_{xy}\colon [0, T]\to [0, \infty)$ with $K_{xy}(0)=0$ such that for every (x_1, y_1), $(x_2, y_2) \in U_{xy}$ and $0\leqslant\alpha<t\leqslant T$ we have

(4) $$|f(\cdot, x_1, y_1)-f(\cdot, x_2, y_2)|_{\alpha t} \leqslant K_{xy}(t-\alpha)[\|x_1-x_2\|_0+|y_1-y_2|_0].$$

In the sequel we will need the following Lemma.

Lemma 1. *For every, locally Lipschitzean $f \in \mathscr{F}$, $\lim_{\beta\to\alpha} \rho_{\alpha\beta}(f, 0)=0$.*

Proof. Suppose $\rho_{\alpha\beta}(f, 0)$ does not converge to zero as $\beta\to\alpha$. Then there exists $\varepsilon_0>0$ such that $\varepsilon_0<\rho_{\alpha\beta}(f, 0)\leqslant 1$ for each $0\leqslant\alpha<\beta\leqslant T$. Hence it follows that for every $0\leqslant\alpha<\beta\leqslant T$ there exists $(x_0, y_0) \in C_0\times L_0$ such that

$$0<\frac{\varepsilon_0}{1-\varepsilon_0}<\int_\alpha^\beta |f(t, x_0, y_0)|\, dt.$$

Let U_0 and K_0: $[0, T]\to[0, \infty)$ be a neighborhood of (x_0, y_0) and a continuous increasing function, respectively such that (4) is satisfied for (x_1, y_1), $(x_2, y_2) \in U_1$ and $t \in [0, T]$. Therefore, for every $(x_1, y_1) \in U_0$ we have

$$0<\frac{\varepsilon_0}{1-\varepsilon_0}<K_0(\beta-\alpha)\left[\|x_1-x_0\|_0+|y_1-y_2|_0+\int_\alpha^\beta |f(t, x_1, y_1)|\, dt\right].$$

Hence it follows $\varepsilon_0=0$ and the proof is complete.

Define now a metric ρ for $\mathscr{F}$ by setting $\rho(f_1, f_2)=\rho_{0T}(f_1, f_2)$ for $f_1, f_2 \in \mathscr{F}$.

Lemma 2. *$(\mathscr{F}, \rho)$ is a complete metric space.*

Proof. Let (f_n) be a Cauchy sequence of $\mathscr{F}$. Then for every $k\geqslant 1$ there is $N_k \geqslant 1$ such that

$$\frac{|f_m(\cdot, x, y)-f_n(\cdot, x, y)|_T}{1+|f_m(\cdot, x, y)-f_n(\cdot, x, y)|_T}<\frac{1}{2^{k+1}}$$

for $n, m\geqslant N_k$ and $(x, y) \in G_0\times L_0$. Hence it follows that

$$|f_m(\cdot, x, y)-f_n(\cdot, x, y)|_T<\frac{1}{2^k}$$

for $n, m\geqslant N_k$, $(x, y) \in C_0\times L_0$ and $k\geqslant 1$. Therefore $(f_n(\cdot, x, y))$ is a Cauchy sequence of $\mathscr{L}_T$. Then there exists $f(\cdot, x, y) \in \mathscr{L}_T$ such that $|f_n(\cdot, x, y)-f(\cdot, x, y)|_T\to 0$ as $n\to\infty$, uniformly with respect to $(x, y) \in C_0\times L_0$. Hence and continuity of g_n: $C_0\times L_0 \ni (x, y)\to f_n(\cdot, x, y) \in \mathscr{L}_T$ it follows that a mapping g: $C_0\times L_0 \ni (x, y)\to f(\cdot, x, y) \ni \mathscr{L}_T$ is continuous. Then $f \in \mathscr{F}$. We have of course $\rho(f_n, f)\to 0$ as $n\to\infty$ which completes the proof.

Adopting now the procedure of Lasota and Yorke ([14]) we shall show that every $f \in \mathscr{F}$ can be approximated by locally Lipschitzean elements of this space.

Theorem 3. *For every $f \in \mathscr{F}$ and $\varepsilon>0$ there exists $f_\varepsilon \in \mathscr{F}$, locally Lipschitzean and such that $\rho(f_\varepsilon, f)\leqslant\varepsilon$.*

Proof. Let $\varepsilon>0$ be given. By the continuity of a mapping $g: C_0\times L_0 \ni (x, y) \to f(\cdot, x, y) \in \mathscr{L}_T$, for every $(x, y) \in C_0\times L_0$ there exists a $\delta(x, y, \varepsilon)>0$ such that $|f(\cdot, x, y)-f(\cdot, \bar{x}, \bar{y})|_T<\varepsilon$ for each $(\bar{x}, \bar{y}) \in C_0\times L_0$ satisfying $\|x-\bar{x}\|_0+|y-\bar{y}|_0< \delta(x, y, \varepsilon)$.

Let $N(x, y, \varepsilon)=\{(u, v) \in C_0\times L_0: \|x-u\|_0+|y-y|_0<\delta(x, y, \varepsilon) \text{ for } (x, y) \in C_0\times L_0\}$. We have $C_0\times L_0=\bigcup \{N(x, y, \varepsilon): (x, y) \in C_0\times L_0\}$. By the paracompactness of $C_0\times L_0$ there exists an open locally finite refirement $(Q_j)_{j\in A}$ of $\{N(x, y, \varepsilon)\}_{(x,y)\in C_0\times L_0}$. Then for every $j \in A$ there exists $(\tilde{x}, \tilde{y}) \in C_0\times L_0$ such that $Q_j\subset(N, \tilde{x}, \tilde{y}, \varepsilon)$.

Define, for fixed $(x, y) \in C_0\times L_0$ and $j \in A$,

$$r_j(x, y)=\begin{cases}0 & \text{for } (x, y) \notin Q_j\\ \inf\{\|x-\bar{x}\|_0+|y-\bar{y}|_0: (\bar{x}, \bar{y}) \in \partial Q_j\} & \text{for } (x, y) \in Q_j,\end{cases}$$

where ∂B denote the boundary of a set B. It is not difficult to see that

$$|r_j(x_1, y_1)-r_j(x_2, y_2)|\leqslant\|x_1-x_2\|_0+|y_1-y_2|_0$$

for $(x_1, y_1), (x_2, y_2) \in C_0\times L_0$ and $j \in A$. Hence it follows, in particular, that for each $j \in A$ a family $\{r_j\}_{j\in A}$ of mappings from $C_0\times L_0$ into R is uniformly equicontinuous on $C_0\times L_0$. Therefore, a mapping $w: C_0\times L_0\to R$ defined by

$$w(x, y)=\left[\sum_{j\in A} r_j(x, y)\right]^{-1}$$

for every $(x, y) \in C_0\times L_0$, is continuous on $C_0\times L_0$. Indeed, by the definition of $(Q_j)_{j\in A}$, for every $(x, y) \in C_0\times L_0$, there are a neighborhood U_{xy} of (x, y) and a set $A_{xy} =\{j_1, \cdots, j_{n_{xy}}\}\subset A$ such that $Q_j\cap U_{xy}=\emptyset$ for $j \notin A_{xy}$. Therefore $r_j(\tilde{x}, \tilde{y})=0$ for $(\tilde{x}, \tilde{y}) \in U_{xy}$ and $j \notin A_{xy}$. For $j \notin A_{xy}$ we have $Q_j\cap U_{xy}\neq\emptyset$. Then

$$\sum_{j\in A_{xy}} r_j(\tilde{x}, \tilde{y})=\sum_{j\in A} r_j(\tilde{x}, \tilde{y})>0 \qquad \text{for } (\tilde{x}, \tilde{y}) \in C_0\times L_0.$$

Therefore, we have

$$w(\tilde{x}, \tilde{y})=\left[\sum_{j\in A_{xy}} r_j(\tilde{x}, \tilde{y})\right]^{-1} \qquad \text{for } (\tilde{x}, \tilde{y}) \in U_{xy}.$$

Hence it follows that w is continuous on $C_0\times L_0$.

Let $v_j(x, y)=r_j(x, y)\cdot w(x, y)$ for $(x, y) \in C_0\times L_0$ and $j \in A$. We shall show that for every $(x, y) \in C_0\times L_0$ there are a neighborhood U_{xy} of (x, y) and a number $L_{xy}>0$ such that

$$|v_j(x_1, y_1)-v_j(x_2, y_2)|\leqslant L_{xy}(\|x_1-x_2\|_0+|y_1-y_2|_0)$$

for each $(x_1, y_1), (x_2, y_2) \in U_{xy}$ and $j \in A$.

Let $j \in A$ be fixed. By the continuity of a mapping $w\colon C_0 \times L_0 \to R$ there exists a neighborhood, say again U_{xy}, of (x, y) such that $w(\tilde{x}, \tilde{y}) < 1 + w(x, y)$ for $(\tilde{x}, \tilde{y}) \in U_{xy}$. Suppose U_{xy} is such taken, that $w(\tilde{x}, \tilde{y}) = [\sum_{j \in A_{xy}} r_j(\tilde{x}, \tilde{y})]^{-1}$ and $w(\tilde{x}, \tilde{y}) < 1 + w(x, y)$ for $(\tilde{x}, \tilde{y}) \in U_{xy}$. For $(x_1, y_1), (x_2, y_2) \in U_{xy}$, we have

$$\begin{aligned}|v_j(x_1, y_1) - v_j(x_2, y_2)| &\leqslant w(x_1, y_1)[|r_j(x_1, y_1) - r_j(x_2, y_2)| \\ &\quad + \sum_{j \in A_{xy}} |r_j(x_1, y_1) - r_j(x_2, y_2)|] \\ &\leqslant M_{xy}(1 + n_{xy})[\|x_1 - x_2\|_0 + |y_1 - y_2|_0],\end{aligned}$$

where $M_{xy} = 1 + w(x, y)$.

Let us define now a desired function $f_\varepsilon\colon [0, T] \times C_0 \times L_0 \to E$ by setting

$$f_\varepsilon(t, x, y) = \sum_{j \in A} v_j(x, y) \cdot f(t, x_j, y_j)$$

for $(x, y) \in C_0 \times L_0$, where $(x_j, y_j) \in Q_j$. Suppose a neighborhood U_{xy} of (x, y), $L_{xy} > 0$ and $A_{xy} = \{j_1, \cdots, j_{n_{xy}}\} \subset A$ are such that

$$f_\varepsilon(t, x, y) = \sum_{j \in A_{xy}} v_j(\tilde{x}, \tilde{y}) \cdot f(t, x_j, y_j)$$

and

$$|v_j(x_1, y_1) - v_j(x_2, y_2)| \leqslant L_{xy}[\|x_1 - x_2\|_0 + |y_1 - y_2|_0]$$

for $0 \leqslant t \leqslant T$ and $(\tilde{x}, \tilde{y}), (x_1, y_1), (x_2, y_2) \in U_{xy}$. Hence it follows that f_ε is measurable with respect to $t \in [0, T]$ and satisfies

$$(5) \qquad |f_\varepsilon(s, x_1, y_1) - f_\varepsilon(s, x_2, y_2)| \leqslant k_{xy}(s)[\|x_1 - x_2\|_0 + |y_1 - y_2|_0]$$

for $s \in [0, T]$ and $(x_1, y_1), (x_2, y_2) \in U_{xy}$, where

$$k_{xy}(s) = n_{xy} L_{xy} \max\{|f(s, x_j, y_j)| : j \in A_{xy}\}.$$

Therefore, $f_\varepsilon(\cdot, x, y) \in \mathscr{L}_T$ for fixed $(x, y) \in C_0 \times L_0$ and

$$\begin{aligned}\int_\alpha^t |f_\varepsilon(s, x_1, y_1) - f_\varepsilon(s, x_2, y_2)|\, ds &\leqslant \int_\alpha^t k_{xy}(s)[\|x_1 - x_2\|_0 + |y_1 - y_2|_0]\, ds \\ &= K_{xy}(t - \alpha)[\|x_1 - x_2\|_0 + |y_1 - y_2|_0]\end{aligned}$$

for $(x_1, y_1), (x_2, y_2) \in U_{xy}$ and $0 \leqslant \alpha \leqslant t \leqslant T$, where

$$K_{xy}(t - \alpha) = \int_\alpha^t k_{xy}(s) ds.$$

Thus f_ε is locally Lipschitzean. Hence, in particular follows that f_ε satisfies condition (ii) and therefore $f_\varepsilon \in \mathscr{F}$.

We shall show now that $\rho(f_\varepsilon, f) \leqslant \varepsilon$. Suppose now, a neighborhood U_{xy} of $(x, y) \in C_0 \times L_0$ and a set $A_{xy} = \{j_1, \cdots, j_{n_{xy}}\} \subset A$ are such that $(x, y) \in \bigcap_{j \in A_{xy}} Q_j$ and $(x, y) \notin Q_j$ for $j \in A \backslash A_{xy}$. For fixed $(x, y) \in C_0 \times L_0$ we have

$$\int_0^T |f_\varepsilon(s, x, y) - f(s, x, y)|\, ds \leqslant \sum_{j \in A_{xy}} v_j(x, y) \int_0^T |f(s, x_j, y_j) - f(s, x, y)|\, ds$$
$$\leqslant \varepsilon \cdot \sum_{j \in A_{xy}} v_j(x, y) = \varepsilon.$$

Hence it follows that $\rho(f_\varepsilon, f) \leqslant \varepsilon$ and the proof is complete.

§ 3. Existence, uniqueness and continuous dependence of solutions.

Let us consider an initial value problem

$$(6) \qquad \begin{cases} \dot{x}(t) = f(t, x_t, \dot{x}_t) & \text{for a.e. } t \in [0, T] \\ x(t) = \varphi(t) & \text{for } t \in [-r, 0], \end{cases}$$

where $f \in \mathscr{F}$ and $\varphi \in \mathscr{A}_0$.

Adopting the classical methods of successive approximations ([19], Th. 1), we can prove that if $f \in \mathscr{F}$ is locally Lipschitzean, then (6) has exactly one solution. Similarly, as in [19] it can be proved that this solution continuously depends on $(\varphi, f) \in \mathscr{A}_0 \times \mathscr{F}$.

Let us observe that in the theory of functional-differential equations of the form (6) it is needed to use the well-known fact that the translation operator is continuous in spaces of continuous and locally integrable functions. It will be convenient to formulate the following lemma

Lemma 4. *For every $x \in C_T$ and $y \in L_T$,*

$$\lim_{t \to 0+} \|x_t - x_0\|_0 = 0 \quad \textit{and} \quad \lim_{t \to 0+} |y_t - y_0|_0 = 0.$$

Theorem 5. *Let $f \in \mathscr{F}$ be locally Lipschitzean. Then, for every $\varphi \in \mathscr{A}_0$ there exists exactly one solution of* (6).

Proof. We can present only sketch of the proof. Suppose U_0 and K_0: $[0, T] \to R$ are a neighborhood of $(\varphi, \dot{\varphi})$ and a continuous increasing function, respectively such that

$$|f(\cdot, x_1, y_1) - f(\cdot, x_2, y_2)|_t \leqslant K_0(t)[\|x_1 - x_2\|_0 + |y_1 - y_2|_0]$$

for $(x_1, y_1), (x_2, y_2) \in U_0$ and $t \in [0, T]$.

Let

$$x^0(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r, 0] \\ \varphi(0) & \text{for } t \in [0, T] \end{cases}$$

and let S_0 be a closed ball of $C_0 \times L_0$ with the center $(\varphi, \dot{\varphi})$ and a radius $r_0 > 0$ such that $S_0 \subset U_0$. Select $T_0 \in (0, T]$ such that $K_0(T_0) < 1/2$, $\rho_{T_0}(f, 0) < 1$,

$$\frac{\rho_{T_0}(f, 0)}{1 - \rho_{T_0}(f, 0)} \leqslant r_0/6 \quad \text{and} \quad \| x_t^0 - x_0^0 \|_0 < r_0/3.$$

It is possible, because $\rho_t(f, 0) \to 0$, $K_0(t) \to 0$ and $\| x_t^0 - x_0^0 \|_0 \to 0$ as $t \to 0+$. Hence it follows that

$$\int_0^{T_0} |f(t, x, y)|\, dt \leqslant r_0/6 \qquad \text{for each } (x, y) \in C_0 \times L_0.$$

Define now a sequence (x^n) of $\mathscr{A}_T$ by setting

$$x^n(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r, 0] \\ \varphi(0) + \int_0^t f(s, x_s^{n-1}, \dot{x}_s^{n-1}) ds & \text{for } t \in [0, T] \end{cases}$$

for $n = 1, 2, \cdots$, where x^0 was defined above. We have $x_n \in \mathscr{A}_T$, $\| x_t^n - \varphi \|_0 \leqslant r_0/2$ and $| \dot{x}_t^n - \dot{\varphi}_0 |_0 \leqslant r_0/2$ for each $t \in [0, T_0]$ and $n = 1, 2, \cdots$. Therefore $\| x_t^n - \varphi \|_0 \leqslant r_0$ and $(x_t^n, \dot{x}_t^n) \in S_0 \subset U_0$ for each $t \in [0, T_0]$ and $n = 1, 2, \cdots$. Hence, similarly as in [19], it follows the existence of $x_1 \in \mathscr{A}_{T_0}$ which satisfies (6) for $t \in [-r, T_0]$. Continuing this process we can define a unique function $x \in \mathscr{A}_T$ satisfying (6) on the whole interval $[-r, T]$. This completes the proof.

Let us denote by $\Lambda(\varphi, f)$ the set of all solutions of (6) corresponding to $(\varphi, f) \in \mathscr{A}_0 \times \mathscr{F}$.

Theorem 6. *Let $x \in \Lambda(\varphi, f)$, where f is locally Lipschitzean and suppose $(\varphi_n, f_n) \in \mathscr{A}_0 \times \mathscr{F}$ are such that $\Lambda(\varphi_n, f_n) \neq \emptyset$ for each $n = 1, 2, \cdots$ and $\| \varphi_n - \varphi \|_0 + \rho(f_n, f) \to 0$ as $n \to \infty$. Then $\| x^n - x \|_T \to 0$ as $n \to \infty$, where $x^n \in \Lambda(\varphi_n, f_n)$.*

Proof. (Sketch of the proof)

Let U_0 and K_0 be a neighborhood of $(\varphi, \dot{\varphi})$ and a continuous increasing function, respectively such that

$$|f(\cdot, x_1, y_1) - f(\cdot, x_2, y_2)|_t \leqslant K_0(t)[\| x_1 - x_2 \|_0 + | y_1 - y_2 |_0]$$

for (x_1, y_1), $(x_2, y_2) \in U_0$ and $t \in [0, T]$.

Let

$$x^0(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r, 0] \\ \varphi(0) & \text{for } t \in [0, T] \end{cases}$$

and suppose S_0 is a closed ball with the center $(\varphi, \dot{\varphi})$ and a radius $r_0 > 0$ such that $S_0 \subset U_0$. Select $T_0 \in [0, T]$ and $N \geqslant 1$ such that $\| \varphi_n - \varphi \|_0 \leqslant r_0/3$, $\rho(f_n, f) < 1$,

$$\frac{\rho(f_n,f)}{1-\rho(f_n,f)}\leqslant r_0/12,\quad \rho_T(f,0)<1,\quad \frac{\rho_{T_0}(f,0)}{1-\rho_{T_0}(f,0)}\leqslant r_0/12,\quad K_0(T_0)<1/2$$

and $\|\,x_t^0-x_0^0\,\|_0<r_0/3$ for $t\in[0,T_0]$. This is possible, because $\|\,\varphi_n-\varphi\,\|_0\to 0$, $\rho(f_n,f)\to 0$, $\rho_t(f,0)\to 0$, $K_0(t)\to 0$ and $\|\,x_t^0-x_0^0\,\|_0\to 0$ as $n\to\infty$ and $t\to 0+$, respectively. Hence it follows that

$$\int_0^{T_0}|f_n(t,x,y)-f(t,x,y)|\,dt\leqslant\frac{\rho(f_n,f)}{1-\rho(f_n,f)}\leqslant r_0/12$$

and

$$\int_0^{T_0}|f(t,x,y)|\,dt\leqslant\frac{\rho_{T_0}(f,0)}{1-\rho_{T_0}(f,0)}\leqslant r_0/12$$

for $(x,y)\in C_0\times L_0$ and $n\geqslant N$. Hence it follows that $(x_t^n,\dot{x}_t^n)\in S_0\subset U_0$ for $n\geqslant N$ and $t\in[0,T_0]$. Thus, for $n,m\geqslant N$ we get

$$\|x^n-x^m\|_{T_0}+|\,\dot{x}^n-\dot{x}^m\,|_{T_0}\leqslant\frac{1}{1-2K_0(T_0)}\Big[\|\,\varphi_n-\varphi_m\,\|_0$$
$$+2\cdot\frac{\rho(f_n,f)}{1-\rho(f_n,f)}+2\cdot\frac{\rho(f_m,f)}{1-\rho(f_m,f)}\Big].$$

Therefore, there exists a $x_0\in\mathscr{A}_{T_0}$ such that $\|\,x^n-x_0\,\|_{T_0}\to 0$ as $n\to\infty$ and

$$\begin{cases} x_0(t)=\varphi(t) & \text{for } t\in[-r,0]\\ \dot{x}_0(t)=f(t,(x_0)_t,(\dot{x}_0)_t) & \text{for a.e. } t\in[0,T_0]. \end{cases}$$

Continuing this process, we can define a function $\bar{x}\in\mathscr{A}_T$ such that $\bar{x}\in\Lambda(\varphi,f)$ and $\|\,x_n-\bar{x}\,\|_T\to 0$ as $n\to\infty$. But $\Lambda(\varphi,f)=\{x\}$. Then $\|\,x_n-x\,\|_T\to 0$ as $n\to\infty$, which completes the proof.

§ 4. Generic properties of functional-differential equations of neutral type.

We will need in this Section the following lemma presented in [10].

Lemma 7. *Suppose (Z,d) is a Baire space and let $A\subset B\subset Z$. If A is a residual subset of (Z,d), then B is a residual subset of (Z,d), too.*

Furthermore, we will use here the following, not published result of Lasota.

Lemma 8. *Let (X,d) be a complete metric space and S a dense subset of (X,d). Suppose a function $\chi\colon X\to[0,\infty)$ is such that $\chi(x_n)\to 0$ for any sequence $(x_n)\subset X$ such that $x_n\to x\in S$. Then the set $\mathscr{X}=\{x\in X;\,\chi(x)=0\}$ is a residual subset of (X,d).*

Proof. Let us observe that $X\backslash\mathscr{X}=\bigcup_{k=1}^{\infty}\mathscr{X}_k$, where $\mathscr{X}_k=\{x\in X: \chi(x)\geqslant 1/k\}$. Then it suffices only to show that $\operatorname{int}(\bar{\mathscr{X}}_k)=\emptyset$ for $k=1, 2, \cdots$. Suppose, that there exists k such that $\operatorname{int}(\bar{\mathscr{X}}_k)\neq\emptyset$. Then there exists an open ball $B(\hat{x}, \delta)$ with the center $\hat{x}$ and a radius $\delta>0$ such that $B(\hat{x}, \delta)\subset\mathscr{X}_k$. But S is dense in (X, d). Therefore there exists $x_0\in B(\hat{x}, \delta)\cap S$ such that $\chi(x_0)=0$. Furthermore, there exists $\lambda>0$ such that $B(x_0, \lambda)\subset B(\hat{x}, \delta)$ and such that $\chi(x)<1/k$ for $x\in B(x_0, \lambda)$. Indeed, suppose that it is not true. Then for $\lambda=1/n$ there exists $x_n\in B(x_0, 1/n)$ such that $\chi(x_n)\geqslant 1/k$. Of course, $x_n\to x_0\in S$ as $n\to\infty$, which contradicts the assumption on χ. Then $x_0\in B(\hat{x}, \delta)$ and $x_0\in\mathscr{X}_k$. This contradiction completes the proof.

Now we can prove the main results of this paper.

Theorem 9. *Let $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$ denote subset of $\mathscr{A}_0\times\mathscr{F}$ such that*

(a) *for each $(\varphi, f)\in\mathscr{X}_1$ an initial value problem* (6) *has at most one solution,*

(b) *for each $(\varphi, f)\in\mathscr{X}_2$ an initial value problem has at least one solution,*

(c) *for each $(\varphi, f)\in\mathscr{X}_3$ a solution $x(\varphi, f)$ depends continuously on (φ, f), i.e. for every sequence $\{(\varphi_n, f_n)\}$ of $\mathscr{X}_3$ such that $\|\varphi_n-\varphi\|_0+\rho(f_n, f)\to 0$ as $n\to\infty$ and $\Lambda(\varphi_n, f_n)\neq\emptyset$ we have $\|x(\varphi_n, f_n)-x(\varphi, f)\|_T\to 0$, where $x(\varphi_n, f_n)\in\Lambda(\varphi_n, f_n)$ for each $n=1, 2, \cdots$. Then $\mathscr{X}_1\cap\mathscr{X}_2\cap\mathscr{X}_3$ is a residual subset of $\mathscr{A}_0\times\mathscr{F}$.*

Proof. Let $X=\mathscr{A}_0\times\mathscr{F}$ and let us denote by S the set of all (φ, f) with locally Lipschitzean f. We have $\bar{S}=X$. Define $\chi: X\to[0, \infty)$ by setting

$$\chi(\varphi, f)=\lim_{\delta\to 0}\sup\{\|x(\varphi_1, f_1)-x(\varphi_2, f_2)\|_T: (\varphi_1, f_1), (\varphi_2, f_2)\in B((\varphi, f), \delta)\},$$

where $(\varphi, f)\in X$, $x(\varphi_i, f_i)\in\Lambda(\varphi_i, f_i)$ $i=1, 2$ and $B((\varphi, f), \delta)$ denotes an open ball of X with the center (φ, f) and a radius $\delta>0$.

For each $(\varphi, f)\in X$ and $(\varphi_1, f_1), (\varphi_2, f_2)\in S\cap B((\varphi, f), \delta)$ we have $\Lambda(\varphi_i, f_i)\neq\emptyset$ for $i=1, 2$. Therefore, $\chi(\varphi, f)$ is defined for each $(\varphi, f)\in X$.

Now let us observe that

(i) $(\chi(\varphi, f)=0)\Rightarrow(\Lambda(\varphi, f)$ has at most one point),

(ii) $(\chi(\varphi, f)=0)\Rightarrow(\Lambda(\varphi, f)\neq\emptyset)$,

(iii) $[\chi(\varphi, f)=0$ and $\lim_{n\to\infty}(\|\varphi_n-\varphi\|_0+\rho(f_n, f))=0]\Rightarrow(\lim_{n\to\infty}\chi(\varphi_n, f_n)=0)$,

(iv) $(\chi(\varphi, f)=0)\Rightarrow$(a solution of (6) depends continuously on (φ, f)),

and

(v) $(\varphi, f)\in S\Rightarrow\chi(\varphi, f)=0$.

Indeed, suppose $\chi(\varphi, f)=0$ and let $\Lambda(\varphi, f)=\{x, y\}$, where $\|x-y\|_T>0$. Let $\varepsilon_0=\|x-y\|_T$. Since

$$\lim_{\delta\to 0}\sup\{\|x(\varphi_1, f_1)-x(\varphi_2, f_2)\|_T: (\varphi_i, f_i)\in B((\varphi, f), \delta)\ i=1, 2\},$$

then there exists a $\delta_0>0$ such that $||x(\varphi_1,f_1)-x(\varphi_2,f_2)||_T<\varepsilon_0/3$ for each (φ_1,f_1), (φ_2,f_2) $\in B((\varphi,f),\delta)$. But

$$\varepsilon_0=||x-y||_T\leqslant||x-x(\varphi_1,f_1)||_T+||x(\varphi_1,f_1)-x(\varphi_2,f_2)||_T+||x(\varphi_2,f_2)-y||_T<\varepsilon_0.$$

Then $||x-y||_T=0$. Suppose $\chi(\varphi,f)=0$ and let $\{(\varphi,f_n)\}$ be a sequence of S such that $\rho(f_n,f)\to0$ as $n\to\infty$ and let $x_n=x(\varphi,f_n)\in\Lambda(\varphi,f_n)$ for $n=1,2,\cdots$. For every $\delta>0$ there is a $N\geqslant1$ such that (φ,f_n), $(\varphi,f_m)\in B((\varphi,f),\delta)$ and $\rho(f_n,f)<1$ for $n,m\geqslant N$. Therefore, we have

$$||x_n-x_m||_T\leqslant\sup\{||x(\varphi_1,f_1)-x(\varphi_2,f_2)||_T:(\varphi_i,f_i)\in B((\varphi,f),\delta),\ i=1,2\}$$

for $n,m\geqslant N$. From this and $\chi(\varphi,f)=0$ it follows that $||x_n-x_m||_T\to0$ as $n,m\to\infty$. Then, there exists a $x\in\mathscr{A}_T$ such that $||x_n-x||_T\to0$ as $n\to\infty$. Since $x(t)=\varphi(t)$ for $t\in[-r,0]$ and

$$\left|x(t)-\varphi(0)-\int_0^t f(s,x_s,\dot{x}_s)\,ds\right|\leqslant||x-x_n||_T$$
$$+\frac{\rho(f_n,f)}{1-\rho(f_n,f)}+\int_0^t|f(s,x_s,\dot{x}_s)-f(s,(x_n)_s,(\dot{x}_n)_s)|\,ds$$

for $t\in[0,T]$ and $n\geqslant N$, then $x\in\Lambda(\varphi,f)$.

For the proof of (iii), suppose $\chi(\varphi,f)=0$, $||\varphi_n-\varphi||_0+\rho(f_n,f)\to0$ as $n\to\infty$ and that $\{\chi(\varphi_n,f_n)\}$ is not converging to 0 as $n\to\infty$. Then, there are $\eta>0$ and a sequence $(\psi_n,g_n)\in X$ such that $||\psi_n-\varphi||_0+\rho(g_n,f)\to0$ as $n\to\infty$ and so that $\chi(\psi_n,g_n)\geqslant\eta$. Consequently, we can find subsequence (ψ^i_{nk},g^i_{nk}) $(i=1,2)$ of X such that

$$||\psi^i_{nk}-\psi_n||_0+\rho(g^i_{nk},g_n)\to0 \qquad \text{as } k\to\infty$$

and so that $x^i_{nk}=x(\psi^i_{nk},g^i_{nk})$ $(i=1,2)$, satisfy $||x^1_{nk}-x^2_{nk}||_T\geqslant\eta/2$. But

$$||\psi^i_{nk}-\varphi||_0+\rho(g^i_{nk},f)\to0 \qquad \text{as } k\to\infty.$$

Hence and $\chi(\varphi,f)=0$, similarly as in the proof of (ii), we get

$$\lim_{k\to\infty}||x^1_{nk}-x^2_{nk}||_T=||x^1-x^2||_T,$$

where x^1, $x^2\in\Lambda(\varphi,f)$. But, by virtue of (i), we have $x^1=x^2$. Therefore

$$\eta/2\leqslant||x^1_{nk}-x^2_{nk}||_T\to0 \qquad \text{as } k\to\infty,$$

which contradicts $\eta>0$.

Suppose $\chi(\varphi,f)=0$ and let (φ_n,f_n) be a sequence of X such that

$$||\varphi_n-\varphi||_0+\rho(f_n,f)\to0 \qquad \text{as } n\to\infty$$

and such that $\Lambda(\varphi_n,f_n)\neq\emptyset$ for each $n=1,2,\cdots$. By virtue of (i) and (ii), $\chi(\varphi,f)=0$

implies that there exists $x \in \mathscr{A}_T$ such that $\Lambda(\varphi, f)=\{x\}$. Using the techniques mentioned above in the proof of (ii), we can show that for every sequence $x_n \in \mathscr{A}_T$ such that $x_n \in \Lambda(\varphi_n, f_n)$, we have $\|x_n - x\|_T \to 0$ as $n \to \infty$.

Suppose (v) is not true. Then there exists $(\tilde{\varphi}, \tilde{f}) \in S$ such that $\chi(\tilde{\varphi}, \tilde{f}) \geqslant \eta > 0$. Therefore, for each $n=1, 2, \cdots$ there are (φ_n^1, f_n^1), $(\varphi_n^2, f_n^2) \in B((\tilde{\varphi}, \tilde{f}), 1/n)$ such that $\|x_n^1 - x_n^2\|_T \geqslant \eta/2$, where $x_n^i \in \Lambda(\varphi_n^i, f_n^i)$ for $i=1, 2$ and $n=1, 2, \cdots$. Since

$$\|\varphi_n^i - \tilde{\varphi}\|_0 + \rho(f_n^i, \tilde{f}) \to 0 \qquad \text{for } i=1, 2,$$

as $n \to \infty$ and $(\tilde{\varphi}, \tilde{f}) \in S$, then by virtue of Theorem 6, we have $\eta/2 \leqslant \|x^1 - x^2\|_T = 0$, because $x^1, x^2 \in \Lambda(\tilde{\varphi}, \tilde{f})$ and $\Lambda(\tilde{\varphi}, \tilde{f})$ has exactly one point. Therefore, (v) holds.

Let $\Omega = \{(\varphi, f) \in X : \chi(\varphi, f) = 0\}$. By virtue of (iii) and Lemma 7, Ω is a residual subset of X. But (i), (ii) and (iv) imply that $\Omega \subset \mathscr{X}_1 \cap \mathscr{X}_2 \cap \mathscr{X}_3$. Therefore, in virtue of Lemma 6, this completes the proof.

Let $(\varphi, f) \in X$ and let $(x_n^{(\varphi, f)})$ be a sequence of $\mathscr{A}_T$ defined by

$$(7) \qquad x_n^{(\varphi, f)}(t) = \begin{cases} \varphi(t) & \text{for } t \in [-r, 0] \\ \varphi(0) + \int_0^t f(s, (x_{n-1}^{(\varphi, f)})_s, (\dot{x}_{n-1}^{(\varphi, f)})_s) ds & \text{for } t \in [0, T], \end{cases}$$

where $x_0^{(\varphi, f)}(t) = \varphi(t)$ for $t \in [-r, 0)$ and $x_0^{(\varphi, f)}(t) = \varphi(0)$ for $t \in [0, T]$.

Similarly to the proof of Theorem 5, it can be proved that for every $(\varphi, f) \in S$, $(x_n^{(\varphi, f)})$ is converging in $\mathscr{A}_T$.

Let us denote by $\mathscr{F}_\alpha$ a closed ball of $\mathscr{F}$ with the center 0 and a radius $\alpha \in (0, 1)$ and let $X_\alpha = \mathscr{A}_0 \times \mathscr{F}_\alpha$. It is not difficult to verify that the set S_α of all $(\varphi, f) \in X_\alpha$ with locally Lipschitzean f, is a dense subset of X_α.

We shall show now that non-convergence of a sequence $(x_n^{(\varphi, f)})$ is in any sense a rare case if $(\varphi, f) \in X_\alpha$.

Theorem 10. *The set $\mathscr{X}$ of all $(\varphi, f) \in X_\alpha$ for which a sequence $(x_n^{(\varphi, f)})$ defined by* (7) *is converging in $\mathscr{A}_T$ is a residual subset of X_α for each $\alpha \in (0, 1)$.*

Proof. Let $\chi: X_\alpha \to [0, \infty)$ be defined by $\chi(\varphi, f) = \lim_{m \to \infty} \operatorname{diam} E_m^{(\varphi, f)}$, where $E_m^{(\varphi, f)} = \{x_m^{(\varphi, f)}, x_{m+1}^{(\varphi, f)}, \cdots\}$ and diam $E_m^{(\varphi, f)}$ denotes the diameter of $E_m^{(\varphi, f)}$. We have $E_{m+1}^{(\varphi, f)} \subset E_m^{(\varphi, f)}$ and then $\operatorname{diam} E_{m+1}^{(\varphi, f)} \leqslant \operatorname{diam} E_m^{(\varphi, f)}$ for each $m=1, 2, \cdots$. Since

$$0 \leqslant \operatorname{diam} E_1^{(\varphi, f)} \leqslant \frac{4\rho(f, 0)}{1-\rho(f, 0)} \qquad \text{for } (\varphi, f) \in X_\alpha,$$

then $\lim_{m \to \infty} \operatorname{diam} E_m^{(\varphi, f)}$ exists for each $(\varphi, f) \in X_\alpha$. We have of course $\chi(\varphi, f) = 0$ iff $(x_m^{(\varphi, f)})$ is converging in $\mathscr{A}_T$.

We shall show now that $\chi(\varphi_n, f_n) \to 0$ for every sequence (φ_n, f_n) of X_α such that

$||\varphi_n-\varphi||_0+\rho(f_n,f)\to 0$ as $n\to\infty$, where $(\varphi,f)\in S_\alpha$.

Let us observe first that $||x_m^{(\varphi_n,f_n)}-x_m^{(\varphi,f)}||_T\to 0$ as $n\to\infty$, uniformly with respect to $m\geqslant 1$. Indeed, select a neighborhood U_0 of $(\varphi,\dot\varphi)$ and a continuous increasing function $K_0\colon[0,T]\to[0,\infty)$ such that

$$|f(\cdot,x_1,y_1)-f(\cdot,x_2,y_2)|_t\leqslant K_0(t)[||x_1-x_2||_0+|y_1-y_2|_0]$$

for $(x_1,y_1),(x_2,y_2)\in U_0$ and $t\in[0,T]$. Let B_0 be a closed ball of $C_0\times L_0$ with the center $(\varphi,\dot\varphi)$ and a radius $t_0>0$ such that $B_0\subset U_0$. Similarly to the proof of Theorem 7, we can select $T_0\in[0,T]$ and $N\geqslant 1$ such that

$$((x_m^{(\varphi,f)})_t,(\dot x_m^{(\varphi,f)})_t),\qquad((x_m^{(\varphi_n,f_n)})_t,(\dot x_m^{(\varphi_n,f_n)})_t)\in B_0\subset U_0$$

for $n\geqslant N$, $t\in[0,T]$ and $m\geqslant 1$. Therefore

$$||x_m^{(\varphi_n,f_n)}-x_m^{(\varphi,f)}||_{T_0}\leqslant\frac{1}{1-2K_0(T_0)}\left[||\varphi_n-\varphi||_0+2\cdot\frac{\rho(f_n,f)}{1-\rho(f_n,f)}\right]$$

for $n\geqslant N$, which proves that

$$||x_m^{(\varphi_n,f_n)}-x_m^{(\varphi,f)}||_{T_0}\to 0\qquad\text{as } n\to\infty,$$

uniformly with respect to $m\geqslant 1$. Continuing this process we can easily get

$$||x_m^{(\varphi_n,f_n)}-x_m^{(\varphi,f)}||_T\to 0\qquad\text{as } n\to\infty,$$

uniformly with respect to $m\geqslant 1$

Suppose now that (φ_n,f_n) is a sequence of X_α converging to $(\varphi,f)\in S_\alpha$ such that $\chi(\varphi_n,f_n)$ is not converging to zero. Then there are $\eta>0$ and a subsequence (φ_k,f_k) of (φ_n,f_n) such that $||\varphi_k-\varphi||_0+\rho(f_k,f)\to 0$ as $k\to\infty$ and $\chi(\varphi_k,f_k)\geqslant\eta$ for each $k\geqslant 1$. Hence, in particular follows that $\operatorname{diam}E_m^{(\varphi_k,f_k)}\geqslant\eta$ for $k,m\geqslant 1$. But a sequence $(x_m^{(\varphi_k,f_k)})$ is converging in $\mathscr{A}_T$, uniformly with respect to $m\geqslant 1$ to $x_m^{(\varphi,f)}$. Therefore, for $n,m\geqslant 1$ and sufficiently large k, say $k\geqslant N_k$, we have

$$||x_m^{(\varphi_k,f_k)}-x_n^{(\varphi_k,f_k)}||_T\leqslant\frac{1}{k}+||x_m^{(\varphi,f)}-x_n^{(\varphi,f)}||_T.$$

Thus

$$\eta\leqslant\operatorname{diam}E_m^{(\varphi_k,f_k)}\leqslant\frac{1}{k}+\operatorname{diam}E_m^{(\varphi,f)}=\frac{1}{k}$$

for $k\geqslant N_k$, because $(\varphi,f)\in S_\alpha$. This contradicts $\eta>0$.

Now, similarly to the proof of Theorem 9, we can see that a set

$$\Omega_\alpha = \{(\varphi, f) \in X_\alpha : \chi(\varphi, f) = 0\}$$

is a residual subset of X_α. Therefore, $\mathscr{X}$ is a residual subset of X_α, too. The proof is complete.

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