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On Cauchy Problem for Linear Partial Differential Equations with Constant Coefficients

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1. Introduction

We shall consider the Cauchy problem for a system of linear partial differential equations for a system of unknown functions $u_\mu = u_\mu(t, x)$ $(\mu = 1, \cdots, k)$ of independent real variables (t, x), with $t \in \boldsymbol{R}^1$ and $x = (x_1, \cdots, x_m) \in \boldsymbol{R}^m$:

$$\partial_t u_\mu = \sum_{\nu=1}^{k} p_{\mu\nu}(D_x) u_\nu \quad (\mu = 1, \cdots, k),$$

where $\partial_t = \partial/\partial t$, $D_x = (\partial/\partial x_1, \cdots, \partial/\partial x_m)$ and $p_{\mu\nu}(\zeta)$ are polynomials in $\zeta = (\zeta_1, \cdots, \zeta_m)$ with constant coefficients. Using vector-matrix notations we can write the above system of equations as

$$(1) \qquad \partial_t u^{\downarrow} = \boldsymbol{P}(D_x) u^{\downarrow},$$

where $u^{\downarrow} = (u_\mu \ \mu \downarrow 1, \cdots, k)$ and $P = (p_{\mu\nu} \ {}^{\mu \downarrow 1, \cdots k}_{\nu \downarrow 1, \cdots k})$.

Let $\mathscr{S}$ be the set of all complex valued rapidly decreasing C^∞ functions on $\boldsymbol{R}^m$ with the usual well known topology and $\mathscr{S}'$ be the dual space of $\mathscr{S}$. By $\mathscr{S}^k$ we denote the product space $\mathscr{S} \times \cdots \times \mathscr{S}$ (k-times) and by $\mathscr{S}'^k$ the product space $\mathscr{S}' \times \cdots \times \mathscr{S}' = (\mathscr{S}^k)'$. Now let $\mathscr{G}$ be a linear space of (generalized) complex vector valued functions on $\boldsymbol{R}^m$ such that $\mathscr{S}^k \subset \mathscr{G} \subset \mathscr{S}'^k$, where the topology of the space on the left side of $\subset$ is finer than that of the space on the right side of $\subset$.

The Cauchy problem for the equation (1) is said to be **forward $\mathscr{G}$-well posed** on the interval $[0, \tau]$ $(\tau > 0)$ if and only if the following two conditions are satisfied:
1) (*Unique existence of the solution*) For any $\mathring{u}^{\downarrow} \in \mathscr{G}$ there exists a unique $\mathscr{G}$-valued solution $u^{\downarrow} = u^{\downarrow}(t, x)$ of (1) for $t \in [0, \tau]$ with the initial condition $u^{\downarrow}(0, x) = \mathring{u}^{\downarrow}(x)$.
2) (*Continuity of solution with respect to the initial value*) If $\mathring{u}^{\downarrow}(x)$ tends to zero in $\mathscr{G}$, then the solution $u^{\downarrow} = u^{\downarrow}(t, x)$ of (1) with the initial value $u^{\downarrow}(0, x) = \mathring{u}^{\downarrow}(x)$ also tends to zero in $\mathscr{G}$ uniformly for $t \in [0, \tau]$.

Since the operator $\boldsymbol{P}(D_x)$ does not depend on the time variable t, we can easily see that the forward $\mathscr{G}$-well posedness does not depend on $\tau > 0$, hence we can simply say of the forward $\mathscr{G}$-well posedness without mentioning the interval $[0, \tau]$.

Making use of the Fourier transform with respect to the space variable x

$$v^{\natural}(\xi)=(2\pi)^{-m/2}\int_{\boldsymbol{R}^m} e^{-i\xi\cdot x}u^{\natural}(x)dx, \qquad \xi=(\xi, \cdots, \xi_m)\in \boldsymbol{R}^m,$$

the Cauchy problem of the equation (1) can be formally reduced to that of the ordinary differential equation for the $\tilde{\mathscr{G}}$-valued unknown function of t, $v^{\natural}=v^{\natural}(t, \xi)$

$$\partial_t v^{\natural}=\boldsymbol{P}(i\xi)v^{\natural}, \tag{2}$$

where ξ is a parameter and $\tilde{\mathscr{G}}$ is the Fourier transform of $\mathscr{G}$.

It is known that for some function spaces, for example $\mathscr{G}=\mathscr{S}^k$ or $(\mathscr{D}_{L^2})^k$, the necessary and sufficient condition for the forward $\mathscr{G}$-well posedness of the equation (1) is given by the **Petrovski correctness** "*The real part of all eigen-values of the matrix* $\boldsymbol{P}(i\xi)$ *are bounded above for* $\xi\in\boldsymbol{R}^m$".

In this note we shall show that the *Petrovski correctness is necessary* for the forward $\mathscr{G}$-well posedness provided that

$$\mathscr{S}^k\subset\mathscr{G}\subset\mathscr{S}'^k.$$

When the independent space variable is one dimensional, $x\in\boldsymbol{R}^1$, we have already published the same result [3], using the property of algebraic function of a single independent variable. In this note we do not use the property of algebraic functions, but we use the complex integration formula

$$\exp(t\boldsymbol{P}(i\xi))=\frac{1}{2\pi i}\int_{\mathscr{C}} e^{t\zeta}(\zeta\boldsymbol{I}-\boldsymbol{P}(i\xi))^{-1}d\zeta.$$

Here we have also to mention that in 1974 Master Jun'ichi Sato has obtained the same result for the case of single equation

$$\partial_t u=P(D_x)u \quad (x\in\boldsymbol{R}^m,\ k=1),$$

which has not been published in printed journal.

2. Proposition and Lemmas

Let $\lambda_j(\xi)$ $(j=1, \cdots, k)$ be eigenvalues of the $k\times k$ matrix $\boldsymbol{P}(i\xi)$ and $\hat{\lambda}(\xi)$ be an eigen-value of $\boldsymbol{P}(i\xi)$ such that

$$\mathscr{R}e\,\hat{\lambda}(\xi)=\operatorname*{Max}_{1\leq j\leq k}\mathscr{R}e\,\lambda_j(\xi).$$

Then, *the Petrovski correctness of the equation* (1) *is equivalent to the existence of constants* C_0 *and* C_1 *such that*

$$\mathscr{R}e\,\hat{\lambda}(\xi)\leq C_0+C_1\log(1+|\xi|) \qquad \text{for all } \xi\in\boldsymbol{R}^m \text{ (cf. [1]).} \tag{3}$$

As the Fourier transform is an isomorphic and homeomorphic mapping of $\mathscr{S}$ onto $\mathscr{S}$ and of $\mathscr{S}'$ onto $\mathscr{S}'$, we have only to prove the following proposition.

Proposition. *If the equation* (1) *is not Petrovski correct, we can construct a sequence* $\{\mathring{v}_n^{\downarrow}(\xi)\}_{n\in \boldsymbol{N}}\subset C_0^\infty(\boldsymbol{R}^m)$ *such that* $\mathring{v}_n^{\downarrow}\to 0$ *in* $\mathscr{S}^k$ *as* $n\to\infty$, *but for the solution* $v_n^{\downarrow}=v_n^{\downarrow}(t,\xi)$ *of the equation* (2) *with the initial condition* $v_n^{\downarrow}(0,\xi)=\mathring{v}_n^{\downarrow}(\xi)$ *we have* $v_n^{\downarrow}(t,\xi)\not\to 0$ *in* $\mathscr{S}'^k$ *as* $n\to\infty$ *for* $t>0$, *where* $\boldsymbol{N}$ *denotes the set of all natural numbers and* C_0^∞ *denotes the set of all* C^∞*-functions with compact supports.*

For the proof of this proposition we shall give some lemmas previously.

Lemma 1. *There exist natural numbers* l_0, l_1 *and constants* C_0, C_1 *such that*

$$\|(\zeta\boldsymbol{1}-\boldsymbol{P}(i\xi))^{-1}\|\leqq C_0(1+|\xi|)^{l_0} \tag{1.1}$$

$$\|D_\xi(\zeta\boldsymbol{1}-\boldsymbol{P}(i\xi))^{-1}\|\leqq C_1(1+|\xi|)^{l_1}, \qquad (\xi\in\boldsymbol{R}^m,\ \zeta\in\boldsymbol{C}) \tag{1.2}$$

if $\mathrm{Min}_{1\leqq j\leqq k}|\zeta-\lambda_j(\xi)|=1$, *where the norm* $\|A\|$ *of a matrix* A *is defined by* $\|A\|=\sup\{|Av^{\downarrow}|;|v^{\downarrow}|=1\}$.

Proof. Cf. [4], p. 157.

Lemma 2. *With the same natural number* l_1 *and constant* C_1 *as in Lemma* 1, *we have*

$$\|D_\xi \exp(t\boldsymbol{P}(i\xi))\|\leqq C_1e^t(1+|\xi|)^{l_1}\|\exp(t\boldsymbol{P}(i\xi))\| \qquad \textit{for } t>0. \tag{2.1}$$

Proof. Let $\mathscr{C}_\xi$ be a closed curve (generally not connected) in the complex plane $\boldsymbol{C}$ with positive orientation such that

$$\mathscr{C}_\xi=\{\zeta\in\boldsymbol{C};\ \underset{1\leqq j\leqq k}{\mathrm{Min}}|\zeta-\lambda_j(\xi)|=1\}. \tag{2.2}$$

Then we have by the complex integration along $\mathscr{C}_\xi$

$$D_\xi\exp(t\boldsymbol{P}(i\xi))=\frac{1}{2\pi i}\int_{\mathscr{C}_\xi}e^{t\zeta}D_\xi(\zeta\boldsymbol{1}-\boldsymbol{P}(i\xi))^{-1}d\zeta \qquad \text{(cf. [4], p. 155).}$$

Thus, by Lemma 1,

$$\|D_\xi\exp(t\boldsymbol{P}(i\xi))\|\leqq\frac{1}{2\pi}C_1(1+|\xi|)^{l_1}\int_{\mathscr{C}_\xi}|e^{t\zeta}||d\zeta|. \tag{2.3}$$

But, as $|\zeta-\lambda_j(\xi)|=1$ with some j if $\zeta\in\mathscr{C}_\xi$, with $a_j^{\downarrow}(\xi)$ a normalized eigen-vector of $\boldsymbol{P}(i\xi)$ corresponding to the eigenvalue $\lambda_j(\zeta)$, we have for, $\zeta\in\mathscr{C}_\xi$, $t>0$,

$$\begin{aligned}|e^{t\zeta}|&\leqq e^{t\mathscr{R}e\lambda_j(\xi)+t}=e^t|e^{t\lambda_j(\xi)}a_j^{\downarrow}(\xi)|\\&=e^t|\exp(t\boldsymbol{P}(i\xi))a_j^{\downarrow}(\xi)|\leqq e^t\|\exp(t\boldsymbol{P}(i\xi))\|.\end{aligned}$$

Hence from (2.3) we get (2.1).

Lemma 3. *With the same natural number l_1 and constant C_1 as in Lemma* 1 *we have*

$$\|\exp(t\boldsymbol{P}(i\xi')) - \exp(t\boldsymbol{P}(i\xi))\| \leqq 2C_1 e^t (2+|\xi|)^{l_1} |\xi'-\xi| \, \|\exp(t\boldsymbol{P}(i\xi))\|, \tag{3.1}$$

if $t>0$ *and* $C_1 e^t (2+|\xi|)^{l_1} |\xi'-\xi| < 1/2$.

Proof. Let $\xi(s)=(1-s)\xi+s\xi'$ for $0\leqq s\leqq 1$ and put $Y(s)=\exp(t\boldsymbol{P}(i\xi(s)))$. Then, by Lemma 2,

$$\begin{aligned} \left\|\frac{d}{ds}Y(s)\right\| &= \|D_\xi \exp(t\boldsymbol{P}(i\xi(s)))(\xi'-\xi)\| \\ &\leqq C_1 e^t (1+|\xi(s)|)^{l_1} \|Y(s)\| \cdot |\xi'-\xi|. \end{aligned} \tag{3.2}$$

But, as $|\xi(s)-\xi|\leqq|\xi'-\xi|$, we have from (3.2) putting $\eta=\mathrm{Max}_{0\leqq s\leqq 1}\|Y(s)-Y(0)\|$,

$$\eta \leqq C_1 e^t (2+|\xi|)^{l_1} |\xi'-\xi| (\|Y(0)\|+\eta).$$

Therefore, if $C_1 e^t (2+|\xi|)^{l_1} |\xi'-\xi| < 1/2$, we get

$$\|Y(1)-Y(0)\| \leqq 2C_1 e^t (2+|\xi|)^{l_1} |\xi'-\xi| \, \|Y(0)\|,$$

which shows (3.1).

Lemma 4. *With the same natural number l_0 and constant C_0 as in Lemma* 1, *we have*

$$\|\exp(t\boldsymbol{P}(i\xi))\| \leqq kC_0 e^t (1+|\xi|)^{l_0} e^{t\mathscr{R}e\hat{\lambda}(\xi)} \qquad \textit{for } t>0. \tag{4.1}$$

Proof. As $\exp(t\boldsymbol{P}(i\xi)) = 1/2\pi i \int_{\mathscr{C}_\xi} e^{t\zeta}(\zeta\boldsymbol{1}-\boldsymbol{P}(i\xi))^{-1} d\xi$ and $|e^{t\xi}| \leqq e^{t\mathscr{R}e\hat{\lambda}(\xi)}+t$ for $\zeta \in \mathscr{C}_\xi$, by Lemma 1 we get (4.1).

3. Proof of Proposition

1^0. If the equation (1) is not correct, i.e., if there do not exist constants C_0 and C_1 such that (3) holds, we can find a sequence of points $\{\xi^{(n)}\}_{n\in\boldsymbol{N}} \subset \boldsymbol{R}^m$ such that

$$\mathscr{R}e\,\hat{\lambda}(\xi^{(n)}) > n^3 \log(1+|\xi^{(n)}|) \quad \text{and} \quad |\xi^{(n+1)}| > |\xi^{(n)}|+2. \tag{4}$$

Then we construct a sequence of initial values $\mathring{v}_n^{\downarrow}(\xi)$ by

$$\mathring{v}_n^{\downarrow}(\xi) = (1+|\xi^{(n)}|)^{-n^2} \rho_{\delta_n}(\xi-\xi^{(n)}) a^{\downarrow}(\xi^{(n)}), \tag{5}$$

where $a^{\downarrow}(\xi)$ is a normalized eigen-vector of $\boldsymbol{P}(i\xi)$ corresponding to the eigen-value

$\hat{\lambda}(\xi)$, and $\rho_\delta(\xi)=\delta^{-m}\rho_1(\delta^{-1}\xi)$ $(\delta>0)$ with $\rho_1(\xi)\in C_0^\infty(\boldsymbol{R}^m)$ such that $\rho_1(\xi)\geqq 0$, supp (ρ_1) $\subset\{\xi\in\boldsymbol{R}^m;|\xi|\leqq 1\}$, $\int\rho_1(\xi)d\xi=1$ and

$$\delta_n=(1+|\xi^{(n)}|)^{-n}. \tag{6}$$

Thus we have $\mathring{v}_n^{\downarrow}\to 0$ in $\mathscr{S}^k$ as $n\to\infty$. Because for any multi-indices α, $\beta\in\boldsymbol{N}_0^m$ ($\boldsymbol{N}_0=\{0\}\cup\boldsymbol{N}$) we have

$$|\xi^\beta D_\xi^\alpha\mathring{v}_n(\xi)|\leqq\sup_{|\xi-\xi^{(n)}|\leqq\delta_n}|\xi^\beta|\cdot(1+|\xi^{(n)}|)^{-n^2}\delta_n^{-m-|\alpha|}C_\alpha,$$

where $C_\alpha=\sup_{|\xi|\leqq 1}|D_\xi^\alpha\rho_1(\xi)|$. Hence as $\delta_n\leqq 1$

$$|\xi^\beta D_\xi^\alpha\mathring{v}_n^{\downarrow}(\xi)|\leqq(1+|\xi^{(n)}|)^{-n^2+|\beta|+(m+|\alpha|)n}C_\alpha\to 0$$

uniformaly on $\boldsymbol{R}^m$ as $n\to\infty$.

2^0. Now to prove that the sequence of the solutions $v^{\downarrow}=v_n^{\downarrow}(t,\xi)=\exp(t\boldsymbol{P}(i\xi))\mathring{v}_n^{\downarrow}(\xi)$ of the equation (2) with the initial values $v_n^{\downarrow}(0,\xi)=\mathring{v}_n^{\downarrow}(\xi)$ does not tend to zero in $\mathscr{S}'^k$ as $n\to\infty$ for $t>0$, we shall construct a function $\varphi^{\downarrow}(\xi)\in\mathscr{S}^k$ such that

$$\left|\int_{\boldsymbol{R}^m}v_n^{\downarrow}(t,\xi)\cdot\varphi^{\downarrow}(\xi)d\xi\right|\to\infty\qquad\text{for } t>0.$$

First consider a vector-valued step function

$$\psi^{\downarrow}(\xi)=(1+|\xi^{(n)}|)^{-n^2}a^{\downarrow}(\xi^{(n)})$$

for $|\xi|<|\xi^{(1)}|+1$ if $n=1$, but for $|\xi^{(n-1)}|+1\leqq|\xi|<|\xi^{(n)}|+1$ if $n\geqq 2$. And define a vector-valued C^∞-function $\varphi^{\downarrow}$ by $\varphi^{\downarrow}(\xi)=\rho_{1/2}*\psi^{\downarrow}(\xi)$. Then

$$\varphi^{\downarrow}(\xi)=(1+|\xi^{(n)}|)^{-n^2}a^{\downarrow}(\xi^{(n)})\ \text{(constant)}\qquad\text{for } |\xi-\xi^{(n)}|<1/2, \tag{7}$$

and $\varphi^{\downarrow}\in\mathscr{S}^k$. Because for $|\xi^{(n-1)}|+1\leqq|\xi|<|\xi^{(n)}|+1$ we have

$$|\xi^\beta D_\xi^\alpha\varphi^{\downarrow}(\xi)|=|\xi^\beta|\cdot|D^\alpha\rho_{1/2}*\psi^{\downarrow}(\xi)|\leqq C'_\alpha|\xi^\beta||\psi^{\downarrow}(\xi)|\leqq C'_\alpha(1+|\xi^{(n)}|)^{-n^2+|\beta|}\to 0$$

as $|\xi|\to\infty$, where $C'_\alpha=\mathrm{Max}_{|\xi|\leqq 1/2}|D^\alpha\rho_{1/2}(\xi)|$.

Now we have

$$\int_{\boldsymbol{R}^m}v_n^{\downarrow}(t,\xi)\cdot\varphi^{\downarrow}(\xi)d\xi=I+II, \tag{8}$$

where

$$I=\int_{\boldsymbol{R}^m}\{\exp(t\boldsymbol{P}(i\xi^{(n)}))\mathring{v}_n^{\downarrow}(\xi)\}\cdot\varphi^{\downarrow}(\xi)d\xi$$

and

$$II=\int_{\boldsymbol{R}^m}\{\exp(t\boldsymbol{P}(i\xi))-\exp(t\boldsymbol{P}(i\xi^{(n)}))\}\mathring{v}_n^{\downarrow}(\xi)\cdot\varphi^{\downarrow}(\xi)d\xi.$$

As $\exp(t\boldsymbol{P}(i\xi^{(n)}))a^{\downarrow}(\xi^{(n)})=e^{t\hat{\lambda}(\xi^{(n)})}a^{\downarrow}(\xi^{(n)})$ from (5) we have

$$\exp(t\boldsymbol{P}(i\xi^{(n)}))\mathring{v}_n^{\downarrow}(\xi)=(1+|\xi^{(n)}|)^{-n^2}\rho_{\delta_n}(\xi-\xi^{(n)})e^{t\hat{\lambda}(\xi^{(n)})}a^{\downarrow}(\xi^{(n)}).$$

Hence by (7), as $\delta_n\leqq 1/2$,

$$(9)\qquad I=\int_{|\xi-\xi^{(n)}|\leqq\delta_n}(1+|\xi^{(n)}|)^{-2n^2}\rho_{\delta_n}(\xi-\xi^{(n)})e^{t\hat{\lambda}(\xi^{(n)})}d\xi=(1+|\xi^{(n)}|)^{-2n^2}e^{t\hat{\lambda}(\xi^{(n)})}.$$

As $|\xi^{(n)}|\geqq n$ and $C_1e^t(2+|\xi^{(n)}|)^{l_1}|\xi-\xi^{(n)}|<1/2$ for $|\xi-\xi^{(n)}|\leqq\delta_n=(1+|\xi^{(n)}|)^{-n}$ if n is sufficiently large, we get by Lemma 3 and (7)

$$\begin{aligned}|II|&\leqq\int\|\exp(t\boldsymbol{P}(i\xi))-\exp(t\boldsymbol{P}(i\xi^{(n)}))\|\,|\mathring{v}_n^{\downarrow}(\xi)|\cdot|\varphi^{\downarrow}(\xi)|d\xi\\&\leqq 2C_1e^t(2+|\xi^{(n)}|)^{l_1}\|\exp(t\boldsymbol{P}(i\xi^{(n)}))\|\,(1+|\xi^{(n)}|)^{-2n^2}\\&\quad\times\int_{|\xi-\xi^{(n)}|\leqq\delta_n}|\xi-\xi^{(n)}|\,\rho_{\delta_n}(\xi-\xi^{(n)})d\xi\\&=2C_1e^t(2+|\xi^{(n)}|)^{l_1}(1+|\xi^{(n)}|)^{-2n^2}\delta_n\,\|\exp(t\boldsymbol{P}(i\xi^{(n)}))\|.\end{aligned}$$

Thus, further by Lemma 4 and (6), if n is sufficiently large,

$$|II|\leqq Ce^{2t}(2+|\xi^{(n)}|)^l(1+|\xi^{(n)}|)^{-2n^2-n}e^{t\mathscr{R}e\hat{\lambda}(\xi^{(n)})},$$

where $C=2kC_1C_0$ and $l=l_1+l_0$. Hence by (4), if n is sufficiently large,

$$\begin{aligned}|I+II|&\geqq|I|-|II|\\&\geqq(1+|\xi^{(n)}|)^{-2n^2}e^{t\mathscr{R}e\hat{\lambda}(\xi^{(n)})}\{1-Ce^{2t}(2+|\xi^{(n)}|)^l(1+|\xi^{(n)}|)^{-n}\}\\&>\tfrac{1}{2}(1+|\xi^{(n)}|)^{tn^3-2n^2}\to\infty\quad\text{as } n\to\infty\quad\text{for } t>0.\end{aligned}$$

Q.E.D.

After all we have established the following theorem.

Theorem. *Let* $\mathscr{S}^k\subset\mathscr{G}\subset\mathscr{S}'^k$, *where the topology of the space on the left side of* $\subset$ *is finer than or equal to that of on the right side of* $\subset$. *Then the Petrovski correctness is necessary for the Cauchy problem of the equation* (1) *to be forward* $\mathscr{G}$*-well posed.*

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