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Non Linear Hyperbolic-Parabolic Partial Differential Equations

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Summary

In this paper it is proved that the mixed problem for the equation

$$k_2(x)u''+k_1(x)u'-\Delta u+|u|^\rho u=f$$

has unique weak solutions, when $0<\rho<2/(n-2)$, n the space dimension. Here, $k_1(x)$ and $k_2(x)$ are assumed to be bounded measurable functions on Ω, where Ω is an open bounded set of R^n, satisfying the conditions $k_2(x)\geq 0$ and $k_1(x)\geq\beta>0$ for all x in Ω.

Introduction

Let Ω be a bounded open set in R^n and $k_1(x)$ and $k_2(x)$ be two real functions defined in Ω. The boundary of Ω we represent by Γ, which we suppose to be regular. For $T>0$, let Q be the cylinder $Q=\Omega\times(0, T)$. When f is a real function defined in Q, one is interested in studying the mixed problem for the non linear partial differential equation:

$$(*) \qquad k_2(x)u''+k_1(x)u'-\Delta u+|u|^\rho u=f \quad \text{in} \quad Q$$

for $k_2(x)\geq 0$ and $k_1(x)\geq\beta>0$. Observe that on the set $k_2(x)=0$ contained in Ω, the equation $(*)$ degenerates into a parabolic case.

This type of equation was studied by Bensoussan-Lions-Papanicolau in [1] for the linear case, that is, without the term $|u|^\rho u$, with non identically vanishing initial data and also in Lions [3]. In [6], Vagrov studied the linear problem when k_2 and k_1 depend on (x, t), that is, $k_2(x, t)$ and $k_1(x, t)$ for (x, t) in Q, but with null initial conditions. In [2], Larkin studied $(*)$ with more general non linearities included also in f, but still with null initial conditions, plus strong restrictions on f.

In this paper we study the mixed problem for $(*)$ in the case of non null initial data and we obtain existence and uniqueness of solutions in one class that has less weak derivatives than that obtained by Larkin, cf. [2].

§ 1. Existence of solutions

Let Ω, Q, $k_2(x)$, $k_1(x)$ be the same as in the Introduction. We suppose $k_2(x)$ and $k_1(x)$ belong to $L^\infty(\Omega)$. By $H^1(\Omega)$ we represent the usual Sobolev space of order one and by $H_0^1(\Omega)$ the closure of the test functions in $H^1(\Omega)$. The dual of $H_0^1(\Omega)$ is represented by $H^{-1}(\Omega)$. By $L^p(\Omega)$ we represent the space of all real functions in Ω which has the power p, integrable in Ω. When $p=2$, we have the Hilbert space $L^2(\Omega)$. We represent by $(\ ,\ )$ and $|\cdot|$, respectively, the inner product and norm in $L^2(\Omega)$; by $\|\cdot\|$ the norm in $H_0^1(\Omega)$. By Sobolev theorems, we know that $H_0^1(\Omega)$ is continuously embedded in $L^q(\Omega)$ for $1/q=1/2-1/n$.

Theorem 1. *Let* $u_0 \in H_0^1(\Omega)$, $v_1 \in L^2(\Omega)$ *and* $0 \leq \rho \leq 2/(n-2)$, *then there exists a unique function* $u(x, t)$, (x, t) *in* Q, *such that*:

$$u \in L^\infty(0, T; H_0^1(\Omega)) \tag{1}$$

$$u' \in L^\infty(0, T; L^2(\Omega)); \qquad \sqrt{k_2}(x)u' \in L^2(0, T, L^2(\Omega)) \tag{2}$$

$$k_2(x)u'' \in L^2(0, T; H^{-1}(\Omega)) \tag{3}$$

$$k_2(x)u''+k_1(x)u'-\Delta u+|u|^\rho u=f \qquad \text{in a weak sense in } Q \tag{4}$$

$$u(0)=u_0; \qquad k_2(x)u'(0)=\sqrt{k_2(x)}\,v_1. \tag{5}$$

Remark 1. Before proving the existence part of Theorem 1, let us observe that the restriction on ρ is only necessary for uniqueness. We can obtain existence for any $\rho>0$, for $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$, $p=\rho+2$, $v_1 \in L^2(\Omega)$. For this case, with $k_2(x)=1$, $k_1(x)=0$, see Lions [4].

Remark 2. Suppose we have proved (1)-(4) of Theorem 1. Let us see that the initial data make sense. In fact, by (1) and (2) it follows, Lions-Magenes [5], that $u \in C^0([0, T], L^2(\Omega))$, therefore $u(0)$ makes sense. By (3), (4) it follows that $k_2(x)u'$ belongs to $C^0([0, T]; H^{-1}(\Omega))$ so that $k_2(x)u'(0)$ makes sense.

Proof of the Existence. The method to be used is to perturb the equation (*) adding the term $\varepsilon u''$ and to obtain estimates to permit passing to the limit when $\varepsilon>0$ goes to zero. After this perturbation, we obtain $k_2(x)+\varepsilon>0$ as the coefficient of u''. Formulating an appropriate mixed problem on Q for the new equation, we obtain a priori estimates for the solutions independent of $\varepsilon>0$, in such way that we can pass to the limit when ε goes to zero, obtaining a function u which is the solution looked for.

In fact, for $\varepsilon>0$, we consider the problem:

$$(k_2(x)+\varepsilon)u_\varepsilon''+k_1(x)u_\varepsilon'-\Delta u_\varepsilon+|u_\varepsilon|^\rho u_\varepsilon=f \qquad \text{in } Q \tag{6}$$

$$u_\varepsilon(0)=u_0, \qquad (k_2(x)+\varepsilon)u'_\varepsilon(0)=\sqrt{k_2(x)+\varepsilon}\, v_1 \tag{7}$$

and $u_\varepsilon=0$ for (x, t) on the lateral boundary of Q.

To prove the existence of solutions for (6) and (7) we use the Faedo-Galerkin method. Let (w_ν) be a sequence of vectors of $H_0^1(\Omega)$ such that for each m, the set $w_1, w_2, \cdots, w_m$ is linearly independent and the set of finite linear combinations are dense in $H_0^1(\Omega)$. Let us consider the linear manifold $V_m=[w_1, w_2, \cdots, w_m]$, generated by $w_1, w_2, \cdots, w_m$ and project equation (5) on V_m. This means that we look for $u_{\varepsilon m}(t)$ in V_m, such that:

$$\begin{aligned} &((k_2(x)+\varepsilon)u''_{\varepsilon m}(t), v)+(k_1(x)u'_{\varepsilon m}(t), v)+a(u_{\varepsilon m}(t), v)+(|u_{\varepsilon m}|^\rho u_{\varepsilon m}, v) \\ &\quad =(f(t), v) \qquad \text{for all } v \in V_m, \end{aligned} \tag{8}$$

$$u_{\varepsilon m}(0)=u_{0m} \quad \text{strongly convergent to } u_0 \quad \text{in } H_0^1(\Omega), \tag{9}$$

$$(k_2(x)+\varepsilon)u'_{\varepsilon m}(0)=\sqrt{k_2(x)+\varepsilon}\, v_{1m},\ (v_{1m}) \quad \text{strongly convergent to } v_1 \quad \text{in } L^2(\Omega). \tag{10}$$

Remark 3. We represent by $a(u, v)$ the Dirichlet form:

$$a(u, v)=\sum_{i=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx \qquad \text{for all } u, v \in H_0^1(\Omega).$$

The quadratic form $a(v, v)$ we represent by $a(v)$.

The equations (8) form a system of ordinary differential equations with initial conditions (9) and (10). There are solutions for this sytem in $[0, t_m)$, $0<t_m<T$. The a priori estimates which shall be obtained, permit us to extend the approximate solutions $u_{\varepsilon m}$ to the interval $[0, T)$ and also pass to the limit in m and ε.

In fact, setting $v=2u'_{\varepsilon m}$ in system (8), we obtain:

$$\frac{d}{dt}((k_2(x)+\varepsilon)u'_{\varepsilon m}, u'_{\varepsilon m})+2(k_1(x)u'_{\varepsilon m}, u'_{\varepsilon m})+\frac{d}{dt}a(u_{\varepsilon m})+2(|u_{\varepsilon m}|^\rho u'_\varepsilon)=2(f, u'_{\varepsilon m}). \tag{11}$$

Integrating (11) from 0 to t, $t<t_m$, we have:

$$\begin{aligned} &|\sqrt{k_2(x)+\varepsilon}\, u'_{\varepsilon m}(t)|^2+2\int_0^t |\sqrt{k_1(x)}\, u'_{\varepsilon m}(s)|^2\, ds+a(u_{\varepsilon m}(t))+\frac{2}{p}\int_\Omega |u_{\varepsilon m}(t)|^p\, dx \\ &\quad =|v_{1m}|^2+a(u_{0m})+\frac{2}{p}\int_\Omega |u_{0m}|^p\, dx+2\int_0^t (f(s), u'_{\varepsilon m}(s))ds. \end{aligned} \tag{12}$$

Remark 4. The Dirichlet form $a(u, v)$ defines in $H_0^1(\Omega)$ a norm equivalent to the norm $\|\cdot\|$ of $H_0^1(\Omega)$. By the embedding theorem, $\|u_{0m}\|_{L^p(\Omega)}$ is uniformly bounded by a constant independent of m, ε and t in $[0, t_m)$. By Remark 4 and the hypothesis on the initial data u_{0m}, u_{1m}, we obtain from (12):

$$
\begin{aligned}
(13)\qquad & |\sqrt{k_2(x)+\varepsilon}\,u'_{\varepsilon m}(t)|^2+2\int_0^t |\sqrt{k_1(x)}\,u'_{\varepsilon m}(s)|^2\,ds+\alpha\,\|u_{\varepsilon m}(t)\|^2+\frac{2}{p}\,\|u_{\varepsilon m}(t)\|_{L^2(\Omega)} \\
& \leq K+2\int_0^t (f(s),\,u'_{\varepsilon m}(s))ds.
\end{aligned}
$$

It follows that:

$$2\int_0^t |\sqrt{k_1(x)}\,u'_{\varepsilon m}(s)|^2\,ds \leq K+2\int_0^t (f(s),\,u'_{\varepsilon m}(s))ds$$

with $0<\beta\leq k_1(x)$. We obtain

$$2\beta\int_0^t |u'_{\varepsilon m}(s)|^2\,ds \leq K+\frac{1}{\lambda}\int_0^T |f(s)|^2\,dx+\lambda\int_0^t |u'_{\varepsilon m}(s)|^2\,ds.$$

Choosing $\lambda=\beta$, we get:

$$\int_0^t |u'_{\varepsilon m}(s)|^2\,ds$$

is bounded by a constant independent of m, ε and t in $[0, t_m)$.

From 13) we obtain:

$$(14)\qquad |\sqrt{k_2(x)+\varepsilon}\,u'_{\varepsilon m}(t)|^2+2\int_0^t |\sqrt{k_1(x)}\,u'_{\varepsilon m}(s)|^2\,ds+\alpha\,\|u_{\varepsilon m}(t)\|^2+\frac{2}{p}\,\|u_{\varepsilon m}(t)\|_{L^p(\Omega)}<C,$$

where C is a positive constant, which is independent of m, ε and t in $[0, t_m)$.

It follows that $\|u_{\varepsilon m}(t)\|$, therefore $|u_{\varepsilon m}(t)|$, is bounded by C, what is sufficient to extend $u_{\varepsilon m}$ to the interval $[0, T)$, with the bound independent of ε and m. Therefore, inequality (14) is true for all t in $[0, T)$ and it follows:

(15) $u_{\varepsilon m}$ belongs to a bounded set of $L^\infty(0, T; H_0^1(\Omega))$, independent of $\varepsilon>0$ and m.

(16) $u'_{\varepsilon m}$ belongs to a bounded set of $L^2(0, T; L^2(\Omega))$ independent of $\varepsilon>0$ and m.

(17) $\sqrt{k_2(x)+\varepsilon}\,u'_{\varepsilon m}$ belongs to a bounded set of $L^\infty(0, T; L^2(\Omega))$ independent of $\varepsilon>0$ and m.

From (15), (16), (17) we obtain a subsequence $u_{\varepsilon\nu}$ such that:

(18) $u_{\varepsilon\nu}$ converges to u_ε weak star in $L^\infty(0, T; H_0^1(\Omega))$

(19) $u'_{\varepsilon\nu}$ converges to u'_ε weakly in $L^2(0, T; L^2(\Omega))$

(20) $\sqrt{k_2(x)+\varepsilon}\,u'_{\varepsilon\nu}$ converges to $\sqrt{k_2(x)+\varepsilon}\,u'_\varepsilon$ weak star in $L^\infty(0, T; L^2(\Omega))$.

Let us study the non linear term. In fact, we obtain from (14):

$$\int_\Omega |u_{\varepsilon\nu}(x, t)|^p \, dx < Cp/2$$

independent of ν and ε. Therefore, if $1/p + 1/p' = 1$,

$$\| |u_{\varepsilon\nu}|^\rho u_{\varepsilon\nu} \|^{p'}_{L^{p'}(\Omega)} = \int_\Omega |u_{\varepsilon\nu}|^{p'\rho + p'} \, dx = \int_\Omega |u_{\varepsilon\nu}(x, t)|^p \, dx \leq Cp/2,$$

which proves that:

(21) $|u_{\varepsilon\nu}|^\rho u_{\varepsilon\nu}$ belongs to a bounded set of $L^\infty(0, T; L^{p'}(\Omega))$ independent of ν and ε, therefore, belongs also to a bounded set of $L^{p'}(Q)$.

From estimates (15), (16) and since the injection of $H^1_0(\Omega)$ in $L^2(\Omega)$ is compact, we have, passing to a subsequence, that $u_{\varepsilon\nu}$ converges to u_ε almost everywhere in Q. Then

(22) $|u_{\varepsilon\nu}|^\rho u_{\varepsilon\nu}$ converges to $|u_\varepsilon|^\rho u_\varepsilon$ almost everywhere in Q.

From (21), (22) and Lions [4], Ch. I, Lemma 1.3, we conclude that:

(23) $|u_{\varepsilon\nu}|^\rho u_\varepsilon$ converges to $|u_\varepsilon|^\rho u_\varepsilon$ weak star in $L^{p'}(Q)$.

From (17) we obtain:

(24) $\sqrt{k^2(x)+\varepsilon}\, u'_{\varepsilon\nu}$ converges to $\sqrt{k_2(x)+\varepsilon}\, u'_\varepsilon$ weak star in $L^\infty(0, T; L^2(\Omega))$.

It follows from (15), (16), (23) and (24), that we can pass to the limit in the approximate equation (8) obtaining:

$$\frac{d}{dt}((k_2(x)+\varepsilon)u'_\varepsilon, v) + (k_1(x)u'_\varepsilon, v) + a(u_\varepsilon, v) + (|u_\varepsilon|^\rho u_\varepsilon, v) = (f, v) \tag{25}$$

for all $v \in H^1_0(\Omega)$, in the weak sense and u^ε satisfies the initial conditions (7).

Observe that the estimates obtained are independent of ε also. Therefore, by the same argument used to obtain u_ε from $u_{\varepsilon\nu}$, which is the solution of (6) and (7), we can pass to the limit when ε goes to zero in u_ε, or subsequence, obtaining a function u independent of ε and ν, such that:

(26) u_ε converges to u, as ε goes to zero, weak star in $L^\infty(0, T; H^1_0(\Omega))$.

(27) u'_ε converges to u', as ε goes to zero, weakly in $L^2(0, T; L^2(\Omega))$.

(28) $\sqrt{k_2(x)+\varepsilon}\, u'_\varepsilon$ converges to $\sqrt{k_2(x)}\, u'$ when ε goes to zero, weak star in $L^\infty(0, T; L^2(\Omega))$, because $\sqrt{k(x)+\varepsilon}$ converges to $\sqrt{k_2(x)}$ strongly in $L^2(\Omega)$ as ε goes to zero.

(29) $|u_\varepsilon|^\rho u_\varepsilon$ converges to $|u|^\rho u$, as ε goes to zero, weak star in $L^{p'}(Q)$.

Therefore, the limit u obtained above satisfies the conditions (1)-(4) of Theorem 1.

In order to check that $u(0)=u_0$, it is sufficient to use (26), (27) followed by an integration by parts. By the same argument, using (28), it can be shown that $k_2(x)u'(0) = \sqrt{k_2(x)}\,v_1$.

To complete the proof of Theorem 1, we need to prove uniqueness, which shall be done in the next section.

§ 2. Uniqueness of solutions

Suppose we have two solutions u and v in the conditions of Theorem 1. It follows that $w=u-v$ is a solution of:

$$k_2(x)w''+k_1(x)w'-\Delta w+|u|^\rho u-|v|^\rho=0, \tag{30}$$

$$w(0)=0, \qquad w'(0)=0, \tag{31}$$

we must prove that $w=0$ on $[0, T)$.

In fact, let $0<s<T$ and $z(t)$ be defined by:

$$z(t)=\begin{cases} -\int_t^s w(\xi)d\xi & \text{if } t\le s \\ 0 & \text{if } t>s. \end{cases}$$

This integral exists and $z(t)\in H_0^1(\Omega)$. If we represent

$$w_1(t)=\int_0^t w(\xi)d\xi,$$

then

$$z(t)=w_1(t)-w_1(s).$$

We have $z(s)=0$, $z'(t)=w(t)$, and it makes sense to evaluate $k_2(x)w''(t)\in H^{-1}(\Omega)$ in $z(t)\in H_0^1(\Omega)$. We obtain:

$$\int_0^s (k_2(x)w'', z)dt+\int_0^s (k_1(x)w', z)dt+\int_0^s a(w, z)dt+\int_0^s (|u|^\rho u-|v|^\rho v, z)dt=0. \tag{32}$$

We have:

$$\int_0^s (k_2w'', z)dt=-\int_0^s (k_2w', w)dt=-\frac{1}{2}\,|\sqrt{k_2(x)}\,w(s)|^2,$$

$$\int_0^s (k_1w', z)dt=-\int_0^s |\sqrt{k_1(x)}\,w(t)|^2\,dt,$$

$$\int_0^s a(w, z)dt=\int_0^s a(z', z)dt=-\frac{1}{2}a(w_1(s)).$$

Then equation (32) can be written as

$$\frac{1}{2}|\sqrt{k_2(x)}\,w(s)|^2+\int_0^s|\sqrt{k_1(x)}\,w(t)|^2\,dt+\frac{1}{2}a(w_1(s))\le\int_0^s|(|u|^\rho u-|v|^\rho v, z)|\,dt. \tag{33}$$

Regarding the non linear term we obtain:

$$|(|u|^\rho u-|v|^\rho v, z)|\le(1+\rho)3^\rho\int_\Omega(|u|^\rho+|v|^\rho)\,|w(t)|\,|z(t)|\,dx.$$

By the embedding theorem of $H_0^1(\Omega)$ in $L^q(\Omega)$ $n\ge3$, for $1/q=1/2-1/n$, we obtain:

$$\frac{1}{n}+\frac{1}{2}+\frac{1}{q}=1.$$

Note that if $n=2$, $H_0^1(\Omega)$ is embedded continuously in $L^q_{\mathrm{loc}}(\Omega)$ for any q. We also have uniqueness. Since $\rho n\le q$, it follows that $|u|^\rho$, $|v|^\rho$ are in $L^n(\Omega)$, because u, v are in $H_0^1(\Omega)$. By the estimates of u, v, in the $H_0^1(\Omega)$ norm and Hölder inequality, we obtain:

$$\begin{aligned}|(|u|^\rho u-|v|^\rho v, z)|&\le(1+\rho)3^\rho\,\||u|^\rho+|v|^\rho\|_{L^n(\Omega)}\,\|w(t)\|_{L^2(\Omega)}\,\|z(t)\|_{L^2(\Omega)}\\&\le C|w(t)|\,(\|w_1(t)\|+\|w_1(s)\|),\end{aligned}$$

where $|\cdot|$ and $\|\cdot\|$ are the norms of $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively.

From (33), since $a(v)\ge\alpha\|v\|^2$, we have:

$$\int_0^s|\sqrt{k_1(x)}\,w(t)|^2\,dx+\frac{\alpha}{2}\|w_1(s)\|^2\le C\int_0^s|w(t)|\,(\|w_1(t)\|+\|w_1(s)\|)dt. \tag{34}$$

By hypothesis, $0<\beta\le k_1(x)$, then (34) can be written as:

$$\begin{aligned}&\int_0^s|\sqrt{k_1(x)}\,w(t)|^2\,dt+\frac{\alpha}{2}\|w_1(s)\|^2\\&\le\frac{C}{\sqrt{\beta}}\int_0^s|\sqrt{k_1(x)}\,w(t)|\,\|w_1(t)\|\,dt+\frac{C}{\sqrt{\beta}}\int_0^s|\sqrt{k_1(x)}\,w(t)|\,\|w_1(s)\|\,dt.\end{aligned} \tag{35}$$

Let $\lambda>0$ be a number to be fixed later. The inequality (35) takes the form:

$$\begin{aligned}&\left(1-\frac{\lambda}{4}\left(1+\frac{C^2}{\beta}\right)\right)\int_0^s|\sqrt{k_1(x)}\,w(t)|^2\,dx+\left(\frac{\alpha}{2}-\frac{4s}{\lambda}\right)\|w_1(s)\|^2\\&\le\frac{4C^2}{\lambda\beta}\int_0^s\|w_1(t)\|\,dt.\end{aligned} \tag{36}$$

Choose λ such that

$$1-\frac{\lambda}{4}\left(1+\frac{C^2}{\beta}\right)=\frac{1}{2}.$$

We obtain $\lambda=2\beta/(\beta+C^2)$. Let s_0 be such that

$$\frac{\alpha}{2}-\frac{4}{\lambda}s_0=\frac{\alpha}{4}.$$

We obtain $s_0=\alpha\beta/8(\beta+C^2)>0$ and for $0\le s\le s_0$,

$$\frac{1}{2}\int_0^s |\sqrt{k_1(x)}\,w(t)|^2\,dx+\frac{\alpha}{2}\,\|w_1(s)\|^2\le K\int_0^s \|w_1(t)\|^2\,dt, \tag{37}$$

or

$$\|w_1(s)\|\le K_0\int_0^s \|w_1(t)\|^2\,dt.$$

This inequality implies $w_1(s)=0$ for all $0\le s\le s_0$, or $w(s)=0$ on $0\le s\le s_0$, or $w=0$ on $[0, T)$, which prove the uniqueness.

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