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On two conjectures by Hájek

By

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§ 1. Introduction

In this paper we are concerned with the control system in the Euclidean n-space $\boldsymbol{R}^n$, represented by

$$\text{(U)} \qquad \dot{x}-Ax=u \qquad u(t)\in U$$

under the following assumptions.

The dimension n of state space is at least 1, the coefficient matrix A is a real n-square, the constraint set U is compact and convex and contains the origin; the control system is proper, in the sense that $R(t)$ is a neighborhood of 0 for each $t>0$; we suppose to start with the system in the origin at time $t=0$, and call $R(t)$ the reachable set at time $t\geqslant 0$:

$$\text{(1)} \qquad \begin{aligned} R(t)&=\left\{\int_0^t e^{sA}\cdot u(s)ds\colon \text{measurable } u\colon [0,t]\to U\right\} \\ &=\int_0^t e^{sA}\cdot U ds \text{ (in the sense of Aumann).} \end{aligned}$$

For any $x\in R=\bigcup_{t\geqslant 0} R(t)$, the minimal time function T is defined by:

$$\text{(2)} \qquad T(x)=\inf\{t\colon x\in R(t)\}.$$

Referring to Hájek's paper [2] we give here a proof for his two conjectures:

Conjecture 1. If U is a neighborhood of the origin, then T has directional derivatives at each point $x\in R$.

Conjecture 2. If in addition the boundary of U is a differentiable $(n-1)$-manifold (i.e. for each point of the boundary there is a unique support hyperplane), then T is C^1 in $R-\{0\}$.

§ 2. Some preliminary notions and lemmas

The proofs given here are based upon the well-known representation of compact convex sets by means of their support function. Dealing with the family $\mathscr{P}_c(\boldsymbol{R}^n)$ of compact subsets of $\boldsymbol{R}^n$, we always use the Hausdorff metric. We set:

$$\psi(\Omega, y)=\sup_{x\in\Omega} y^*x \tag{3}$$

for any compact convex $\Omega\subseteq\boldsymbol{R}^n$ and $y\in\boldsymbol{R}^n$. ψ is continuous with respect to both arguments.

For the control system (U) we have:

$$R(t)=\{x\in\boldsymbol{R}^n: y^*x\leqslant\psi(R(t), y),\ \forall y\in\boldsymbol{R}^n\}, \tag{4}$$

and

$$\psi(R(t), y)=\int_0^t \psi(U, e^{sA^*}\cdot y)ds=\int_0^t \psi(e^{sA}\cdot U, y)ds. \tag{5}$$

For any $x_0\in R$, if $t_0=T(x_0)$ we set:

$$\nu(x_0)=\{y\in\boldsymbol{R}^n: \|y\|=1, y^*x_0=\psi(R(t_0), y)\}. \tag{6}$$

Indeed, $\nu(x_0)$ is the set of outer normals to the boundary of $R(t_0)$ at x_0. Equality (4) can now be improved by:

$$R(t)=\{x: x\in R, y^*x=\psi(R(t), y),\ \forall y\in\nu(x)\}. \tag{4'}$$

Namely, let $x\notin R(t)$, so that $t<T(x)$. Since the mappings $t\to\psi(R(t), y)$, for $y\neq 0$, are strictly increasing, for any $y\in\nu(x)$ we have:

$$y^*x=\psi(R(T(x)), y)>\psi(R(t), y).$$

Lemma 1. *For all $x_0\in R$ and $\varepsilon>0$, there exists a $\delta>0$ such that $\|x-x_0\|\leqslant\delta$ implies $\nu(x)\subseteq B(\nu(x_0), \varepsilon)$. ($B(\nu(x_0), \varepsilon)$ stands for the set: $\{y\in\boldsymbol{R}^n: \|y\|=1, d(y, \nu(x_0))\leqslant\varepsilon\}$, with $d(y, \nu(x_0))=\inf_{y_0\in\nu(x_0)}\|y-y_0\|$).*

Proof. Suppose a sequence $(x_n, y_n)_{n\geqslant 1}$ be given, such that $y_n\in\nu(x_n)$ and $d(y_n, \nu(x_0))\geqslant\varepsilon/2$ for each $n\geqslant 1$, and $\lim_{n\to\infty} x_n=x_0$. Taking a subsequence, we may assume $\lim_{n\to\infty} y_n=y_0$ for some y_0. Indeed $d(y_0, \nu(x_0))\geqslant\varepsilon/2$, but $y_0^*x_0=\lim_{n\to\infty} y_n^*x_n$ $=\lim_{n\to\infty}\psi(R(T(x_n)), y_n)=\psi(R(T(x_0)), y_0)$, so that $y_0\in\nu(x_0)$, which is absurd.

Lemma 2. *The minimal-time function T for the control system* (U) *is locally lipschitz at a reachable point x_0 if and only if $\psi(e^{t_0A}\cdot U, y)>0$ for every $y\in\nu(x_0)$ (we set $t_0=T(x_0)$).*

Proof. If $\psi(e^{t_0A}\cdot U, y_0)=0$ for some $y_0\in\nu(x_0)$ then, given $\varepsilon>0$ we can find a $\delta>0$ such that $|t-t_0|<\delta$ implies $\psi(e^{tA}\cdot U, y_0)<\varepsilon$. Using (5) we get:

$$\psi(R(t_0+\lambda), y_0)-\psi(R(t_0), y_0)<\varepsilon\lambda,$$

thus $x_0+\varepsilon\lambda y_0 \notin R(t_0+\lambda)$ for any λ such that $0<\lambda<\delta$.

This implies $T(x_0+\varepsilon\lambda y_0)-T(x_0)>\lambda=\|(x_0+\varepsilon\lambda y_0)-x_0\|\cdot\varepsilon^{-1}$, and shows that the lipschitz condition fails at x_0.

Suppose now that $\psi(e^{t_0A}\cdot U, y)>0$ for every $y \in \nu(x_0)$.

The continuity of ψ and the compactness of $\nu(x_0)$ imply the existence of δ, ρ and $m>0$ such that, if $|t-t_0|\leqslant\delta$, $y \in B(\nu(x_0),\rho)$ then $\psi(e^{tA}\cdot U, y)\geqslant m$.

Choose a neighborhood V of x_0 such that $|T(x)-t_0|\leqslant\delta$ and $\nu(x)\subseteq B(\nu(x_0),\rho)$ for every $x \in V$ (this is possible because of Lemma 2 and the continuity of T). We claim that

$$(7) \qquad |T(x_1)-T(x_2)|\leqslant m^{-1}\cdot\|x_1-x_2\| \qquad \text{for any } x_1, x_2 \in V.$$

To see this, let $t_1=T(x_1)\leqslant T(x_2)=t_2$. For any $y \in B(\nu(x_0),\rho)$, $t\geqslant t_1$, using (5) we get:

$$\psi(R(t), y)=\psi(R(t_1)y)+\int_{t_1}^{t}\psi(e^{sA}\cdot U, y)ds\geqslant y^*x_1+m(t-t_1).$$

Hence $m(t-t_1)\geqslant\|x_2-x_1\|$ implies $\psi(R(t), y)\geqslant y^*x_2$; so that by (4′) $x_2 \in R(t)$. Thus $T(x_2)\leqslant T(x_1)+m^{-1}\|x_2-x_1\|$, which proves (7).

For fixed $x_0 \in R$ and $t_0=T(x_0)$, we say that a vector $w \in \boldsymbol{R}^n$ is

a) *interior* iff the half-line $S=\{x_0+\alpha w: \alpha\geqslant 0\}$ contains interior points of $R(t_0)$,

b) *tangent* iff w is not interior and $\lim_{\alpha\to 0+} d(x_0+\alpha w, R(t_0))/\alpha=0$

c) *exterior* iff $\lim_{\alpha\to 0+} d(x_0+\alpha w, R(t_0))/\alpha>0$.

These definitions imply the following

Lemma 3. *A vector w is exterior, tangent, or interior to $R(T(x_0))$ at point x_0 iff $\max_{y\in\nu(x_0)} y^*w \gtreqless 0$ respectively.*

We can now prove the main result:

§ 3. Proof of Conjecture 1

Theorem 1. *The minimal-time function T for control system* (U) *has finite directional derivatives at a reachable point x_0 if and only if T is locally lipschitz at x_0. In the positive case, $t_0=T(x_0)$ and $w \in \boldsymbol{R}^n$ imply*

$$(8) \qquad DT(x_0, w)=\lim_{\alpha\to 0+}\frac{T(x_0+\alpha w)-T(x_0)}{\alpha}=\max_{y\in\nu(x_0)}\frac{y^*w}{\psi(e^{t_0A}\cdot U, y)}.$$

Proof. We may suppose $\|w\|=1$ and consider three cases:

Case 1. w is tangent. T is locally lipschitz, hence $|T(x_1)-T(x_2)|\leqslant k\,\|x_1-x_2\|$, for x_1, x_2 in a suitable neighborhood of x_0 and for some k. We have then

$$0 \leqslant \frac{T(x_0+\alpha w)-t_0}{\alpha} \leqslant k \frac{d(x_0+\alpha w, R(t_0))}{\alpha}$$

Setting $\alpha \to 0+$, since w is tangent, we obtain $DT(x_0, w)=0$, which implies (8) in Case 1 because of Lemma 3.

Case 2. w is exterior. We know that the function T is strictly increasing on the half line $S=\{x_0+\alpha w : \alpha \geqslant 0\}$. For each $t \geqslant t_0$ there is exactly one $\alpha(t)$ such that $T(x_0+\alpha(t)\cdot w)=t$. If we show that

$$\lim_{t \to t_0+} \frac{\alpha(t)}{t-t_0} = \min_{y \in \nu(x_0)} \frac{\psi(e^{t_0 A}\cdot U, y)}{y^* w} > 0, \tag{9}$$

we are done, because then:

$$\lim_{\alpha \to 0+} \frac{T(x_0+\alpha w)-T(x_0)}{\alpha} = \left[\lim_{t \to t_0+} \frac{\alpha(t)}{t-t_0}\right]^{-1},$$

which gives exactly (8).

For any $y \in \boldsymbol{R}^n$ with $\|y\|=1$ and $y^* w>0$, and any $t \geqslant t_0$, we define $\alpha_y(t)$ by means of the formula

$$y^*(x_0+\alpha_y(t)\cdot w) = \sup_{x \in R(t)} y^* x = \psi(R(t), y). \tag{10}$$

From a geometrical point of view, $x_0+\alpha_y(t)\cdot w$ is the intersection point of S with the support hyperplane of $R(t)$ normal to y. Note that $\alpha_y(t)$ is always $\geqslant 0$, and that $\alpha_y(t_0)=0$ iff $y \in \nu(x_0)$.

Using (4), we recognize that for any $t \geqslant t_0$

$$\alpha(t) = \inf\{\alpha_y(t) : \|y\|=1, y^* w>0\}. \tag{11}$$

By (10) and (5)

$$\alpha_y(t) = \frac{\psi(R(t), y)-y^* x_0}{y^* w} = \left[\int_0^t \psi(e^{sA}\cdot U, y)ds - y^* x_0\right] \Big/ y^* w. \tag{12}$$

Hence

$$\alpha_y'(t) = \frac{\partial \alpha_y(t)}{\partial t} = \frac{\psi(e^{tA}\cdot U, y)}{y^* w}. \tag{13}$$

Choose positive δ_0, ρ_0, m and M such that $|t-t_0| \leqslant \delta_0$ and $y \in B(\nu(x_0), \rho_0)$ imply $0<m \leqslant \psi(e^{tA}U, y) \leqslant M$. By Lemma 3, there is an $y_0 \in \nu(x_0)$ for which $y_0^* w=\mu>0$. Substituting $\alpha_y'(s)$ as given by (13) inside the equality:

$$\alpha_y(t) = \alpha_y(t_0) + \int_{t_0}^t \alpha_y'(s)ds$$

we obtain

$$\alpha_{y_0}(t) \leqslant \frac{M(t-t_0)}{\mu} \leqslant \alpha_y(t),$$

for any y, t as soon as $y \in B(\nu(x_0), \rho_0)$, $y^*w \leqslant m\mu/M$ and $|t-t_0| \leqslant \delta_0$. If $|t-t_0| \leqslant \delta_0$, we can thus write:

$$\alpha(t) = \min_{y \in K} \alpha_y(t), \quad \text{with} \quad K = \{y \in \boldsymbol{R}^n : \|y\| = 1, y^*w \geqslant m\mu/M\}.$$

Note that K is compact.

The continuity of the function $(y, t) \to \psi(U, e^{tA^*} \cdot y)/y^*w$ implies that given any $\varepsilon > 0$ we can find $\delta, \rho > 0$ such that $\delta \leqslant \delta_0$, $\rho \leqslant \rho_0$ and such that, if $y \in K$, $d(y, \nu(x_0)) \leqslant \rho$, $0 \leqslant t - t_0 \leqslant \delta$ and $K_0 = K \cap \nu(x_0)$, then

$$\frac{\psi(e^{tA} \cdot U, y)}{y^*w} \geqslant \min_{y_0 \in K_0} \frac{\psi(e^{t_0 A} \cdot U, y_0)}{y_0^* w} - \varepsilon.$$

Because of Lemma 1 and the continuity of T, there is a neighborhood V of x_0 such that $x \in V$ implies both $\nu(x) \subseteq B(\nu(x_0), \rho)$ and $|T(x) - t_0| \leqslant \delta$. This means that

$$\alpha(t) = \inf\{\alpha_y(t) : y \in K \cap B(\nu(x_0), \rho)\}$$

as long as $x_0 + \alpha(t) \cdot w \in V$. We have the estimate:

$$\alpha_y(t) \geqslant \left[\inf_{y \in \nu(x_0)} \frac{\psi(e^{t_0 A} \cdot U, y)}{y^*w} - \varepsilon\right](t - t_0), \quad \forall t \in [t_0, t_0 + \delta], \quad \forall y \in B(\nu(x_0), \rho).$$

Thus, for any $\varepsilon > 0$,

$$\liminf_{t \to t_0+} \frac{\alpha(t)}{t - t_0} \geqslant \min_{y \in \nu(x_0)} \frac{\psi(e^{t_0 A} \cdot U, y)}{y^*w} - \varepsilon. \tag{14}$$

On the other hand it is obvious from (11) that

$$\limsup_{t \to t_0+} \frac{\alpha(t)}{t - t_0} \leqslant \min_{y \in \nu(x_0)} \alpha_y'(t_0) = \min_{y \in \nu(x_0)} \frac{\psi(e^{t_0 A} \cdot U, y)}{y^*w}. \tag{15}$$

Combining (14) and (15) we obtain the desired result.

Case 3. w is interior. We can find an $\alpha_0 > 0$ such that the function $\alpha \to T(x_0 + \alpha w)$ is strictly decreasing for $0 \leqslant \alpha \leqslant \alpha_0$. For all y and t, if $\|y\| = 1$, $y^*w < 0$ and $T(x_0 + \alpha_0 w) < t \leqslant t_0$, we set:

$$\alpha_y(t) = \frac{\psi(R(t), y) - y^*x_0}{y^*w} \quad \text{and} \quad \alpha(t) = \sup_{y^*w < 0} \alpha_y(t).$$

The same arguments used in Case 2 show that

$$\lim_{t\to t_0-}\frac{\alpha(t)}{t-t_0}=-\max_{y\in\nu(x_0)}\frac{-\psi(e^{t_0A}\cdot U,y)}{y^*w}=\min_{y\in\nu(x_0)}\frac{\psi(e^{t_0A}\cdot U,y)}{y^*w}.$$

Since $T(x_0+\alpha(t)w)=t$, we obtain formula (8) again.

To see the converse, note that the existence of finite directional derivatives implies

$$\max_{y\in\nu(x_0)}\frac{y^*w}{\psi(e^{t_0A}\cdot U,y)}<+\infty,$$

thus $\psi(e^{t_0A}\cdot U,y)>0$ for any $y\in\nu(x_0)$. Apply then Lemma 2.

If U is a neighborhood of the origin, T is locally lipschitz at every reachable point ([2], Prop. 4), thus Conjecture 1 is proved.

Corollary 1. *If T is locally lipschitz at x_0, $t_0=T(x_0)<+\infty$, and if the boundary of $R(t_0)$ has only one outer normal at x_0, that is $\nu(x_0)=\{y_0\}$, then T is differentiable at x_0, and its derivative is*:

$$DT(x_0)=\left[\max_{u\in U}y_0^*e^{t_0A^*}\cdot u\right]^{-1}\cdot y_0.$$

§ 4. Proof of Conjecture 2

In order to prove the second statement, we need only show that if the constraint set U is smooth, so are all reachable sets $R(t)$ for $t>0$. To this purpose, for each compact convex $\Omega\subseteq \boldsymbol{R}^n$ we define a relation $\sigma(\Omega)$ by setting: $S^{n-1}=\{y\in\boldsymbol{R}^n:\|y\|=1\}$, $\sigma(\Omega)\subseteq S^{n-1}\times S^{n-1}$ and $(y_1,y_2)\in\sigma(\Omega)$ iff there exists some $x\in\Omega$ for which both $y_1^*x=\psi(\Omega,y_1)$ and $y_2^*x=\psi(\Omega,y_2)$ hold.

Geometrically, this means that y_1 and y_2 are exterior normals to the boundary of Ω at a same point x. Indeed this boundary is smooth iff $\sigma(\Omega)$ reduces to the diagonal $\Delta(S^{n-1}\times S^{n-1})$.

Lemma 4. *Let $\Omega(t)$ be a compact convex set in $\boldsymbol{R}^n$ for each $t\in[0,t_0]$, $t_0>0$, and let the mapping $t\to\Omega(t)$ be continuous relative to the Hausdorff metric. Then*:

$$\sigma\left(\int_0^{t_0}\Omega(t)dt\right)=\bigcap_{t\in[0,t_0]}\sigma(\Omega(t)) \tag{17}$$

Proof. It is a simple geometrical fact that $(y_1,y_2)\in\sigma(\Omega)$ iff

$$\psi(\Omega,y_1)+\psi(\Omega,y_2)=\psi(\Omega,y_1+y_2). \tag{18}$$

In any case

$$\psi(\Omega, y_1)+\psi(\Omega, y_2)\geqslant\psi(\Omega, y_1+y_2). \tag{19}$$

It follows that

$$(y_1, y_2)\in\sigma\left(\int_0^{t_0}\Omega(t)dt\right)$$

iff

$$\psi\left(\int_0^{t_0}\Omega(t)dt, y_1\right)+\psi\left(\int_0^{t_0}\Omega(t)dt, y_2\right)=\psi\left(\int_0^{t_0}\Omega(t)dt, y_1+y_2\right),$$

that is

$$\int_0^{t_0}[\psi(\Omega(t), y_1)+\psi(\Omega(t), y_2)]dt=\int_0^{t_0}\psi(\Omega(t), y_1+y_2)dt. \tag{20}$$

From (19) and the continuity of the integrands in (20) we deduce that (20) holds iff these integrands are equal for all t, that is iff $(y_1, y_2)\in\sigma(\Omega(t))$ for every $t\in[0, t_0]$.

Theorem 2. *If the constraint set U of the control system* (U) *is a neighborhood of the origin and has a smooth boundary, then the minimal time function T is C^1 in $R-\{0\}$.*

Proof. Using (1) and (17) we obtain

$$\sigma(R(t))=\bigcap_{s\in[0,t]}\sigma(e^{sA}\cdot U)\subseteq\sigma(U)=\Delta(S^{n-1}\times S^{n-1}).$$

Thus $R(t)$ is smooth for every $t>0$, and $\nu(x)$ consists of just one point, for any $x\in R-\{0\}$. Then by Corollary 1, DT exists, and hence, by Theorem 5 in [2], DT is continuous.

References

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