Funkcialaj Ekvacioj, **23** (1980) 327–334

On Some Partial Integral Inequalities in Two Independent Variables

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§ 1. Introduction

During the last few years the area of applications of partial differential and integral inequalities has greatly expanded, and now encompass not only many questions in the theory of partial differential and integral equations but also contain areas of physics, technology and biological sciences. A large number of papers dealing with partial differential and integral inequalities have appeared during the last few years, (see, [1, 2, 10, 17–20] and some of the references given therein). Recently, in a series of papers, Bondge and Pachpatte [3]–[6], Bondge, Pachpatte and Walter [7] and Pachpatte [14]–[16] have established several new partial integral inequalities which can be used as handy tools in the theory of partial differential and integral equations of the more general type. Our objective here is to establish two independent variable generalizations of the integral inequalities established by Gollwitzer [9] and Pachpatte [11, 12] which can be used in some applications in the theory of partial differential and integral equations of the more general type.

§ 2. Main results

In this section we state and prove our main results on two independent variable generalizations of some of the integral inequalities established by Gollwitzer [9] and Pachpatte [11, 12]. An elementary method used by Snow [17] to obtain a generalization of Gronwall's inequality in two independent variables will be used to establish our results.

Our first result deals with the two independent variable generalization of the integral inequality established by Gollwitzer [9, Theorem 1].

Theorem 1. *Suppose* $\phi(x, y)$, $b(x, y)$, *and* $c(x, y)$ *are real-valued nonnegative continuous functions defined on a domain D. Let* $G(r)$ *be continuous, strictly increasing, convex and submultiplicative function for* $r \geq 0$, $G(0)=0$, $\lim_{r\to\infty} G(r)=\infty$ *for all* (x, y) *in D,* $\alpha(x, y)$, $\beta(x, y)$ *be positive continuous functions defined on D, and* $\alpha(x, y)+\beta(x, y)=1$. *Let* $P_0(x_0, y_0)$ *and* $P(x, y)$ *be two points in D such that* $(x-x_0)(y-y_0)>0$ *and R be the rectangular region whose opposite corners are the points* P_0 *and P. Let* $v(s, t; x, y)$

be the solution of the characteristic initial value problem

$$(1)\qquad L[v]=v_{st}-c(s,t)\beta(s,t)G(b(s,t)\beta^{-1}(s,t))v=0,\quad v(x,t)=v(s,y)=1,$$

and let D^{+} *be a connected subdomain of* D *which contains* P *and on which* $v>0$, (See Fig. 1). *If* $R\subset D^{+}$ *and* $\phi(x,y)$ *satisfies*

$$(2)\qquad \phi(x,y)\leq a(x,y)+b(x,y)G^{-1}\left(\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)G(\phi(s,t))\,dsdt\right),$$

then $\phi(x,y)$ *also satisfies*

$$(3)\qquad \phi(x,y)\leq a(x,y)+b(x,y)G^{-1}\left(\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)\alpha(s,t)G(a(s,t)\alpha^{-1}(s,t))\right.$$
$$\left.\cdot v(s,t;x,y)dsdt\right).$$

The proof of this theorem is obtained by reducing the integral inequality to a differential inequality and then integrating it by Riemann's method for hyperbolic partial differential equations [8, p. 120]. The function $v(s, t; x, y)$ involved in Theorem 1 is a Riemann function relative to the point $P(x, y)$ for the self adjoint operator L. There is such a function and a domain D^{+} on which $v>0$ since $v=1$ on the vertical and horizontal lines through P and since V is continuous.

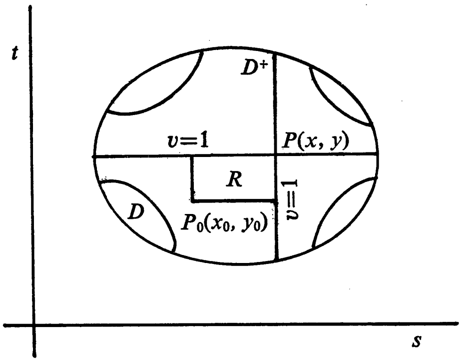

Fig. 1

Proof. Rewrite (2) as

$$\phi(x,y)\leq\alpha(x,y)a(x,y)\alpha^{-1}(x,y)$$
$$+\beta(x,y)b(x,y)\beta^{-1}(x,y)G^{-1}\left(\int_{x_0}^{x}\int_{y_0}^{y}c(s,t)G(\phi(s,t))\,dsdt\right).$$

Since G is convex, submultiplicative and monotonic we have

$$G(\phi(x, y)) \leq \alpha(x, y)G(a(x, y)\alpha^{-1}(x, y)) + \beta(x, y)G(b(x, y)\beta^{-1}(x, y)) \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)G(\phi(s, t))dsdt. \tag{4}$$

Define

$$u(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)G(\phi(s, t))dsdt, \quad u(x, y_0) = u(x_0, y) = 0, \tag{5}$$

then

$$u_{xy}(x, y) = c(x, y)G(\phi(x, y)),$$

which in view of (4) implies

$$\begin{aligned} L[u] &= u_{xy}(x, y) - c(x, y)\beta(x, y)G(b(x, y)\beta^{-1}(x, y))u(x, y) \\ &\leq c(x, y)[\alpha(x, y)G(a(x, y)\alpha^{-1}(x, y))]. \end{aligned} \tag{6}$$

The operator L is self-adjoint and hyperbolic. For any twice continuously differentiable u and v the operator L satisfies the identity

$$vL[u] - uL[v] = -(uv_y)_x + (vu_x)_y. \tag{7}$$

Let P_0 and P be any points as in the theorem and lebel the directed sides and corners of the rectangle R as shown in Fig. 2.

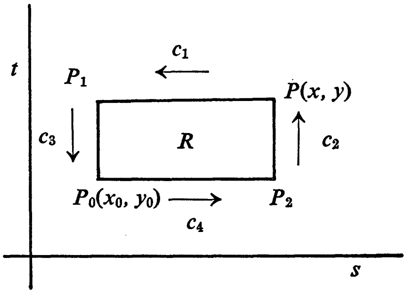

Fig. 2

Using s and t as the independent variables, we integrate the identity (7) over R and use Green's Theorem to obtain

$$\iint_R (vL[u] - uL[v])dsdt = -\int_{c_1+c_2+c_3+c_4} (vu_s ds + uv_t dt).$$

This holds for any functions in C^2.

For the particular function u defined earlier we have $u=0$ on c_3 and $u=u_s=0$ on c_4, so the right hand side of the above identity reduces to

$$-\int_{c_1} vu_s ds - \int_{c_2} uv_t dt. \tag{8}$$

Now suppose v satisfies

$$L[v]=v_{st}-c(s,t)\beta(s,t)G(b(s,t)\beta^{-1}(s,t))v=0, \tag{9}$$

$$v=1 \quad \text{on } c_1, \tag{10}$$

$$v_t=0 \quad \text{on } c_2. \tag{11}$$

Then (10) and (11) imply that

$$v=1 \quad \text{on } c_2. \tag{12}$$

Since $v \geq 0$ on R and $u(P_1)=0$, by using (6) identity (8) becomes

$$u(x,y) \leq \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\alpha(s,t)G(a(s,t)\alpha^{-1}(s,t))v(s,t;x,y)dsdt. \tag{13}$$

The conclusion (3) of the theorem follows from (4) and (13).

In Theorem 2 given below we establish the following two independent variable generalization of the integral inequality recently established by Pachpatte [12, Theorem 2].

Theorem 2. *Suppose $\phi(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, and $k(x,y)$ be real-valued nonnegative continuous functions defined on a domain D. Let $G(r)$, $\alpha(x,y)$, $\beta(x,y)$ be the same functions as defined in Theorem 1. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)(y-y_0)>0$ and R be the rectangular region whose opposite corners are the points P_0 and P. Let $E(s,t;x,y)$ be the solution of the characteristic initial value problem*

$$\begin{gathered} L[E]=E_{st}(s,t)-\beta(s,t)G(b(s,t)\beta^{-1}(s,t))[c(s,t)+k(s,t)]E(s,t)=0, \\ E(x,t)=E(s,y)=1, \end{gathered} \tag{14}$$

and let D^+ be a connected subdomain of D which contains P and on which $v>0$. If $R\subset D^+$ and $\phi(x,y)$ satisfies

$$\begin{aligned} \phi(x,y) \leq a(x,y)+b(x,y)G^{-1}\Big[&\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)G(\phi(s,t))dsdt \\ &+\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\beta(s,t)G(b(s,t)\beta^{-1}(s,t)) \\ &\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t} k(m,n)G(\phi(m,n))dmdn\Big)dsdt\Big], \end{aligned} \tag{15}$$

then $\phi(x, y)$ *also satisfies*

$$\begin{aligned}(16)\qquad \phi(x, y) \leq a(x, y) + b(x, y)G^{-1}\Big[&\int_{x_0}^{x}\int_{y_0}^{y} Q(s, t)\,dsdt \\ &+\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\beta(s, t)G(b(s, t)\beta^{-1}(s, t)) \\ &\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t} Q(m, n)E(m, n, s, t)dmdn\Big)dsdt\Big],\end{aligned}$$

where

$$\begin{aligned}(17)\qquad Q(x, y) = c(x, y)\alpha(x, y)G(a(x, y)\alpha^{-1}(x, y)) + c(x, y)\beta(x, y)G(b(x, y)\beta^{-1}(x, y))& \\ \cdot\int_{x_0}^{x}\int_{y_0}^{y} k(m, n)\alpha(m, n)G(a(m, n)\alpha^{-1}(m, n))dmdn.&\end{aligned}$$

Proof. Rewrite (15) as

$$\begin{aligned}&\phi(x, y) \leq \alpha(x, y)a(x, y)\alpha^{-1}(x, y) \\ &\quad + \beta(x, y)b(x, y)\beta^{-1}(x, y)G^{-1}\Big[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)G(\phi(s, t))\,dsdt \\ &\quad +\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\beta(s, t)G(b(s, t)\beta^{-1}(s, t))\Big(\int_{x_0}^{s}\int_{y_0}^{t} k(m, n)G(\phi(m, n))dmdn\Big)dsdt\Big].\end{aligned}$$

Since G is convex, submultiplicative and monotonic we have

$$\begin{aligned}(18)\qquad G(\phi(x, y)) \leq\ & \alpha(x, y)G(a(x, y)\alpha^{-1}(x, y)) \\ &+ \beta(x, y)G(b(x, y)\beta^{-1}(x, y))\cdot\Big[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)G(\phi(s, t))dsdt \\ &+\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\beta(s, t)G(b(s, t)\beta^{-1}(s, t)) \\ &\cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t} k(m, n)G(\phi(m, n))dmdn\Big)dsdt\Big].\end{aligned}$$

Define

$$\begin{aligned}u(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)G(\phi(s, t))dsdt + \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\beta(s, t)G(b(s, t)\beta^{-1}(s, t))& \\ \cdot\Big(\int_{x_0}^{s}\int_{y_0}^{t} k(m, n)G(\phi(m, n))dmdn\Big)dsdt, \qquad u(x, y_0) = u(x_0, y) = 0,&\end{aligned}$$

then

$$\begin{aligned}u_{xy}(x, y) = c(x, y)\Big[&G(\phi(x, y)) + \beta(x, y)G(b(x, y)\beta^{-1}(x, y)) \\ &\cdot\int_{x_0}^{x}\int_{y_0}^{y} K(m, n)G(\phi(m, n))dmdn\Big],\end{aligned}$$

which in view of (18) implies

$$(19)\quad u_{xy}(x, y)\leq Q(x, y)+c(x, y)\beta(x, y)G(b(x, y)\beta^{-1}(x, y))$$
$$\cdot\left\{u(x, y)+\int_{x_0}^{x}\int_{y_0}^{y} k(m, n)G(b(m, n)\beta^{-1}(m, n))u(m, n)dmdn\right\}.$$

Define

$$(20)\quad r(x, y)=u(x, y)+\int_{x_0}^{x}\int_{y_0}^{y} k(m, n)\beta(m, n)G(b(m, n)\beta^{-1}(m, n))u(m, n)dmdn,$$
$$r(x, y_0)=r(x_0, y)=0,$$

then

$$(21)\quad r_{xy}(x, y)=u_{xy}(x, y)+k(x, y)\beta(x, y)G(b(x, y)\beta^{-1}(x, y))u(x, y).$$

Using the facts that

$$u_{xy}(x, y)\leq Q(x, y)+c(x, y)\beta(x, y)G(b(x, y)\beta^{-1}(x, y))r(x, y),$$

from (19) and $u(x, y)\leq r(x, y)$ from (20) in (21) we have

$$L[r]=r_{xy}(x, y)-\beta(x, y)G(b(x, y)\beta^{-1}(x, y))[c(x, y)+k(x, y)]r(x, y)\leq Q(r, y).$$

Now by following the last argument as in the proof of Theorem 1 we obtain the bound on $r(x, y)$ such that

$$r(x, y)\leq\int_{x_0}^{x}\int_{y_0}^{y} Q(s, t)E(s, t; x, y)dsdt.$$

Using this bound on $r(x, y)$ in (19) we have

$$u_{xy}(x, y)\leq Q(x, y)+c(x, y)\beta(x, y)G(b(x, y)\beta^{-1}(x, y))\int_{x_0}^{x}\int_{y_0}^{y} Q(s, t)E(s, t; x, y)dsdt.$$

Integrating both sides of the above inequality first with respect to y from y_0 to y and then with respect to x from x_0 to x we have

$$(22)\quad u(x, y)\leq\int_{x_0}^{x}\int_{y_0}^{y} Q(s, t)dsdt+\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\beta(s, t)G(b(s, t)\beta^{-1}(s, t))$$
$$\cdot\left(\int_{x_0}^{s}\int_{y_0}^{t} Q(m, n)E(m, n, s, t)dmdn\right)dsdt.$$

The desired bound in (16) follows from (18) and (22).

To this end we establish the two independent variable generalization of the integral inequality established by Pachpatte in [11, Theorem 2].

Theorem 3. *Suppose* $\phi(x, y)$, $a(x, y)$, $b(x, y)$, $c(x, y)$, *and* $k(x, y)$ *be real-valued nonnegative continuous functions defined on a domain* D. *Let* $N(r)$ *be a positve, continuous, strictly increasing, subadditive and submultiplicative function for* $r\geq 0$ *and* N^{-1} *is the inverse function of* N. *Let* $P_0(x_0, y_0)$ *and* $P(x, y)$ *be two points in* D *such that* $(x-x_0)(y-y_0)>0$ *and* R *be the rectangular region whose opposite corners are the points* P_0 *and* P. *Let* $e(s, t; x, y)$ *be the solution of the characteristic initial value problem*

$$L[e]=e_{st}(s, t)N(b(s, t))[c(s, t)+k(s, t)]e(s, t)=0,\qquad e(x, t)=e(s, y)=1, \tag{23}$$

and let D^+ *be a connected subdomain of* D *which contains* P *and on which* $e>0$. *If* $R\subset D^+$ *and* $\phi(x, y)$ *satisfies*

$$\phi(x, y)\leq a(x, y)+b(x, y)N^{-1}\left[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)N(\phi(s, t))dsdt + \int_{x_0}^{x}\int_{y_0}^{y} c(s, t)N(b(s, t))\left(\int_{x_0}^{s}\int_{y_0}^{t} k(m, n)N(\phi(m, n))dmdn\right)dsdt\right], \tag{24}$$

then $\phi(x, y)$ *also satisfies*

$$\phi(x, y)\leq N^{-1}\left[N(a(x, y))+N(b(x, y))\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)\left\{N(a(s, t)) + N(b(s, t))\int_{x_0}^{s}\int_{y_0}^{t} [c(m, n)+k(m, n)]N(b(m, n))e(m, n; s, t)dmdn\right\}dsdt\right]. \tag{25}$$

The proof of this theorem follows by the similar argument as in the proof of Theorems 1 and 2 with suitable modifications (see, also [11, Theorem 2]). We omit the details.

In concluding this paper we note that the inequalities presented here can be extended very easily to the corresponding vector problems as in [18]. We also note that there is no essential difficulty in obtaining *n* independent variable generalizations of the inequalities established in Theorems 1–3 by using the technique used by Young in [19]. Since this translation is quite straight forward in view of the results of this paper and we omit the details.

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(Ricevita la 24-an de julio, 1979)