

Partial Difference Equations Analogous to the Cauchy-Riemann Equations

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§ 1. Introduction

For complex valued functions $f: R \times R \rightarrow C$ of two real variables (or of one complex variable) the Cauchy-Riemann partial differential equations can be written as

$$(1) \quad \partial f / \partial x = -i \partial f / \partial y.$$

It is quite natural to ask what happens if one or both sides of (1) are replaced by the partial difference quotients

$$\frac{f(x+t, y) - f(x, y)}{t} \quad \text{or} \quad -i \frac{f(x, y+t) - f(x, y)}{t},$$

respectively. This leads to the equations

$$(2) \quad \frac{f(x+t, y) - f(x, y)}{t} = -i \frac{\partial f(x, y)}{\partial y},$$

$$(3) \quad i \frac{\partial f(x, y)}{\partial x} = \frac{f(x, y+t) - f(x, y)}{t}$$

and

$$(4) \quad \frac{f(x+t, y) - f(x, y)}{t} = -i \frac{f(x, y+t) - f(x, y)}{t}.$$

In this paper we will show, among others, that these equations do not lead essentially beyond linear functions. We will be able to generalize our results from R to arbitrary monoids (associative algebraic systems with unit elements) and from C to rings or even groups. Further generalizations will introduce several unknown functions (three in each equation) instead of one and determine all of them, showing again the remarkable phenomenon that one functional equation can determine several unknown functions.

§ 2. Equations (2), (3) and their generalizations

We replace (2) by

$$(5) \quad f(x+t, y) - g(x, y) = h(x, y)t$$

or, since y is the same parameter in each term of (5), by

$$(6) \quad f(x+t) - g(x) = h(x)t$$

for $f, g, h: R \rightarrow R$ where R is a ring.

In order to solve (6), we put $x=0$ into (6) and get, with $g(0)=b$, $h(0)=c$,

$$(7) \quad f(t) = b + ct.$$

Substituting (7) back into (6), we have

$$(8) \quad b + cx + ct = g(x) + h(x)t.$$

With $t=0$, (8) goes over into

$$(9) \quad g(x) = b + cx.$$

Putting this back into (8), we get

$$ct = h(x)t,$$

that is,

$$(10) \quad h(x) = c$$

if, in our ring R ,

$$(ax=0 \text{ for all } x \in R) \text{ implies } a=0.$$

Such rings are called *faithful* [5].

One can check immediately that (7), (9) and (10) always satisfy (6). So we have the following.

Theorem 1. *The general solution of (6), in an arbitrary faithful ring, is given by (7), (9) and (10) where b and c are arbitrary constants.*

Corollary 1. *The general solution of (5), where x, t are in an arbitrary faithful ring and y in an arbitrary set, are given by*

$$f(x, y) = g(x, y) = b(y) + c(y)x, \quad h(x, y) = c(y).$$

Corollary 2. *The general solution of*

$$(11) \quad f(x, y+t) - g(x, y) = h(x, y)t,$$

where y, t are in an arbitrary set, is given by

$$f(x, y) = g(x, y) = b(x) + c(x)y, \quad h(x, y) = c(x).$$

(Notice that x in (11) is an unchanged parameter, so (11) leads to (6) too.)

Proposition 1. *The general solution $f: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{C}$ of (2) for $t \neq 0, x, y \in \mathbf{R}$ is given by*

$$f(x, y) = c(x + iy) + a,$$

where a, c are arbitrary complex constants.

Proof. If we write (2) as

$$(12) \quad f(x+t, y) - f(x, y) = -i \frac{\partial f(x, y)}{\partial y} t,$$

we see that (12) remains true for $t=0$ and that it is of the form (5). So, by Corollary 1,

$$(13) \quad \begin{aligned} f(x, y) &= b(y) + c(y)x, \\ -i \frac{\partial f(x, y)}{\partial y} &= h(x, y) = c(y), \end{aligned}$$

thus

$$-ib'(y) - ic'(y)x = c(y)$$

or

$$c'(y) = 0, \quad c(y) = c \quad (\text{constant})$$

and

$$-ib'(y) = c, \quad b(y) = icy + a. \quad \square$$

The following proposition follows in a similar way from Corollary 2.

Proposition 2. *The general solution of (3) for $t \neq 0, x, y \in \mathbf{R}$ is given by*

$$f(x, y) = b(x + iy) + a,$$

where a, b are arbitrary complex constants.

Remark 1. Equations (5) and (11), after proof of $g(x, y) = f(x, y)$, read

$$(14) \quad \frac{f(x+t, y) - f(x, y)}{t} = h(x, y)$$

or

$$\frac{f(x, y+t) - f(x, y)}{t} = h(x, y),$$

that is, since the right hand sides are independent of t , the respective partial derivatives of f exist and

$$(15) \quad h(x, y) = \frac{\partial f(x, y)}{\partial x}$$

or

$$h(x, y) = \frac{\partial f(x, y)}{\partial y},$$

respectively. If we compare, say, equations (15) and (14) to (2), we get the Cauchy-Riemann equation (1).

§ 3. Equation (4)

We write (4) as

$$(16) \quad f(x+t, y) + if(x, t+y) = (1+i)f(x, y)$$

and suppose it satisfied for all x, y, t in a monoid M . Putting $x=0$ (the unit element) into (16) and writing $a(y) = f(0, y)$, we get

$$(17) \quad f(t, y) = (1+i)a(y) - ia(t+y).$$

Substituting this back into (16) gives

$$(1+i)a(y) - ia(x+t+y) + (i-1)a(t+y) + a(x+t+y) = 2ia(y) + (1-i)a(x+y)$$

or

$$(1-i)a(x+t+y) = (1-i)a(x+y) + (1-i)a(t+y) - (1-i)a(y).$$

Putting here $y=0$, we have

$$(18) \quad a(x+t) = a(x) + a(t) - a(0).$$

Writing $a(0) = b$ and defining A by $A(x) = -ia(x) + ib$, that is,

$$(19) \quad a(x) = iA(x) + b,$$

equation (18) goes over into

$$(20) \quad A(x+t) = A(x) + A(t),$$

that is, A is a homomorphism from M into the additive group of complex numbers. From (17), (19) and (20), we get finally

$$(21) \quad f(x, y) = A(x) + iA(y) + b.$$

Since (21) satisfies (16) whenever A is a homomorphism, we have proved the following.

Theorem 2. *The general solution $f: M \times M \rightarrow C$ of (16) in an arbitrary monoid M is given by (21), where A is an arbitrary solution of (20), that is, an arbitrary homomorphism from M into the additive group of C and b is an arbitrary (complex) constant.*

Remark 2. If the monoid M is commutative or at least $f(0, t+y) = f(0, y+t)$, then Theorem 2 remains valid with

$$f(x+t, y) - f(x, y) = -i[f(x, y+t) - f(x, y)],$$

that is,

$$f(x+t, y) + if(x, y+t) = (1+i)f(x, y)$$

instead of (16).

By splitting f , A and b into real and imaginary parts

$$(22) \quad f(x, y) = u(x, y) + iv(x, y),$$

$$(23) \quad A(x) = a_1(x) + ia_2(x), \quad b = b_1 + ib_2$$

we get the following Corollary to Theorem 2.

Corollary 3. *The general solutions $u, v: M \times M \rightarrow R$ of the system*

$$u(x+t, y) - u(x, y) = v(x, t+y) - v(x, y)$$

$$v(x+t, y) - v(x, y) = u(x, y) - u(x, t+y)$$

are given by

$$(24) \quad u(x, y) = a_1(x) - a_2(y) + b_1$$

$$(25) \quad v(x, y) = a_2(x) + a_1(y) + b_2,$$

where a_1, a_2 are arbitrary homomorphism from M into R and b_1, b_2 are real constants.

Remark 3. A proof similar to that of Theorem 2 shows that even if u, v (and a_1, a_2) map into an arbitrary abelian group rather than R , the general solutions of the

same equations are of the same form (24), (25).

A consequence of (21), if $M=G$ is a group, is that

$$(26) \quad A(x) = \frac{f(x, y) + f(x, -y)}{2} - b,$$

so that (most) regularity conditions imposed upon $x \mapsto f(x, y_0)$ ($y_0 \in G$ fixed) "inherit" to A . If, for instance, $M=G$ is a topological group, then the homomorphism $A: G \rightarrow \mathbf{C}$ will be continuous.

Using known theorems on regular solutions of (20) (cf. Rubin [7, Theorem 4.12, p. 80], Sierpiński [8]), we obtain the following Corollaries.

Corollary 4. *Let H be a Hilbert space and $f: H \times H \rightarrow \mathbf{C}$. The general continuous solution of (4) is given by*

$$f(x, y) = (x, u) + i(y, u),$$

where $u \in H$ is arbitrary, but uniquely determined by f and (\cdot, \cdot) is the inner product of H .

Corollary 5. *The general solution $f(x, y) = f(z)$ of the functional equation*

$$(27) \quad f(z+t) - f(z) = -i[f(z+it) - f(z)] \quad (z \in \mathbf{C}, t \in \mathbf{R}),$$

for which $x \mapsto f(x, y_0)$ is Lebesgue measurable for a fixed $y_0 \in \mathbf{R}$, is linear.

Proof. Equation (27) is clearly the same as (4). With $f(\cdot, y_0)$ and A , also a_1 and a_2 are measurable [cf. (26) and (23)]. Since they also satisfy

$$a_j(x+t) = a_j(x) + a_j(t) \quad (j=1, 2; x, t \in \mathbf{R}),$$

we have (cf. Aczél [1, p. 217])

$$a_j(x) = c_j x \quad (c_j \in \mathbf{R} \text{ real}; j=1, 2).$$

From (22), (24) and (25)

$$\begin{aligned} f(z) = f(x, y) &= c_1 x - c_2 y + i c_2 x + i c_1 y + b_1 + i b_2 \\ &= (c_1 + i c_2)(x + i y) + b_1 + i b_2 = cz + b, \end{aligned}$$

which indeed satisfies (27).

§ 4. A generalization

In similar vein to the generalization (5) of (2) we consider the generalization

$$(28) \quad f(x+t, y) + g(x, t+y) = h(x, y)$$

of (16). While the variables, here too, may be in a monoid M , this time it is sufficient to have the function values in a group G (not necessarily abelian, but written additively). In order to solve (28), first choose $x=0$ and get, with $a(y)=g(0, y)$, $b(y)=h(0, y)$,

$$(29) \quad f(t, y) = b(y) - a(t+y).$$

Now we put into (28) $y=0$ and get

$$g(x, t) = -f(x+t, 0) + h(x, 0)$$

or, writing $h(x, 0) = c(x)$, $b(0) = k$ and with (29),

$$(30) \quad g(x, t) = a(x+t) - k + c(x).$$

Finally, with $t=0$, we get from (28)

$$h(x, y) = f(x, y) + g(x, y)$$

or, with (29) and (30),

$$(31) \quad h(x, y) = b(y) - k + c(x).$$

Since (29), (30) and (31) always satisfy (28)

$$b(y) - a(x+t+y) + a(x+t+y) - k + c(x) = b(y) - k + c(x),$$

we have proved the following.

Theorem 3. *The general solutions $f, g, h: M \times M \rightarrow G$ of (28) are given by (29), (30) and (31), where $a, b, c: M \rightarrow G$ are arbitrary functions.*

Of course, Theorem 2 follows from Theorem 3 but, as often happens with functional equations, it is about as difficult to specialize the solution of (28) to that of (16) as to solve (16) in the first place.

Final remark. A "discrete analytic function theory", which replaces, in (1), partial derivatives by partial differences with *fixed spans*, has been developed by Issacs [4], Duffin [2], Kurowski [6], Hayabara [3] and others.

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