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On the Semigroup Treatment of the Hamilton-Jacobi Equation in Several Space Variables

By

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Dedicated to Professor Taro Ura in commemoration of His Sixtieth Birthday

§ 1. Introduction

In this paper we improve our previous work [2] and present a semigroup treatment of the Cauchy problem (hereafter called (CP)) for the Hamilton-Jacobi equation

$$\text{(DE)} \qquad u_t+f(u_x)=0, \quad x \in R^n, \quad t>0,$$

under a less restrictive assumption on f. Here $u(x, t)$ is a real-valued unknown function, $f\colon R^n \to R^1$, $u_t=\partial u/\partial t$, and u_x denotes the gradient $(\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ in the space variables $x=(x_1, \cdots, x_n)$.

The semigroup approach to solving (CP) was taken in [2], [4] and, under the assumption of convexity and C^2-regularity on f, a semigroup of contractions in $L^\infty(R^n)$ was constructed on the closed subspace $BU(R^n)$ of bounded and uniformly continuous functions on R^n. This semigroup was shown to give a unique generalized solution in the sense of Kružkov [7] if the initial data lie in the set $E(R^n)$ consisting of semiconcave, bounded and uniformly Lipschitz continuous functions on R^n.

The purpose of the present paper is to develop a similar semigroup approach to (CP) under a single assumption of convexity (and no further regularity assumption) on f.

We start, in Section 2, with a definition of an operator $A_0\colon v \to f(v_x)$ in $L^\infty(R^n)$ that may be associated with (CP). We shall define this operator simply to be $A_0 v= f(v_x)$ for $v \in E(R^n)$ and require no additional condition on $A_0 v$ (cf. [2, Definition 2.1, p. 16] and [4, Definition 5.1, p. 119]). In Section 3, we shall show that the operator A_0 defined in this way is accretive and that, for all $\lambda>0$, $D(A_0) \subset R(I+\lambda A_0)$. The establishment of the accretive property for A_0 under our assumption on f is the main result of this paper, and this property will be proved by first deriving an inequality involving a weighted L^p-norm and then taking the limit as $p \to \infty$. In Section 4, the generation theorem of M. G. Grandall and T. M. Liggett [5] is applied to the closure of A_0 to yield a semigroup of contractions on $BU(R^n)$, and the properties of this semigroup relating to (CP) is studied.

§ 2. Definition of the operator A_0: $v \to f(v_x)$

Throughout the present paper we shall, as before [2], work in the Banach space $L^\infty(R^n)$ of real-valued, bounded and measurable functions with norm $\|\cdot\|_\infty$.

$W_k^\infty(R^n)$ denotes the subspace of $L^\infty(R^n)$ consisting of measurable functions whose distribution derivatives of order at most k lie in $L^\infty(R^n)$. Thus, in particular, $W_1^\infty(R^n)$ is the subspace of bounded and uniformly Lipschitz continuous functions on R^n. For $v \in W_1^\infty(R^n)$ we set

$$\|v_x\|_\infty = (\textstyle\sum_{i=1}^n \|\partial v/\partial x_i\|_\infty^2)^{1/2}.$$

We shall assume that the function f: $R^n \to R^1$ appearing in (DE) is convex. (Note that a finite convex function on R^n is necessarily locally Lipschitz continuous.) Corresponding to this assumption, we need a subclass of $W_1^\infty(R^n)$:

$E(R^n)$ denotes the subset of $L^\infty(R^n)$ consisting of bounded and uniformly Lipschitz continuous functions v on R^n satisfying the semiconcavity condition

$$\text{(SC)} \qquad v(x+y)+v(x-y)-2v(x) \leqq k|y|^2, \qquad x, y \in R^n,$$

for some constant $k \geqq 0$.

A function v: $R^n \to R^1$ that satisfies the semiconcavity condition (SC) will be called a semiconcave function, and we let $|v|_E$ denote the infimum of such constants k.

As was stated in the introduction, our definition below is simpler than the previous one (cf. [2], [4]).

Definition 2.1. *Let f: $R^n \to R^1$ be convex. A_0 is the operator in $L^\infty(R^n)$ defined by*: $D(A_0)=E(R^n)$ *and for* $v \in D(A_0)$, $A_0 v = f(v_x)$.

The following remark follows from the fact that every function in $W_2^\infty(R^n)$ is semiconcave and so belongs to $E(R^n)$.

Remark 2.1. $W_2^\infty(R^n) \subset D(A_0)$.

Definition 2.2. *A is the closure of A_0, i.e., $v \in D(A)$ and $w \in Av$ if there is a sequence $\{v^m\} \subset D(A_0)$ such that $v^m \to v$ and $A_0 v^m \to w$ in $L^\infty(R^n)$.*

§ 3. The equation $u+f(u_x)=h$

The semigroup approach to solving (CP) requires us to verify that the operator A_0 of Definition 2.1 is accretive and that, for all $\lambda>0$, $R(I+\lambda A_0) \supset D(A_0)$. To this end, under a single assumption of convexity on f, we shall study certain bounded generalized solutions of a time independent equation of the form

$$(3.1) \qquad u+f(u_x)=h(x), \qquad x \in R^n,$$

where h is a given function belonging to $L^\infty(R^n)$.

For simplicity the normalization

$$f(0)=0 \tag{3.2}$$

will be assumed throughout this section, for this can always be achieved by introducing the new unknown $\bar{u}=u+f(0)$.

Definition 3.1. *Let f: $R^n \to R^1$ be convex. For given $h \in L^\infty(R^n)$, a function u will be called a bounded generalized solution of* (3.1) *if u belongs to $E(R^n)$ and satisfies* (3.1) *at almost all points of R^n.*

When the given function h on the right belongs to $E(R^n)$, a bounded generalized solution of (3.1) has been obtained as the limit of bounded C^2-solutions of the regularized elliptic equation

$$u+f(u_x)-\varepsilon\Delta u=h, \qquad x \in R^n, \tag{3.3}$$

as $\varepsilon \downarrow 0$. The following theorem is a result of our previous paper [2, Theorem 3.1, p. 17].

Theorem 3.1. *Let f: $R^n \to R^1$ be convex. Then, for each $h \in E(R^n)$, there is a bounded generalized solution of* (3.1) *such that*

$$\|u\|_\infty \leqq \|h\|_\infty, \qquad \|u_x\|_\infty \leqq \|h_x\|_\infty, \tag{3.4}$$

and, for the semiconcavity constant of u,

$$|u|_E \leqq |h|_E. \tag{3.5}$$

From Theorem 3.1 we immediately have:

Corollary 3.1. $R(I+\lambda A_0) \supset D(A_0)$ *for all* $\lambda > 0$.

We now turn to the proof of the accretive property for A_0. The method hitherto used ([2], [4]) to show this property originated from [1] and was based upon (i) the proof of uniqueness of bounded generalized solutions of (3.1) under additional C^2-regularity assumption on f, and (ii) the fact that, for $h \in E(R^n)$, the unique bounded generalized solution could be obtained as the limit of bounded C^2-solutions of the regularized elliptic equation (3.3) as $\varepsilon \downarrow 0$, and for these solutions, a similar accretive property could be verified via a variant of the maximum principle for second order elliptic equations. Below we shall use an entirely different method and give a direct proof of accretiveness of A_0 without assuming C^2-regularity of f. Our procedure will be to derive an inequality involving a weighted L^p-norm and then take the limit as $p \to \infty$ in the inequality.

The following theorem is the core of our development.

Theorem 3.2. *Let f: $R^n \to R^1$ be convex. If u and v belong to $E(R^n)$ and satisfy the equations*

$$u+f(u_x)=h, \qquad v+f(v_x)=g \tag{3.6}$$

respectively at almost all points of R^n, where h and g are in $L^\infty(R^n)$, then we have

$$\|u-v\|_\infty \leqq \|h-g\|_\infty. \tag{3.7}$$

Proof. Our proof below is divided into two steps. In the first step we shall show (3.7) assuming C^2-regularity of f.

The First Step. Suppose $f \in C^2(R^n)$. For u and v, let U denote a common absolute bound in R^n, let P be a common Lipschitz constant, and let k be a common semiconcavity constant. We set

$$K_1=\sup \{(\textstyle\sum_{i=1}^n (f_{p_i}(p))^2)^{1/2} \mid |p| \leqq P\}$$

and

$$K_2=\sup \{\textstyle\sum_{i=1}^n f_{p_i p_i}(p) \mid |p| \leqq P\}.$$

Since (3.6) hold a.e. in R^n, the difference $w=u-v$ satisfies the equation

$$w+\textstyle\sum_{i=1}^n G_i w_{x_i}=h-g$$

a.e. in R^n, where

$$G_i=G_i(u, v)=\int_0^1 f_{p_i}(v_x+\theta(u_x-v_x))d\theta, \qquad i=1, \cdots, n.$$

Multiplying the last equation by $qw^{q-1}e^{-|x|}$, where q is an even integer, and setting $W=w^q$, we obtain

$$qWe^{-|x|}+\textstyle\sum_{i=1}^n G_i W_{x_i} e^{-|x|}=q(h-g)w^{q-1}e^{-|x|}. \tag{3.8}$$

Convolving u and v with mollifying kernels, we can find two approximate sequences $\{u^m\}$ and $\{v^m\}$ of C^2-functions, each having the same absolute bound U, Lipschitz constant P and semiconcavity constant k as u and v, such that the sequences $\{u_x^m\}$ and $\{v_x^m\}$ converge a.e. in R^n to u_x and v_x respectively. If we set

$$G_i^m=G_i(u^m, v^m), \qquad i=1, \cdots, n,$$

then equation (3.8) can be written as

$$\begin{aligned} &qWe^{-|x|}+\textstyle\sum_{i=1}^n (G_i^m We^{-|x|})_{x_i} \\ &\quad =\textstyle\sum_{i=1}^n (G_i^m-G_i)W_{x_i}e^{-|x|}+W\sum_{i=1}^n G_i^m(e^{-|x|})_{x_i} \\ &\qquad +W\textstyle\sum_{i=1}^n (G_i^m)_{x_i}e^{-|x|}+q(h-g)w^{q-1}e^{-|x|}. \end{aligned} \tag{3.9}$$

Let r be an arbitrary positive number, and we integrate the both sides of (3.9) over the ball $|x| \leqq r$. We thus get

$$
\begin{aligned}
&q\int_{|x|\leqq r} We^{-|x|}dx+\int_{|x|=r} We^{-r} \textstyle\sum_{i=1}^{n} G_i^m \cos(n, x)dS \\
(3.10)\qquad &=\int_{|x|\leqq r} \textstyle\sum_{i=1}^{n}(G_i^m-G_i)W_{x_i}e^{-|x|}dx+\int_{|x|\leqq r} W\textstyle\sum_{i=1}^{n} G_i^m(e^{-|x|})_{x_i}dx \\
&+\int_{|x|\leqq r} W\textstyle\sum_{i=1}^{n}(G_i^m)_{x_i}e^{-|x|}dx+q\int_{|x|\leqq r}(h-g)w^{q-1}e^{-|x|}dx \\
&=I_1+I_2+I_3+I_4,
\end{aligned}
$$

where n is the outer normal to the sphere S: $|x|=r$ and dS is the surface element.

The second term on the left-hand side of (3.10) tends to zero as $r\to\infty$, since $|\sum_{i=1}^{n} G_i^m \cos(n, x_i)|\leqq K_1$ by Schwarz' inequality.

We shall estimate the four integrals I_i on the right. For the first integral I_1 it is shown by using the Lebesgue convergence theorem that $I_1\to 0$ if we let first $r\to\infty$ and then $m\to\infty$, since $G_i^m\to G_i$ a.e. in R^n and $|G_i^m|\leqq K_1$ for $i=1, \cdots, n$.

Using $(e^{-|x|})_{x_i}=(-x_i/|x|)e^{-|x|}$ and Schwarz' inequality, we have

$$
|I_2|\leqq K_1\int_{|x|\leqq r} We^{-|x|}dx. \tag{3.11}
$$

For the third integral I_3 we obtain

$$
I_3\leqq kK_2\int_{|x|\leqq r} We^{-|x|}dx, \tag{3.12}
$$

since

$$
\textstyle\sum_{i=1}^{n}(G_i^m)_{x_i}=\sum_{i,j=1}^{n}\left(u_{x_ix_j}^m\int_0^1 \theta f_{p_ip_j}(\cdots)d\theta+v_{x_ix_j}^m\int_0^1(1-\theta)f_{p_ip_j}(\cdots)d\theta\right)
$$

where $(\cdots)=(v_x^m+\theta(u_x^m-v_x^m))$ and, by virtue of the convexity of f and the semiconcavity condition (SC),

$$
\textstyle\sum_{i,j=1}^{n}u_{x_ix_j}^m f_{p_ip_j}(\cdots)=tr[(M-kI)F]+k\sum_{i=1}^{n}f_{p_ip_i}(\cdots)\leqq k\sum_{i=1}^{n}f_{p_ip_i}(\cdots),
$$

M and F denoting the matrices $(u_{x_ix_j}^m)$ and $(f_{p_ip_j}(\cdots))$ respectively.

Finally, Hölder's inequality gives

$$
|I_4|\leqq q\left(\int_{|x|\leqq r}|h-g|^q e^{-|x|}dx\right)^{1/q}\left(\int_{|x|\leqq r} We^{-|x|}dx\right)^{(q-1)/q}. \tag{3.13}
$$

Substituting (3.11)–(3.13) into (3.10) and letting first $r\to\infty$ and then $m\to\infty$, we obtain

$$(1-(K_1+kK_2)/q)\left(\int_{R^n} We^{-|x|}dx\right)^{1/q} \leqq \left(\int_{R^n} |h-g|^q e^{-|x|}dx\right)^{1/q}$$

for $W=w^q$, provided q is a sufficiently large even integer.

Therefore, letting again $q\to\infty$ and noting the well known fact that, in a finite measure space, $\|w\|_{L^\infty}=\lim_{p\to\infty}\|w\|_{L^p}$ for $w \in L^\infty$, we conclude that

$$\|w\|_\infty=\|u-v\|_\infty \leqq \|h-g\|_\infty,$$

which is the desired estimate (3.7).

The Second Step. Let f be convex and let P be a common Lipschitz constant for u and v. Convolving f with mollifying kernels, we can find a sequence $\{f_m\}$ of convex C^2-functions satisfying

$$|f_m(p)-f(p)| \leqq 1/m, \qquad m=1, 2, \cdots$$

for every p such that $|p| \leqq P$. Then, since

$$u+f_m(u_x)=h+f_m(u_x)-f(u_x),$$
$$v+f_m(v_x)=g+f_m(v_x)-f(v_x),$$

and

$$\|h-g+(f_m(u_x)-f(u_x))-(f_m(v_x)-f(v_x))\|_\infty \leqq \|h-g\|_\infty+2/m,$$

we have by the result of the first step

$$\|u-v\|_\infty \leqq \|h-g\|_\infty+2/m$$

and hence, letting m tend to infinity, we obtain (3.7). Thus the proof is complete.

Immediate consequences of Theorem 3.2 are:

Corollary 3.2. *Let f: $R^n\to R^1$ be convex. Then, for each $h \in L^\infty(R^n)$, there exists at most one bounded generalized solution of* (3.1).

Corollary 3.3. *Let f: $R^n\to R^1$ be convex. The operator A_0 of Definition* 2.1 *is accretive in $L^\infty(R^n)$, that is, we have*

$$\|(u+\lambda A_0 u)-(v+\lambda A_0 v)\|_\infty \geqq \|u-v\|_\infty$$

for every $\lambda>0$ and every u, $v \in D(A_0)$.

§ 4. The semigroup of contractions associated with (CP)

The Cauchy problem (CP) consists of (DE) and the initial condition

(IC) $$u(x, 0)=u^0(x), \qquad x \in R^n,$$

where u^0 is a given function on R^n.

It is assumed throughout the section that $f\colon R^n \to R^1$ is convex and satisfies the normalization (3.2), for the latter can always be achieved by introducing the new unknown $\bar{u}=u+f(0)t$.

We shall choose $L^\infty(R^n)$ as the Banach space associated with (CP) and regard the unknown function u as a map: $[0, \infty) \ni t \to u(\cdot, t) \in L^\infty(R^n)$. Let A be given by Definition 2.2. Then (CP) can be rewritten in the abstract form

(ACP) $$du/dt+Au \ni 0, \qquad u(0)=u^0.$$

(Note that A may be multi-valued.)

In order to apply the abstract theory to (ACP), we shall state the generation theorem of M. G. Crandall and T. M. Liggett in a form suitable for our use. Let X be a real Banach space and A be an operator in X (that is allowed to be multi-valued). A is said to be accretive in X if

$$\|(u+\lambda w)-(v+\lambda z)\| \geqq \|u-v\|$$

for $\lambda>0$, $u, v \in D(A)$, $w \in Au$ and $z \in Av$, where $\|\cdot\|$ denotes the norm in X.

The following generation theorem is a result of M. G. Crandall and T. M. Liggett [5].

Generation Theorem. *Let A be an accretive operator in a real Banach space X. If $R(I+\lambda A) \supset \overline{D(A)}$ for all sufficiently small positive λ, then*

(4.1) $$\lim_{n\to\infty} \left(I+\frac{t}{n}A\right)^{-n} u^0$$

exists for $u^0 \in \overline{D(A)}$ and $t \geqq 0$. Moreover, if $S(t)u^0$ is defined as the limit in (4.1), *then $S(t)$ is a semigroup of contractions on $\overline{D(A)}$:*

(i) *We have $S(t)\colon \overline{D(A)} \to \overline{D(A)}$ for $t \geqq 0$; $S(t)S(\tau)=S(t+\tau)$ for $t, \tau \geqq 0$; $\|S(t)v-S(t)w\| \leqq \|v-w\|$ for $v, w \in \overline{D(A)}$ and $t \geqq 0$; $S(0)=I$ and $S(t)v$ is continuous in (t, v).*

(ii) *If $v \in D(A)$, then $S(t)v$ is locally Lipschitz continuous in t.*

(iii) *For each $\varepsilon>0$ and each $u^0 \in \overline{D(A)}$, the problem*

(4.2) $$\begin{cases} \varepsilon^{-1}(u^\varepsilon(t)-u^\varepsilon(t-\varepsilon))+Au^\varepsilon(t) \ni 0, & t>0, \\ u^\varepsilon(t)=u^0, & t \leqq 0, \end{cases}$$

has a unique solution $u^\varepsilon(t)$ on $[0, \infty)$ and $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t)=S(t)u^0$ uniformly in t on compact sets.

We have to verify the hypotheses of the Generation Theorem for the A of Definition 2.2.

From Corollary 3.3 we easily have:

Proposition 4.1. *A is accretive in* $L^\infty(R^n)$, *that is,*

$$\|(u+\lambda w)-(v+\lambda z)\|_\infty \geqq \|u-v\|_\infty$$

for every $\lambda>0$ *and every* u, $v \in D(A)$, $w \in Au$ *and* $z \in Av$.

In what follows, $BU(R^n)$ denotes the closed linear subspace of $L^\infty(R^n)$ consisting of all bounded and uniformly continuous functions on R^n.

Now we shall give another definition of bounded generalized solutions of (3.1).

Definition 4.1. *Let* $h \in BU(R^n)$. *A function* $u \in BU(R^n)$ *will be called a bounded generalized solution of* (3.1) *provided* $u \in D(A)$ *and* $h \in u+Au$.

It follows from Theorem 3.1 that $R(I+\lambda A)\supset BU(R^n)$ for $\lambda>0$, since $R(I+\lambda A_0)\supset E(R^n)$, $E(R^n)$ is dense in $BU(R^n)$ and A is the closure of A_0 (Note that $R(I+\lambda A)$ is closed for $\lambda>0$ when A is closed and accretive.). By virtue of Remark 2.1 we have $\overline{D(A)}=BU(R^n)$. Therefore we have proved the following theorem.

Theorem 4.1. *Let* f: $R^n\rightarrow R^1$ *be convex. Then the operator* A *of Definition* 2.2 *satisfies the assumptions of the Generation Theorem with* $\overline{D(A)}=BU(R^n)$. *In particular,* $u=(I+A)^{-1}h$ *is the unique bounded generalized solution of* (3.1) *for* $h \in BU(R^n)$.

According to Theorem 4.1 and the Generation Theorem, a semigroup of contractions $S(t)$ on $BU(R^n)$ is determined by the operator A. Concerning the properties of this semigroup we first have:

Theorem 4.2. *Let* f: $R^n\rightarrow R^1$ *be convex and* $f(0)=0$, *and let* $S(t)$ *be the semigroup of contractions on* $BU(R^n)$ *obtained from* A *through the Generation Theorem. If* $v \in BU(R^n)$ *and* $t\geqq 0$, *then we have*

$$\sup_{x\in R^n}|S(t)v(x+y)-S(t)v(x)|\leqq \sup_{x\in R^n}|v(x+y)-v(x)|$$

for $y \in R^n$. *Moreover, if* $v \in E(R^n)$, *then* $S(t)v \in E(R^n)$ *and*

$$\|S(t)v\|_\infty\leqq\|v\|_\infty, \quad \|(S(t)v)_x\|_\infty\leqq\|v_x\|_\infty, \quad |S(t)v|_E\leqq|v|_E.$$

Proof. The solution $u^\varepsilon(t)$ of (4.2) is given by $u^\varepsilon(t)=(I+\varepsilon A)^{-[t/\varepsilon]-1}u^0$, where $[t/\varepsilon]$ is the greatest integer in t/ε. Since $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t)=S(t)u^0$ uniformly in t on compact sets, the proof follows immediately from Proposition 4.1 and Theorem 3.1.

When f: $R^n\rightarrow R^1$ is convex, a Lipschitz continuous function $u(x, t)$ defined on $R^n\times[0, \infty)$ is called a generalized solution of (CP) if i) u satisfies (DE) a.e. as well as (IC), and ii) for each level $t>0$, u satisfies a semiconcavity condition

$$u(x+y,t)+u(x-y,t)-2u(x,t)\leqq k(t)|y|^2, \qquad x, y \in R^n,$$

where $k(t)\leqq k_\delta$ for $t\geqq\delta>0$. Below we shall show that the semigroup $S(t)$ obtained above provides a bounded generalized solution $S(t)u^0$ of (CP) if u^0 lies in $E(R^n)$.

Theorem 4.3. *Let f: $R^n\to R^1$ be convex and $f(0)=0$, and let $S(t)$ be the semigroup of contractions on $BU(R^n)$ obtained from A through the Generation Theorem. If $u^0 \in E(R^n)$, then:*

(i) $\|S(t)u^0\|_\infty\leqq\|u^0\|_\infty$, $\|(S(t)u^0)_x\|_\infty\leqq\|u^0_x\|_\infty$, *and for the semiconcavity constant of $S(t)u^0$*, $|S(t)u^0|_E\leqq|u^0|_E$,

(ii) $S(t)u^0(x)$ *is uniformly Lipschitz continuous on $R^n\times[0,\infty)$ and satisfies* (DE) *a.e. as well as* (IC).

Proof. It suffices to prove (ii). That $S(t)u^0(x)$ satisfies (IC) follows from the continuity of $S(t)u^0$ in t, i.e., $\lim_{t\downarrow 0}\|S(t)u^0-u^0\|_\infty=0$. Since $E(R^n)=D(A_0)\subset D(A)$, the Generation Theorem, (ii), asserts that $S(t)u^0$ is locally Lipschitz continuous in t. Therefore it remains to prove that $S(t)u^0(x)$ satisfies (DE) a.e.. For $u^0 \in E(R^n)$, let $u^\varepsilon(t)$ satisfy

$$\begin{aligned}&\varepsilon^{-1}(u^\varepsilon(t)-u^\varepsilon(t-\varepsilon))+A_0u^\varepsilon(t)=0, && t>0,\\ &u^\varepsilon(t)=u^0, && t\leqq 0.\end{aligned}$$

Then, by Definition 2.1, $u^\varepsilon(t)$ satisfies the equation

$$\varepsilon^{-1}(u^\varepsilon(t)-u^\varepsilon(t-\varepsilon))+f((u^\varepsilon(t))_x)=0 \tag{4.3}$$

a.e. in R^n for each $t\geqq0$. Let $T>0$. Since $S(t)u^0$ is Lipschitz continuous for $0\leqq t\leqq T$ and $S(t)u^0 \in W_1^\infty(R^n)$ for each $t\geqq0$, $S(t)u^0(x)$ is Lipschitz continuous in (x,t) and hence, (totally) differentiable a.e. in $R^n\times[0,T]$. Moreover, by a result (see, for instance [6, Theorem 2.3, p. 11]) concerning the convergence of a sequence of semiconcave functions, the sequence $\{(u^\varepsilon(t))_x\}$ converges a.e. in $R^n\times[0,T]$ to $(S(t)u^0)_x$ as $\varepsilon\downarrow0$. Multiply (4.3) by $\phi \in C_0^\infty(R^n\times(0,T))$ and integrate over $R^n\times[0,T]$. Integration by parts and letting $\varepsilon\downarrow0$ yield

$$\int_0^T\int_{R^n}\{-(S(t)u^0)\phi_t+f((S(t)u^0)_x)\phi\}dxdt=0,$$

which can be rewritten as

$$\int_0^T\int_{R^n}\{(S(t)u^0)_t+f((S(t)u^0)_x)\}\phi dxdt=0.$$

But this implies that $S(t)u^0(x)$ satisfies (DE) a.e. in $R^n\times(0,T)$, which in turn shows that $S(t)u^0(x)$ is uniformly Lipschitz continuous on $R^n\times[0,\infty)$ by Theorem 4.2. Thus the proof is complete.

Remark 4.1. It has been observed in [10] that the generalized solution of (CP) is uniquely determined by its initial values under a single condition of convexity on f: $R^n \to R^1$. Hence, if $u^0 \in E(R^n)$, $S(t)u^0$ provides a unique bounded generalized solution of (CP).

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