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Local Equivalence of Linear Pfaffian Systems with Simple Poles

By

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Dedicated to Professor Taro Ura in commemoration of the sixtieth anniversary of his birth

§ 1. Introduction.

In this paper we study completely integrable linear Pfaffian systems with simple poles on a divisor in a neighborhood of a point at which many irreducible components of the divisor intersect transversally or contact.

Let U be an open neighborhood of the origin of C^n and S the union of non-singular irreducible divisors $\{S_j\}_{1\le j\le\mu}$ through the origin. Let $x=(x_1, \cdots, x_n)$ be a system of coordinates of C^n and let $f_j(x)=0$ $(1\le j\le\mu)$ be a local equation of S_j, namely,

$$S_j=\{x \in U; f_j(x)=0\},$$

where $f_j(x)$ is a holomorphic function in $U, f_j(0)=0$ and $df_j(x)\neq 0$ for every $x \in U$. The case where $\{S_j\}_{1\le j\le\mu}$ are in general position has been completely investigated in the paper [5] by M. Yoshida and the author. Therefore in this paper we suppose that $\{S_j\}_{1\le j\le\mu}$ are not in general position.

We want to find canonical forms of linear Pfaffian systems in U having simple poles on S. For this purpose, we introduce a notion of *holomorphic equivalence*. Consider completely integrable linear Pfaffian systems of the form

$$du=\Omega u, \qquad \Omega=\sum_{j=1}^{\mu} A_j(x)df_j/f_j+\Theta, \tag{1.1}$$

$$dv=\Omega' v, \qquad \Omega'=\sum_{j=1}^{\mu} B_j(x)df_j/f_j+\Theta'. \tag{1.2}$$

Here u and v are complex m dimensional column vectors, all $A_j(x)$ and $B_j(x)$, $1\le j\le\mu$, are m by m matrices of which the components are holomorphic functions in U, Θ and Θ' are m by m matrices of which the components are holomorphic differential 1-forms in U. The integrability conditions for (1.1) and (1.2) are $d\Omega=\Omega\wedge\Omega$ and $d\Omega'=\Omega'\wedge\Omega'$ respectively. We say that systems (1.1) and (1.2) are *holomorphically equivalent* if there exists a holomorphic invertible matrix $P(x)$ in U such that $u=P(x)v$ transforms system (1.1) to system (1.2). It is easy to see that if systems (1.1) and (1.2) are holomorphically equivalent then there exists an invertible constant matrix P_0 such that

$$B_j(0)=P_0^{-1}A_j(0)P_0$$

for every $1\leq j\leq\mu$. Therefore we can suppose without loss of generality that the conditions

$$A_j(0)=B_j(0), \qquad 1\leq j\leq\mu$$

are satisfied for holomorphically equivalent systems (1.1) and (1.2).

In Section 2, we state our main results; Theorems 1 and 2. In Theorem 1, we give some sufficient conditions for systems (1.1) and (1.2) to be holomorphically equivalent. The conditions as those appearing in Theorem 1 are called eigenvalue conditions. As a corollary of Theorem 1, we obtain Theorem 2. It is a reduction theorem for a system of the form (1.1). We note that it gives an answer to the question of finding a canonical form of system (1.1). In Section 3, we prove Theorem 1 by establishing preliminary propositions in the case where irreducible divisors are in general position and by solving a kind of connection problem.

§ 2. Statement of results.

Let U and S be those given in Section 1. Note that each irreducible component S_j $(1\leq j\leq\mu)$ of S is nonsingular.

Denote by $\Sigma(S)$ the analytic set of points at which the irreducible components of S are not in general position. By assumption, $\Sigma(S)$ is not empty, therefore we make a Hopf transformation. It is well known that there exists a finite sequence $\sigma_1,\cdots,\sigma_s$ such that

(i) each $\sigma_{s'}^{-1}$ $(1\leq s'\leq s)$ is a quadric transformation from $U^{s'-1}$ to $U^{s'}=\sigma_{s'}^{-1}(U^{s'-1})$ along an irreducible component of $\Sigma(S^{s'-1})$, where $U^{s'-1}=\sigma_{s'-1}^{-1}(U^{s'-2})$, $S^{s'-1}=\sigma_{s'-1}^{-1}(S^{s'-2})$,

(ii) the irreducible components of $\sigma^{-1}(S)$ $(\subset\sigma^{-1}(U))$ are in general position where $\sigma=\sigma_1\cdots\sigma_s$.

We call σ^{-1} a Hopf transformation.

Denote $\sigma^{-1}(U)$ and $\sigma^{-1}(S)$ by U' and S' respectively. Decompose S' into the union of irreducible components as

$$S'=\bigcup_{j=1}^{\mu+\nu}S'_j$$

where we suppose that $\sigma(S'_j)=S_j$ for each $1\leq j\leq\mu$. We call S'_j $(\mu+1\leq j\leq\mu+\nu)$ an exceptional divisor. Then the residues of $\sigma^*\Omega$ and $\sigma^*\Omega'$ on S'_j $(1\leq j\leq \mu+\nu)$ are defined and will be denoted by $\mathrm{Res}_{S'_j}\sigma^*\Omega$ and $\mathrm{Res}_{S'_j}\sigma^*\Omega'$. It is easy to see that $\mathrm{Res}_{S'_j}\sigma^*\Omega$ and $\mathrm{Res}_{S'_j}\sigma^*\Omega'$ are holomorphic on S'_j. Denote $S'_j\cap\sigma^{-1}(0)$ by S''_j. Then, we can assume $S''_j\neq\emptyset$ without loss of generality, because for sufficiently small U, this condition is satisfied. The restrictions of $\mathrm{Res}_{S'_j}\sigma^*\Omega$ and $\mathrm{Res}_{S'_j}\sigma^*\Omega'$ on S''_j denoted by $\mathrm{Res}_{S'_j}\sigma^*\Omega|_{S''_j}$ and $\mathrm{Res}_{S'_j}\sigma^*\Omega'|_{S''_j}$ are constant matrices. We can verify that, for every $1\leq j\leq \mu$,

$$\mathrm{Res}_{S'_j}\sigma^*\Omega|_{S''_j}=A_j(0), \qquad \mathrm{Res}_{S'_j}\sigma^*\Omega'|_{S''_j}=B_j(0),$$

and for every $\mu+1\leq j\leq \mu+\nu$, $\mathrm{Res}_{S'_j}\sigma^*\Omega|_{S''_j}$ and $\mathrm{Res}_{S'_j}\sigma^*\Omega'|_{S''_j}$ are linear combinations of $\{A_l(0)\}_{1\leq l\leq\mu}$ and $\{B_l(0)\}_{1\leq l\leq\mu}$ respectively with nonnegative integer coefficients. Now we state our results.

Theorem 1. *Consider two completely integrable linear Pfaffian systems* (1.1) *and* (1.2) *where all $A_j(x)$ and $B_j(x)$, $1\leq j\leq\mu$, are holomorphic in U satisfying*

$$A_j(0)=B_j(0), \qquad 1\leq j\leq\mu,$$

Θ and Θ' are matrices of which the components are holomorphic differential 1-forms in U. Let σ^{-1} be a Hopf transformation as above. Suppose that no two eigenvalues of $A_j(0)=B_j(0)$ $(1\leq j\leq\mu)$ differ by a positive integer and no two eigenvalues of $\mathrm{Res}_{S'_j}\sigma^*\Omega|_{S''_j}=\mathrm{Res}_{S'_j}\sigma^*\Omega'|_{S''_j}$ $(\mu+1\leq j\leq\mu+\nu)$ differ by an integer. Then systems (1.1) *and* (1.2) *are holomorphically equivalent.*

As an immediate consequence of Theorem 1, we have

Theorem 2. *Consider a completely integrable linear Pfaffian system of the form* (1.1) *satisfying the same eigenvalue conditions as those in Theorem* 1. *If further the system*

$$dw=\left(\sum_{j=1}^{\mu}A_j(0)df_j/f_j\right)w \tag{2.1}$$

is completely integrable, then systems (1.1) *and* (2.1) *are holomorphically equivalent, namely, there exists a holomorphic invertible matrix $P(x)$ in U such that $u=P(x)w$ transforms system* (1.1) *to system* (2.1).

Remark. In the case where each S_j $(1\leq j\leq\mu)$ is a complex hyperplane through the origin of $\boldsymbol{C}^n$, namely, each $f_j(x)$ is a homogeneous linear form of $x_1,\cdots,x_n$, it can be verified that, if (1.1) is integrable, then (2.1) is integrable. A reduction theorem in this case was obtained by a different method in [4]. We can also verify that the integrability for (2.1) follows from the integrability for (1.1) in the case where $n=\mu=2$, $f_1(x_1,x_2)=x_2$, $f_2(x_1,x_2)=x_2-cx_1^l$, c and l being a nonzero constant and a positive integer respectively.

§ 3. Proof of Theorem 1.

In this section, we shall prove Theorem 1. We shall first study linear Pfaffian systems in the case where the irreducible components of a divisor are in general position and where there is an analytic subset of an irreducible component along which the coefficients of the systems are constant.

3.1. Preliminary propositions. We explain the notation used in this part. We denote by $D^n(\rho)$ an open polydisk in $\boldsymbol{C}^n$ defined by

$$\{x \in \boldsymbol{C}^n; |x_i|<\rho_i, 1\leq i\leq n\},$$

where $\rho=(\rho_1, \cdots, \rho_n)$, $\rho_i>0$ $(1\leq i\leq n)$. I denotes the identity matrix of rank m. For a multi-index $k=(k_1, \cdots, k_n)$ where each k_i $(1\leq i\leq n)$ is an integer, we denote $x_1^{k_1}\cdots x_n^{k_n}$ and $k_1+\cdots+k_n$ by x^k and $|k|$ respectively. For two multi-indices $k=(k_1, \cdots, k_n)$ and $l=(l_1, \cdots, l_n)$, $k\geq l$ means $k_i\geq l_i$ for each $1\leq i\leq n$, and $k>l$ means $k\geq l$ and $k_i>l_i$ for some $1\leq i\leq n$.

Proposition 1. *Let*

$$du=\left(\sum_{i=1}^{n} A_i(x)dx_i/x_i\right)u \tag{3.1}$$

be a completely integrable linear Pfaffian system where all $A_i(x)$, $1\leq i\leq p$, *and* $A_i(x)/x_i$, $p+1\leq i\leq n$, *are holomorphic in a polydisk* $D^n(\rho)$. *Assume that no two eigenvalues of* $A_{i_0}(0)$ *for some* $1\leq i_0\leq p$ *differ by an integer and no two eigenvalues of* $A_i(0)$ $(1\leq i\leq p,$ $i\neq i_0)$ *differ by a positive integer. Then there exists a unique holomorphic invertible matrix* $P(x)$ *in* $D^n(\rho)$ *with* $P(0)=I$ *such that the change of variables* $u=P(x)v$ *transforms system* (3.1) *to the system*

$$dv=\left(\sum_{i=1}^{p} A_i(0)dx_i/x_i\right)v. \tag{3.2}$$

Let W *be a nonsingular analytic subset of* $D^n(\rho)\cap\{x_{i_0}=0\}$ *through the origin of* $\boldsymbol{C}^n$ *of dimension* $r\geq 1$. *If further*

$$A_i(x)=A_i(0) \tag{3.3}$$

holds for every $x \in W$ *and for every* $1\leq i\leq n$, *then the matrix* $P(x)$ *satisfies*

$$P(x)=I \tag{3.4}$$

for every $x \in W$.

Proof. Under the above eigenvalue conditions, we can verify by Theorem 3 and Theorem 5 in the paper [5], that there exists a unique holomorphic invertible matrix

$P(x)$ in $D^n(\rho)$ with $P(0)=I$ such that $u=P(x)v$ changes system (3.1) to system (3.2). Therefore we have only to prove the last part of this proposition.

We shall prove that $P(x)$ is independent of $x \in W$ under condition (3.3). Let us expand $P(x)$ and each $A_i(x)$ $(1\leq i\leq n)$ into convergent power series of $x=(x_1, \cdots, x_n)$ as

$$P(x)=I+\sum_{k>0} P_k x^k, \tag{3.5}$$

$$A_i(x)=\sum_{k\geq 0} A_{ik} x^k. \tag{3.6}$$

The integrability condition for system (3.1) implies

$$A_{i0}A_{i'0}-A_{i'0}A_{i0}=0$$

for every $1\leq i, i'\leq n$. Therefore, by the assumption that $A_{i_0}(0)=A_{i_0 0}$ has distinct eigenvalues, we can suppose without loss of generality that each A_{i0} $(1\leq i\leq p)$ is of diagonal form. Note that $A_{i0}=0$ for every $p+1\leq i\leq n$. Since $u=P(x)v$ transforms system (3.1) to system (3.2), we have the equation

$$dP=\Big(\sum_{i=1}^{n} A_i(x)dx_i/x_i\Big)P-P\Big(\sum_{i=1}^{p} A_i(0)dx_i/x_i\Big). \tag{3.7}$$

Then, by substituting (3.5) and (3.6) in (3.7) and by identifying the coefficients of like powers of x, we have the following recursion formulas

$$k_i P_k - A_{i0}P_k + P_k A_{i0} = \sum_{k'>0,\, k'+k''=k} A_{ik'}P_{k''}, \qquad k>0,\ 1\leq i\leq n. \tag{3.8}$$

We note that the first half of this proposition asserts that P_k's are determined by (3.8) compatibly and uniquely.

Take a system of local coordinates $t=(t_1, \cdots, t_r) \in \boldsymbol{C}^r$ of W so that

$$W=\{x=x(t);\, t \in D^r(\tau)\},$$

where each $x_i(t)$ $(1\leq i\leq n)$ is holomorphic in t in a polydisk $D^r(\tau)$ with $x_i(0)=0$. Note that $x_{i_0}(t)\equiv 0$. Expand $x(t)^k$ $(k>0)$ into a convergent power series of $t=(t_1, \cdots, t_r)$ as

$$x(t)^k=\sum_{h>0} g_h^k t^h \tag{3.9}$$

where $h=(h_1, \cdots, h_r)$, each h_i being a nonnegative integer. It is easy to see that

$$g_h^k=0 \tag{3.10}$$

holds if $|k|>|h|$ or if $k_{i_0}>0$. From the identity

$$x(t)^{k'+k''}=x(t)^{k'}x(t)^{k''}$$

it follows that

(3.11) $$g_h^{k'+k''}=\sum_{h'+h''=h} g_{h'}^{k'}g_{h''}^{k''}.$$

By substituting (3.9) in (3.6), we have

$$A_i(x(t))=A_{i0}+\sum_{h>0}\left(\sum_{k>0} g_h^k A_{ik}\right)t^h,$$

therefore condition (3.3) implies

(3.12) $$\sum_{k>0} g_h^k A_{ik}=0,$$

for every $h>0$ and $1\leq i\leq n$. By (3.10), we see that the summation in the left hand side of (3.12) is a finite summation.

We shall show that $P(x(t))$ is independent of t. For this purpose, we develop $P(x(t))$ into a convergent power series of t as

$$P(x(t))=I+\sum_{h>0} Q_h t^h,$$

then, by (3.5) and (3.9), we have

(3.13) $$Q_h=\sum_{k>0} g_h^k P_k,$$

for every $h>0$. Multiply both sides of equation (3.8) by g_h^k $(h>0)$ and make summation with respect to $k>0$, then we obtain

$$\sum_{k>0} k_i g_h^k P_k - A_{i0}Q_h + Q_h A_{i0} = \sum_{k>0}\ \sum_{k'>0,\, k'+k''=k} g_h^{k'+k''} A_{ik'} P_{k''}.$$

By (3.11), we see that the right hand side of this equation is equal to

$$\sum_{h'+h''=h}\left(\sum_{k'>0} g_{h'}^{k'} A_{ik'}\right)\left(\sum_{k''>0} g_{h''}^{k''} P_{k''}\right),$$

which is equal to zero by virtue of (3.12). Hence we obtain

(3.14) $$\sum_{k>0} k_i g_h^k P_k - A_{i0}Q_h + Q_h A_{i0}=0,$$

for every $h>0$ and $1\leq i\leq n$. Consider this equation for $i=i_0$. Since $g_h^k=0$ for $k_{i_0}>0$, equation (3.14) for $i=i_0$ becomes

$$-A_{i_00}Q_h+Q_hA_{i_00}=0.$$

Therefore, since A_{i_00} is of diagonal form with distinct eigenvalues, Q_h is of diagonal form. Hence, noting again that each A_{i0} $(1\leq i\leq n)$ is of diagonal form, we have from (3.14) that

$$\sum_{k>0} k_i g_h^k P_k = 0, \tag{3.15}$$

for every $h>0$ and $1\leq i\leq n$. From (3.5), we derive

$$x_i \cdot \partial P/\partial x_i = \sum_{k>0} k_i P_k x^k,$$

and then considering (3.15), we obtain

$$(x_i \cdot \partial P/\partial x_i)(t) = \sum_{h>0} \Big(\sum_{k>0} k_i g_h^k P_k\Big) t^h = 0, \tag{3.16}$$

for every $t \in D^r(\tau)$ and $1\leq i\leq n$. Denote by R the set of i's such that $x_i(t)\not\equiv 0$. Then it is easy to see that $i_0 \notin R$ and the cardinal number of R is greater than or equal to r. For each $i \in R$, denote by T_i the analytic set

$$\{t \in D^r(\tau);\, x_i(t)=0\}.$$

Note that $\dim T_i \leq r-1$. Thus (3.16) is reduced to that

$$(\partial P/\partial x_i)(t) = 0$$

for $t \in D^r(\tau) - T_i$ and $i \in R$. Hence, for every $1\leq j\leq r$,

$$\partial P/\partial t_j = \sum_{i\in R} (\partial P/\partial x_i)(\partial x_i/\partial t_j)$$

vanishes on $D^r(\tau) - \bigcup_{i\in R} T_i$, and so on $D^r(\tau)$. Therefore $P(x(t))$ is independent of t, namely, $P(x)$ is constant on W. Since W contains the origin of C^n, we have $P(x) = P(0) = I$ for $x \in W$. Thus we have completed the proof of Proposition 1.

As a consequence of Proposition 1, we have

Proposition 2. *Consider two completely integrable linear Pfaffian systems of the form*

$$du = \Big(\sum_{i=1}^{n} A_i(x) dx_i/x_i\Big) u, \tag{3.17}$$

$$dv = \Big(\sum_{i=1}^{n} B_i(x) dx_i/x_i\Big) v, \tag{3.18}$$

where all $A_i(x)$, $B_i(x)$, $1\leq i\leq p$, and $A_i(x)/x_i$, $B_i(x)/x_i$, $p+1\leq i\leq n$, are holomorphic in a polydisk $D^n(\rho)$. Assume that

(i) $\quad A_i(0) = B_i(0), \qquad 1\leq i\leq p,$

(ii) *no two eigenvalues of* $A_{i_0}(0)=B_{i_0}(0)$ *for some* $1\leq i_0\leq p$ *differ by an integer and no two eigenvalues of* $A_i(0)=B_i(0)$ $(1\leq i\leq p,\ i\neq i_0)$ *differ by a positive integer.*

Then there exists a unique holomorphic invertible matrix $P(x)$ *in* $D^n(\rho)$ *with* $P(0)=I$ *such that the change of variables* $u=P(x)v$ *transforms system* (3.17) *to system* (3.18).

Let W *be a nonsingular analytic subset of* $D^n(\rho)\cap\{x_{i_0}=0\}$ *through the origin of* $\boldsymbol{C}^n$ *of dimension* $r\geq 1$. *If further*

$$\text{(iii)}\qquad A_i(x)=A_i(0)=B_i(x)=B_i(0)$$

holds for every $x\in W$ *and for every* $1\leq i\leq n$, *then the matrix* $P(x)$ *satisfies*

$$P(x)=I \tag{3.19}$$

for every $x\in W$.

Proof. We first show the existence of $P(x)$. From Proposition 1, there exist holomorphic invertible matrices $P'(x)$ and $P''(x)$ in $D^n(\rho)$ with $P'(0)=P''(0)=I$ such that $u=P'(x)w$ and $u=P''(x)w$ change systems (3.17) and (3.18) to the same system

$$dw=\left(\sum_{i=1}^{p} A_i(0)dx_i/x_i\right)w \tag{3.20}$$

respectively. Therefore if $P(x)$ is defined by $P(x)=P'(x)P''(x)^{-1}$, then $P(x)$ is holomorphic in $D^n(\rho)$, $P(0)=I$, and the change of variables $u=P(x)v$ transforms system (3.17) to system (3.18). We next show the uniqueness of $P(x)$. Note that $P'(x)$ and $P''(x)$ are unique by Proposition 1. Let $P(x)$ be any holomorphic invertible matrix in $D^n(\rho)$ with $P(0)=I$ such that $u=P(x)v$ transforms system (3.17) to system (3.18). Then $u=P(x)P''(x)w$ changes system (3.17) to system (3.20). Since $P(0)P''(0)=I$, from the uniqueness property in Proposition 1, $P(x)P''(x)$ must be equal to $P'(x)$, hence $P(x)$ must be equal to $P'(x)P''(x)^{-1}$, which shows the uniqueness of $P(x)$. By Proposition 1, we have $P'(x)=I$ and $P''(x)=I$ for every $x\in W$ under the condition (iii), which proves the latter half of Proposition 2.

3.2. Proof of Theorem 1. In this part, we shall prove Theorem 1. For an open covering $U'=\bigcup_\alpha V_\alpha$, where $U'=\sigma^{-1}(U)$, we define $M(\alpha)$, $N(\alpha)$, $M(\alpha,\beta)$ and $N(\alpha,\beta)$ by

$$\begin{aligned}
M(\alpha)&=\{j;\ 1\leq j\leq\mu+\nu,\ V_\alpha\cap S_j'\neq\emptyset\},\\
N(\alpha)&=\{j;\ \mu+1\leq j\leq\mu+\nu,\ V_\alpha\cap S_j'\neq\emptyset\},\\
M(\alpha,\beta)&=\{j;\ 1\leq j\leq\mu+\nu,\ V_\alpha\cap V_\beta\cap S_j'\neq\emptyset\},\\
N(\alpha,\beta)&=\{j;\ \mu+1\leq j\leq\mu+\nu,\ V_\alpha\cap V_\beta\cap S_j'\neq\emptyset\}.
\end{aligned}$$

Denote by $p(\alpha)$, $q(\alpha)$, $p(\alpha,\beta)$, and $q(\alpha,\beta)$ the cardinal numbers of $M(\alpha)$, $N(\alpha)$, $M(\alpha,\beta)$

and $N(\alpha, \beta)$ respectively. Since σ^{-1} is a Hopf transformation from U to U', we can take a finite open covering $U'=\bigcup_\alpha V_\alpha$ so that

(i) $V_\alpha \cap S'_j$ and $V_\alpha \cap V_\beta \cap S'_j$ are connected for every α, β and $1 \leq j \leq \mu+\nu$,

(ii) $1 \leq q(\alpha) \leq p(\alpha) \leq n$, for every α,

(iii) $p(\alpha, \beta)=q(\alpha, \beta)=1$, for every α, β with $V_\alpha \cap V_\beta \neq \emptyset$.

We denote by $j(\alpha, \beta)$ the unique element of $M(\alpha, \beta)=N(\alpha, \beta)$ for α, β with $V_\alpha \cap V_\beta \neq \emptyset$.

We shall first show the existence of P_α for each α such that $u=P_\alpha v$ changes system (1.1) to system (1.2) in V_α. Let $x^\alpha=(x_1^\alpha, \cdots, x_n^\alpha)$ be a system of coordinates of V_α such that

$$V_\alpha \cap S'=\{x_1^\alpha \cdots x_{p(\alpha)}^\alpha=0\},$$

and each $V_\alpha \cap S''_j$ $(j \in N(\alpha))$ contains the origin $x^\alpha=0$. Then we see that $\sigma^*\Omega$ and $\sigma^*\Omega'$ are written in V_α as

$$\sigma^*\Omega=\sum_{i=1}^n A_i^\alpha(x^\alpha)dx_i^\alpha/x_i^\alpha,$$

$$\sigma^*\Omega'=\sum_{i=1}^n B_i^\alpha(x^\alpha)dx_i^\alpha/x_i^\alpha,$$

where all $A_i^\alpha(x^\alpha)$, $B_i^\alpha(x^\alpha)$, $1 \leq i \leq p(\alpha)$, and $A_i^\alpha(x^\alpha)/x_i^\alpha$, $B_i^\alpha(x^\alpha)/x_i^\alpha$, $p(\alpha)+1 \leq i \leq n$, are holomorphic in V_α. We can verify, moreover, that if $V_\alpha \cap S'_j$ $(j \in M(\alpha)-N(\alpha))$ corresponds to $\{x_{i(j)}^\alpha=0\}$ $(1 \leq i(j) \leq p(\alpha))$, then

$$A_{i(j)}^\alpha(0)=A_j(0)=B_{i(j)}^\alpha(0)=B_j(0),$$

and that if $V_\alpha \cap S'_j$ $(j \in N(\alpha))$ corresponds to $\{x_{i(j)}^\alpha=0\}$ $(1 \leq i(j) \leq p(\alpha))$, then

$$A_{i(j)}^\alpha(0)=\mathrm{Res}_{S'_j}\,\sigma^*\Omega|_{S''_j}=B_{i(j)}^\alpha(0)=\mathrm{Res}_{S'_j}\,\sigma^*\Omega'|_{S''_j}$$

and

$$A_i^\alpha(x^\alpha)=A_i^\alpha(0)=B_i^\alpha(x^\alpha)=B_i^\alpha(0)$$

for $x^\alpha \in V_\alpha \cap S''_j$, and for $1 \leq i \leq n$. Therefore, by Proposition 2 and by the eigen value conditions in Theorem 1, we can find a holomorphic invertible matrix $P_\alpha(x^\alpha)$ in V_α satisfying

$$P_\alpha(x^\alpha)=I, \qquad x^\alpha \in V_\alpha \cap S''_j, \tag{3.21}$$

for each $j \in N(\alpha)$, so that $u=P_\alpha(x^\alpha)v$ transforms system (1.1) to system (1.2) in V_α.

We shall next show that

$$P_\alpha=P_\beta \tag{3.22}$$

in $V_\alpha \cap V_\beta$ for every α, β with $V_\alpha \cap V_\beta \neq \emptyset$. We take a system of coordinates $x^{\alpha\beta}=(x_1^{\alpha\beta}, \cdots, x_n^{\alpha\beta})$ of $V_\alpha \cap V_\beta$ so that

$$V_\alpha \cap V_\beta \cap S'_{j(\alpha,\beta)} = \{x_1^{\alpha\beta}=0\},$$

and $S''_{j(\alpha,\beta)}$ contains the origin $x^{\alpha\beta}=0$. Then $\sigma^*\Omega$ and $\sigma^*\Omega'$ are written in $V_\alpha \cap V_\beta$ as

$$\sigma^*\Omega = \sum_{i=1}^n A_i^{\alpha\beta}(x^{\alpha\beta})dx_i^{\alpha\beta}/x_i^{\alpha\beta},$$

$$\sigma^*\Omega' = \sum_{i=1}^n B_i^{\alpha\beta}(x^{\alpha\beta})dx_i^{\alpha\beta}/x_i^{\alpha\beta},$$

where $A_1^{\alpha\beta}(x^{\alpha\beta})$, $B_1^{\alpha\beta}(x^{\alpha\beta})$ and all $A_i^{\alpha\beta}(x^{\alpha\beta})/x_i^{\alpha\beta}$, $B_i^{\alpha\beta}(x^{\alpha\beta})/x_i^{\alpha\beta}$, $2\leq i\leq n$, are holomorphic in $V_\alpha \cap V_\beta$. Moreover, we can verify that

$$A_1^{\alpha\beta}(0) = \mathrm{Res}_{S'_{j(\alpha,\beta)}}\sigma^*\Omega|_{S''_{j(\alpha,\beta)}} = B_1^{\alpha\beta}(0) = \mathrm{Res}_{S'_{j(\alpha,\beta)}}\sigma^*\Omega'|_{S''_{j(\alpha,\beta)}}.$$

Therefore, from the uniqueness property in Proposition 2 and the eigenvalue conditions in Theorem 1, it follows that the change of variables $u=P_{\alpha\beta}(x^{\alpha\beta})v$ transforming system (1.1) to system (1.2) in $V_\alpha \cap V_\beta$ is uniquely determined under the condition $P_{\alpha\beta}(0)=I$. On the other hand, by (3.21), both P_α and P_β are equal to I on $S''_{j(\alpha,\beta)}$, hence we have (3.22).

Now we define a holomorphic invertible matrix $P(x)$ in $U-\Sigma(S)$ by

$$P(x) = P_\alpha(x^\alpha(x))$$

for $x \in \sigma(V^\alpha)-\Sigma(S)$. By (3.22), $P(x)$ is well defined. Since the codimension of $\Sigma(S)$ in U is strictly greater than one, $P(x)$ can be analytically continued in U and satisfies $\det P(x)\neq 0$ for every $x \in U$. The construction of the matrix makes clear that $u=P(x)v$ transforms system (1.1) to system (1.2). Thus we have completed the proof of Theorem 1.

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