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A General Theory of Invariants for Meromorphic Differential Equations;

Part I, Formal Invariants

By

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§ 0. Introduction

It is well-known that systems of *meromorphic linear differential equations* $x'=A(z)x$ at ∞ have formal fundamental solution matrices of the form $H(z)=F(z)G(z)$, where $F(z)$ is a formal power series in a root of z^{-1} and $G(z)$ is a matrix of elementary functions which consists of exponential polynomials in a root of z, complex powers of z, and positive integral powers of $\log z$. Moreover, in appropriately small sectorial regions, the formal solutions are asymptotic expansions for actual solutions as $z\rightarrow\infty$. These results are usually known as the *Poincaré Theory* for meromorphic differential equations.

Thus the matrix $G(z)$ determines the *asymptotic type* of the singularity of the solutions and as such controls the most important aspect of the solutions. The other essential aspects concern the size of the maximal sectors in which an asymptotic is valid and the connection matrices which describe the change in the asymptotic as the solution is continued around the singularity.

For meromorphic differential equations, only certain asymptotic types are possible and one would wish to know exactly those which can arise. Furthermore, one would like to know the size of the maximal sectors, the complete structure of the connection matrices, and how this characterizes the singularity of the solutions.

In this note we obtain a theory of invariants for meromorphic differential equations which can be used to answer such questions. The equivalences which determine the invariants correspond to various types of linear transformations acting on the differential equation. This theory may be viewed as a completion of the *Birkhoff Theory* [2] to the general case. Our treatment is based upon lectures given by the second author in the Summer of 1977 in Ulm [9] and represents a finalized and more economical presentation of the results. Some topics have been added (e.g., Section 5 of Part I on the regular singular case $r=0$) and certain proofs have been substantially simplified (e.g., the proof of the Uniqueness Theorem, Section 6, Part II),

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which has resulted, in the authors' viewpoint, in a more direct and accessible theory.

In Part I we consider formal transformations and the corresponding (formal) invariants. They are determined from $G(z)$ and their calculation from $A(z)$ is purely an algebraic problem. In Part II we discuss convergent transformations and the corresponding (proper) invariants. They are related to the sectorial regions in which an asymptotic expansion is valid and the associated connection matrices.

The theory we present here is a completion to the general situation of some special cases which we have previously treated (see [1], [7], and [8]). In particular, when A_0, the leading coefficient matrix in the expansion of $A(z)$ at ∞, has all distinct eigenvalues, the formal part of the theory is trivial. When A_0 has some equal eigenvalues, the formal structure becomes interesting. We have already treated some such (two-dimensional) cases, which may be viewed as a preliminary attempt at explaining the general structure of the invariants in the equal eigenvalue case.

In discussing formal invariants, it is natural to let $A(z)$ also be a formal meromorphic series, in which case we call $x'=A(z)x$ a *formal meromorphic differential equation.* Our treatment begins with a more detailed factorization and normalization of a formal fundamental solution matrix from which one can first see exactly which $G(z)=G_m(z)$ correspond to formal meromorphic differential equations. We then show that $G_m(z)$ is a complete system of formal meromorphic invariants. The more restrictive classes of invariants (analytic and Birkhoff) are treated by further normalizing a formal solution and adjoining further invariants.

As an application of our results, we show that the logarithmic derivative of $G_m(z)$ is of the form $P(z)z^{-1}$, where $P(z)$ is a matrix polynomial in z whose degree is at most equal to r, the Poincaré rank of the differential equation. This shows that a *Birkhoff reduction* can be achieved through the use of formal meromorphic transformations. The differential equation which $G_m(z)$ satisfies may be considered as a *formal meromorphic canonical form.* These are the simplest meromorphic differential equations having solutions with only *elementary singularities.*

Although the existence of a complete set of formal solutions (first for n^{th} order scalar differential equations and then for first order systems) is a well established fact, the characterization of $G(z)$, its freedom, and its invariance have not been so well understood. Cope [4] recognized that the exponent $Q(z)$ in $G(z)$ must satisfy a closure property with respect to analytic continuation, but did not explicitly exhibit its precise structure and was under the impression that the root which enters into Q is not greater than the dimension of the system, which is false. Several formal invariants have been introduced and calculated in certain cases by Gérard and Levelt [6], however, these invariants are not a complete system of formal invariants. Recently Levelt [9] has shown that the differential operator $z(d/dz)-A(z)$ has a unique formal decomposition as the sum of a diagonalizable operator and a nilpotent operator which commute. He also obtains a bound on the root which enters into the formal solutions. His factorization is related to, but not equivalent to ours, which concerns a factorization of a fundamental solution instead of the operator. In particular, one does not see in his theory which factorizations exactly correspond to meromorphic differential operators.

§ 1. Structure of formal solutions

A *formal meromorphic differential equation* has the form $x'=A(z)x$, where $A(z)$ is a *formal meromorphic series* at ∞, i.e.,

$$A(z)=z^{r-1}\sum_{k=0}^{\infty} A_k z^{-k}, \tag{1.0}$$

where $A_0\neq 0$ and r is an integer (called the Poincaré rank of $A(z)$). It is sometimes convenient to denote $x'=A(z)x$ by $[A]$. From the formal theory of such differential equations, it is well-known that a formal fundamental solution matrix for $[A]$ exists and can be written as

$$H(z)=\Phi(z)z^L \exp[Q(z)], \tag{1.1}$$

where $Q(z)=\sum_{j=1}^{h} Q_j z^{r_j}$. Here h is a non-negative integer ($h=0$ means $Q(z)\equiv 0$), the r_j are distinct positive rationals, the Q_j are constant non-zero diagonal matrices, L is a constant matrix, $\Phi(z)=z^{m_0/g}\sum_{m=0}^{\infty}\Phi_m z^{-m/g}$ is a formal series with g a positive interge, m_0 an integer, Φ_m are constant matrices, and det $\Phi(z)$ is not the zero series. Since multi-valued functions arise, it is convenient to consider them as being defined on the Riemann surface of $\log z$ whose points will also be denoted by z and are given by the "coordinates" $|z|>0$ and $\arg z\varepsilon \in (-\infty, +\infty)$.

Such a formal solution comes about ([15], [17], [9]) through the use of an algorithm which reduces the differential equation in a finite number of steps to one from which $Q(z)$ and L are obtained immediately and the coefficients of $\Phi(z)$ are calculated recursively. From the algorithm it is not easy to see, in general, how the formal meromorphic structure of $A(z)$ is related to the formal solution, i.e., which formal solution matrices of the form (1.1) correspond to formal meromorphic differential equations.

For the purpose of discussing invariants of the differential equation it is important to determine the freedom in the components of $H(z)$ and a normalization which exhibits its essential structure. We now state our first such result as

Theorem I. *To each formal meromorphic differential equation* $x'=A(z)x$ *there exists a formal fundamental solution matrix of the form*

$$H(z)=F(z)G(z), \tag{1.2}$$

where $F^{\pm 1}(z)$ *are formal meromorphic series and*

$$G(z)=z^{J}U\exp[Q(z)]. \tag{1.3}$$

The matrices $Q(z)$, U, J *are simultaneously decomposed into a direct sum of "super-blocks" which will also be denoted by the same letters and each such super-block has the following structure:*

$$Q(z)=\operatorname{diag}\{q(z)I_s, q(ze^{2\pi i})I_s, \cdots, q(ze^{2\pi i(p-1)})I_s\},$$
$$U=[\varepsilon^{(j-1)(k-1)}I_s],\ 1\leq j,\ k\leq p,\ \varepsilon=\exp(2\pi i/p),$$

and

$$J=\operatorname{diag}\{J_s, J_s+(1/p)I_s, \cdots, J_s+((p-1)/p)I_s\};$$

where $q(z)=\hat{q}(z^{1/p})$ *for a polynomial* $\hat{q}$ *without constant term and* p *is the smallest positive integer for which* q *can be so represented,* s *is the multiplicity with which* q *and its analytic continuations appears in the total* Q, *in particular, none of the functions* $q(z)$, $q(ze^{2\pi i})$, $\cdots$, $q(ze^{2\pi i(p-1)})$ *occur in another super-block,* I_s *denotes the* $s\times s$ *identity matrix, and* J_s *is an s-dimensional matrix in Jordan canonical form with eigenvalues which can arbitrarily be selected from residue classes* mod $1/p$.

Conversely, each such $H(z)$ *is a formal fundamental solution matrix of a formal meromorphic differential equation.*

Note that a super-block is uniquely determined by $q(z)$, s, and J_s (since p and U are uniquely determined by $q(z)$). In comparing with (1.1), a possible choice of L is $U^{-1}JU$ in which case $\Phi(z)$ carries no roots.

It can, of course, happen that $\hat{q}(z)$ in a super-block is the zero polynomial in which case we see immediately that $p=1$. If the polynomials corresponding to each super-block are equal to zero, the differential equation is said, traditionally, to have a *regular-singular* point at $z=\infty$ and in this case the convergence of the formal series for $A(z)$ is equivalent to the convergence of the formal series $F(z)$. If $r=0$, the singularity at ∞ is called *simple* or of *first kind* and it is well-known that ∞ is then a regular singular point. In Section 5 we discuss the invariants in the case $r=0$.

§ 2. Proof of Theorem I

We begin with $H(z)$ as in (1.1) and first determine the characteristic property for such matrices $H(z)$ to be a formal fundamental solution of a formal meromorphic differential equation. In doing so, it is first helpful to discuss

a. *Formal logarithmic-exponential expressions.*

A *formal logarithmic-exponential expression* (compare [3; pp. 141-142]) is a finite sum of terms of the form

$$\psi(z)\exp q(z), \tag{2.1}$$

where $q(z)=b_1z^{r_1}+\cdots+b_hz^{r_h}$, the b_i are non-zero complex numbers, the r_i are distinct positive rationals, h is a non-negative integer ($h=0$ means $q(z)\equiv 0$), and $\psi(z)$ is a *formal logarithmic expression,* that is, a finite sum of terms of the form

$$f(z)z^{\lambda}(\log z)^{k}, \tag{2.2}$$

where λ is a complex number, k is a non-negative integer, and $f(z)$ is a formal meromorphic series (at ∞). These expressions are considered on the Riemann surface of $\log z$ and we note that the usual operations of taking sums, products, derivatives, and "analytic continuation" (by which we mean formally replacing z by $ze^{2\pi ik}$, k an integer) are defined formally for such expressions and have the same properties as if the expressions were functions. In particular, the derivative of a formal logarithmic-exponential expression is equal to zero only if the expression is equal to a constant and a formal logarithmic expression is equal to a formal meromorphic series if and only if it remains unchanged under analytic continuation: $z\to ze^{2\pi i}$. Matrices whose entries are expressions of the foregoing types are called by the same names.

b. *The structural equation for $H(z)$.*

Note that $H(z)$ in (1.1) can be written in the form

$$H(z)=\Psi(z)\exp\,[Q(z)], \tag{2.3}$$

where $\Psi^{\pm 1}(z)$ are formal logarithmic matrices and $Q(z)$ is the same as in (1.1), since $\Phi(z)$ can be expressed as the sum of at most g terms and each is a power of z times a formal meromorphic series. Because $A(z)$ is a formal meromorphic series and the operation $z\to ze^{2\pi i}$ commutes with differentiation and formation of the inverse, it follows from $A(z)=H'(z)H^{-1}(z)$ that $H(ze^{2\pi i})$ is also a formal fundamental solution matrix for $[A]$, hence $H^{-1}(z)H(ze^{2\pi i})$ has derivative zero, therefore there exists a constant non-singular matrix C (called the *formal circuit matrix* of $H(z)$) such that

$$H(ze^{2\pi i})=H(z)C. \tag{2.4}$$

Moreover, if $H(z)$ is a matrix of the form (2.3) then

$$H'(z)H^{-1}(z)=\Psi'(z)\Psi^{-1}(z)+\Psi(z)Q'(z)\Psi^{-1}(z),$$

which is clearly a formal logarithmic matrix, and if $H(z)$ satisfies (2.4) then $H'H^{-1}$ is invariant under analytic continuation and is therefore a formal meromorphic series, hence H is a formal solution of a formal meromorphic differential equation. Thus equation (2.4) *characterizes* those $H(z)$ of the form (2.3) which are solutions of formal meromorphic differential equations and we call (2.4) *the structural equation of $H(z)$.*

c. *The super-block structure of* $Q(z)$.

If we replace $H(z)$ by $H(z)C_1$, where C_1 is an arbitrary permutation matrix, then

$$H(z)C_1 = \Psi(z)C_1 \exp \tilde{Q}(z), \qquad \text{where } \tilde{Q}(z) = C_1^{-1}Q(z)C_1,$$

is again of the form (2.3), hence without loss in generality we may assume that

$$Q(z) = \operatorname{diag}\{q_1(z)I_{s_1}, q_2(z)I_{s_2}, \cdots, q_l(z)I_{s_l}\},$$

where I_{s_k} denotes the s_k-dimensional identity matrix and the scalar functions $q_k(z)$ are all distinct. In this way the matrix $Q(z)$ can be blocked and the order of the blocks can still be assigned.

If $C = [C_{jk}]$, $1 \leq j, k \leq l$, is blocked like $Q(z)$, then from (2.3) and (2.4) it follows that

$$\Psi^{-1}(z)\Psi(ze^{2\pi i}) = [C_{jk} \exp (q_j(z) - q_k(ze^{2\pi i}))], \qquad 1 \leq j, k \leq l. \tag{2.5}$$

Since the left-hand side of (2.5) is a formal logarithmic matrix, them for each pair (j, k) either

$$q_j(z) = q_k(ze^{2\pi i}) \quad \text{or} \quad C_{jk} = 0. \tag{2.6}$$

Therefore from (2.5) we obtain

$$\Psi(ze^{2\pi i}) = \Psi(z)C \tag{2.7}$$

and

$$Q(ze^{2\pi i}) = C^{-1}Q(z)C. \tag{2.8}$$

We remark that (2.7) and (2.8) are equivalent to (2.4).

Since C is non-singular, for each fixed k there exists a j such that $C_{jk} \neq 0$. Also, because the $q_j(z)$ are all distinct, then from (2.6) we see that this $C_{jk} \neq 0$ can occur for only one j and C_{jk} must be square and non-singular because the same argument holds for the rows as well. For this pair (j, k) we have $q_j(z) = q_k(ze^{2\pi i})$ and the corresponding blocks in $Q(z)$ must be of the same size. Hence it follows that for each $q_j(z)$, the analytic continuation $q_j(ze^{2\pi im})$, m an integer, likewise appears in $Q(z)$ and with the *same multiplicity*. But recall that the order of the blocks in $Q(z)$ is arbitrary, so we may arrange that all the $q_j(z)$ which are carried into each other by $z \rightarrow ze^{2\pi i}$ occur cyclically. This yields the super-block structure of $Q(z)$, namely to each super-block (which we again denote by Q) there correspond two positive integers, s (the dimension of the sub-blocks) and p (the "period"), such that

$$Q(z) = \operatorname{diag}\{q(z)I_s, q(ze^{2\pi i})I_s, \cdots, q(ze^{2\pi i(p-1)})I_s\}$$

and p is the smallest positive integer such that $q(z) = q(ze^{2\pi ip})$. Moreover, it follows that none of the functions in one super-block appear in any other super-block. In addition, it also follows that $q(z)=\hat{q}(z^{1/p})$ for a polynomial $\hat{q}$ in $z^{1/p}$ without constant term and p is the smallest positive integer such that q can be so represented.

From the above ordering, it follows that to each super-block $Q(z)$ there corresponds a unique $p\times p$-block permutation matrix

$$R=\begin{bmatrix} O_s & O_s & \cdots & O_s & I_s \\ I_s & O_s & \cdot & \cdot\ \cdot & O_s \\ O_s & I_s & \ddots & & \vdots \\ \vdots & & & & \vdots \\ O_s & \cdot & \cdot\ \cdot & I_s & O_s \end{bmatrix}$$

(for $p>1$ and $R=I_s$ for $p=1$) such that

$$Q(ze^{2\pi i})=R^{-1}Q(z)R. \tag{2.9}$$

If the direct sum of the R corresponding to each super-block is again denoted by R, then (2.9) holds as well for the total matrix $Q(z)$.

d. *Normalization of a formal circuit matrix.*

Utilizing (2.9) in (2.8), one sees that C has the form

$$C=DR, \tag{2.10}$$

where D commutes with Q and is, therefore, a constant non-singular matrix which is diagonally blocked like $Q(z)$. At this point, we remark that the structure of $H(z)$ corresponding to formal meromorphic differential equations has already been determined, namely Q satisfies (2.9) and C satisfies (2.10) and (2.7); the meaning of (2.7) will be discussed at the beginning of Section 1.e.

We now proceed to further normalize $H(z)$ so that the invariants of $[A]$ can be determined. We do this by modifying $H(z)$ (and consequently its formal circuit matrix) by right hand constant non-singular factors which leave $Q(z)$ fixed. The admissible factors are constant matrices which are diagonally blocked like $Q(z)$ with arbitrary non-singular diagonal blocks.

If $\tilde{D}$ is such a matrix, then $\tilde{D}^{-1}(DR)\tilde{D}$ is again a circuit matrix for another $\Psi(z)$ but the same $Q(z)$. We now use successively the following four such similarity transformations to bring about the desired normalization of $C=DR$. We describe these for each super-block separately and note that the total matrix is the direct sum of those corresponding to the super-blocks.

(i) Let $D=\operatorname{diag}\{D_1,\cdots,D_p\}$ and define

$$\tilde{D}_1=\operatorname{diag}\{I_s, D_2, D_3D_2, \cdots, D_pD_{p-1}\cdots D_2\}.$$

Then $\tilde{D}_1^{-1}(DR)\tilde{D}_1 = D^{(1)}R$, with $D^{(1)} = \operatorname{diag}\{W, I_s, \cdots, I_s\}$ and where $W = D_1 D_p D_{p-1} \cdots D_2$.

(ii) Define $\tilde{D}_2 = \operatorname{diag}\{I_s, W^{-1/p}, W^{-2/p}, \cdots, W^{-(p-1)/p}\}$, which exists since W is non-singular. Then $\tilde{D}_2^{-1}(D^{(1)}R)\tilde{D}_2 = D^{(2)}R$, where $D^{(2)} = \operatorname{diag}\{W^{1/p}, \cdots, W^{1/p}\}$.

(iii) Define $D_3 = \operatorname{diag}\{T_s, \cdots, T_s\}$, where T_s is selected so that $T_s^{-1}W^{1/p}T_s = \exp(2\pi i J_s)$ and J_s is in any of its Jordan canonical forms. Then $\tilde{D}_3^{-1}(D^{(2)}R)\tilde{D}_3 = D^{(3)}R$, where $D^{(3)} = \operatorname{diag}\{\exp(2\pi i J_s), \cdots, \exp(2\pi i J_s)\}$.

It is easy to see that the eigenvalues of J_s can be arbitrarily adjusted modulo one; however, we can do more, namely, the objective of the final transformation is to adjust them arbitrarily modulo $1/p$.

(iv) Let $J_s = \operatorname{diag}\{J^{(1)}, \cdots, J^{(m)}\}$, where each $J^{(k)}$ is a single Jordan block, and let

$$K_s = \operatorname{diag}\{k_1 I_{t_1}, \cdots, k_m I_{t_m}\},$$

where the k_l are integers and the dimension of I_{t_l} is equal to the dimension of $J^{(l)}$, $1 \le l \le m$. Then define

$$\tilde{D}_4 = \operatorname{diag}\{e^{(2\pi i/p)K_s}, \cdots, e^{(2\pi i(p-1)/p)K_s}, e^{2\pi i K_s}\}$$

and it follows that

$$\tilde{D}_4^{-1}(D^{(3)}R)\tilde{D}_4 = D^{(4)}R,$$

where

$$D^{(4)} = \operatorname{diag}\{e^{2\pi i(J_s - (1/p)K_s)}, \cdots, e^{2\pi i(J_s - (1/p)K_s)}\}.$$

Hence the eigenvalues of J_s can be arbitrarily chosen modulo $1/p$.

So we may now assume that in each super-block, the D in (2.10) has been normalized to equal

$$D = \operatorname{diag}\{e^{2\pi i J_s}, \cdots, e^{2\pi i J_s}\} = \exp(2\pi i J'), \tag{2.11}$$

with $J' = \operatorname{diag}\{J_s, \cdots, J_s\}$.

e. *Normalization of a formal monodromy matrix.*

If L is defined to be a solution of the equation

$$\exp(2\pi i L) = C \tag{2.12}$$

where C is the formal circuit matrix for $\Psi(z)$, then L is called a *formal monodromy matrix* for $\Psi(z)$. This means that $\Psi(z)z^{-L}$ is a "single-valued" formal logarithmic matrix, i.e., it is a formal meromorphic matrix (and its inverse, too). In order to complete the normalization of $H(z)$ and thereby the proof of Theorem I, we now see how L may be selected once $C=DR$ in (2.12) and D is normalized as in (2.11). Since this D and R commute, an especially simple choice for L can be made by putting R into canonical form as follows: We now discuss one super-block.

Let $U=[\varepsilon^{(j-1)(k-1)}I_s]$, $1\leq j, k\leq p$, with $\varepsilon=\exp(2\pi i/p)$. Then

$$U\bar{U}=pI=\bar{U}U, \qquad (\bar{U} \text{ denotes the conjugate}) \tag{2.13}$$

hence $U^{-1}=(1/p)\bar{U}$, and

$$UR=[\varepsilon^{(j-1)k}I_s]=\operatorname{diag}\{I_s, \varepsilon I_s, \cdots, \varepsilon^{p-1}I_s\}U. \tag{2.14}$$

Defining now

$$U'=\operatorname{diag}\left\{0\cdot I_s, \frac{1}{p}I_s, \cdots, \frac{p-1}{p}I_s\right\},$$

we see from (2.14) that

$$R=U^{-1}\exp(2\pi iU')U.$$

Since the blocks in U are scalar multiples of I_s, then D commutes with U and we therefore have in each super-block

$$DR=U^{-1}\exp(2\pi iJ)U, \qquad \text{where } J=J'+U'. \tag{2.15}$$

Hence for L in each super-block we may choose

$$L=U^{-1}JU \tag{2.16}$$

and, as mentioned before, we build the corresponding total matrices as the direct sums of those for each super-block and use the same letters to denote them. Hence (2.15) and (2.16) hold for the total matrices as well. To complete the proof of Theorem I, we remark that (1.2) and (1.3) follow by taking $F(z)=\Psi(z)z^{-L}U^{-1}$ with L defined by (2.16). By the above discussion, $F^{\pm1}(z)$ are formal meromorphic series and the properties attributed to the matrices in the statement of the theorem have all been verified.

§ 3. Formal meromorphic invariants

A *formal meromorphic transformation* at ∞ of $x'=A(z)x$ has the form $x=T(z)y$, where $T(z)$ is a formal meromorphic series and det $T(z)$ is not the zero series (hence $T^{-1}(z)$ is also a formal meromorphic series). Differential equations which are related by such transformations are called *formally meromorphically equivalent* and the corresponding invariants are called *formal meromorphic invariants.*

We will look for formal meromorphic invariants in $H(z)$ as described in

Theorem I. Since $F(z)$ can be removed by the formal meromorphic transformation $x=F(z)y$, formal meromorphic invariants can first only occur in $G(z)$. We proceed by discussing the freedom which remains in $G(z)$.

a. *Further normalization of $G(z)$ and its invariance.*

(i) The order of the super-blocks in $Q(z)$ can be arbitrarily chosen and in each super-block we can choose any of the scalar functions which appear as the first one, i.e., $q(z)$ can be chosen arbitrarily from the set of cyclically permuted functions in $Q(z)$. To remove this freedom, we now make for each $Q(z)$ an a priori fixed choice for the order of its super-blocks and within each super-block, for the function $q(z)$. We note that any such choice is allowed, but once selected, it remains fixed.

(ii) The ordering of the super-blocks in $Q(z)$ prescribes the corresponding ordering for the super-blocks of J. Within each super-block, we have already remarked (see 2(d)) that the eigenvalues of J_s can be changed arbitrarily modulo $1/p$. We now make an a priori fixed, but arbitrary, choice for a system of representatives modulo $1/p$ for the complex numbers (for example, we could require $0 \leq \operatorname{Re} \lambda < 1/p$) and we require that the eigenvalues of J_s come from this system of representatives. We also make a fixed, a priori, choice for the ordering of the Jordan blocks within J_s (by 2(d) (iii)).

If $Q(z)$ and J satisfy these requirements, we say that H is *meromorphically normalized* and we write $Q=Q_A$, $J=J_A$, $G=G_A$, $F=F_A$, and $H=H_A$.

We do not intend to imply by this notation that the above quantities are uniquely determined by $[A]$. It turns out, however, that J_A and Q_A are unique as a consequence of

Theorem II. *Two formal meromorphic differential equations $[A]$ and $[B]$ are formally meromorphically equivalent if and only if $G_A(z)=G_B(z)$, that is, $Q_A(z)=Q_B(z)$ and $J_A=J_B$. Furthermore, in case of equivalence, all formal meromorphic transformations are given by*

$$T(z)=F_A(z)CF_B^{-1}(z), \tag{3.1}$$

where C is any constant non-singular matrix which commutes with $Q(=Q_A=Q_B)$, R, and J, that is, C is diagonally blocked like $Q(z)$, in each super-block C has equal sub-blocks which commute with J_s; this means that C commutes with G (for the matrices C in question).

Remark 1. If we set $A=B$ in the Theorem, we see that G_A is uniquely determined by A and putting $T=I$ in (3.1) we see that $F_A(z)C=F_B(z)$, i.e., $F(z)$ is determined only up to a right hand constant non-singular factor which commutes with $G(z)$.

Remark 2. The pair (Q, J) is a *complete system of formal meromorphic invariants.* This means that all other formal meromorphic invariants are functions of Q and J. Note that they have become invariant due to our normalization and in applications one should keep this in mind. The normalization of the eigenvalues of $J_s \bmod (1/p)$ can be undone, but the corresponding statement of invariance becomes more complicated. The invariants Q and J are free within their special structure and a priori normalization since in each super-block the polynomial $q(z)$ and matrix J_s can be given arbitrarily to yield a formal meromorphic differential equation having these invariants.

b. *A commutation lemma.*

We now state a lemma which is important in the proof of Theorem II.

Lemma 1. *Let M and M' be matrices in Jordan canonical form, whose eigenvalues λ come from the same system of representatives modulo one (that is, $\lambda_1 \equiv \lambda_2 \bmod 1$ implies $\lambda_1 = \lambda_2$). If C is a non-singular constant matrix such that*

$$z^M C z^{-M'} \tag{3.2}$$

is single-valued, then $MC = CM'$ and it follows that

$$z^M C z^{-M'} = C. \tag{3.3}$$

Proof. The single-valuedness of (3.3) is equivalent to

$$e^{2\pi i M} C e^{-2\pi i M'} = C, \tag{3.4}$$

and it follows that $e^{2\pi i M}$ is similar to $e^{2\pi i M'}$. Since a Jordan form of $e^{2\pi i M}$ can be obtained from M by replacing the eigenvalue λ by $e^{2\pi i \lambda}$, there exists a permutation matrix P of the Jordan blocks of M' such that the off-diagonal elements of M and $P^{-1}M'P$ are the same and the diagonal elements are congruent mod 1 and hence equal (in view of our special assumption); so we have

$$M = P^{-1} M' P. \tag{3.5}$$

Hence from (3.4) and (3.5) we obtain

$$e^{2\pi i M} C' = C' e^{2\pi i M} \tag{3.6}$$

with $C' = CP$, and we now show that $MC' = C'M$. We may assume that $M = \operatorname{diag}\{M_1, \cdots, M_l\}$, where M_j consists of all Jordan blocks corresponding to the jth eigenvalue of M. It follows from (3.6) (see [5, vol. I; pp. 215-219]) that C' is blocked diagonally according to the distinct eigenvalues of $e^{2\pi i M}$, and hence those of M, therefore we may assume that M in (3.6) has only one eigenvalue. Since a scalar multiple of the identity commutes with C', we may assume that M in (3.6) is nilpotent. But

(3.6) then implies that

$$z^{M}C'z^{-M}$$

is single-valued and using

$$z^{\pm M}=I\pm M\log z+\left(\frac{M\log z}{2}\right)^{2}\pm\cdots\qquad\text{(finitely many terms),}$$

we obtain

$$z^{M}C'z^{-M}=C'+(MC'-C'M)\log z+\cdots\qquad\text{(finitely many terms).}$$

Hence $MC'=C'M$, $z^{M}C'z^{-M}=C'$, and using (3.5) the conclusion of the Lemma follows.

Remark. Lemma 1 is true even when M and M' are not in Jordan canonical form.

c. *Proof of Theorem II.*

If $[A]$ is formally meromorphically equivalent to $[B]$, there exists a formal meromorphic transformation $T(z)$ and a non-singular constant matrix C such that

$$T(z)H_{B}(z)=H_{A}(z)C, \tag{3.7}$$

which may be rewritten as

$$U_{A}^{-1}z^{-J_{A}}F_{A}^{-1}(z)T(z)F_{B}(z)z^{J_{B}}U_{B}=\exp\,[Q_{A}(z)]C\exp\,[-Q_{B}(z)]. \tag{3.8}$$

In the same way as in the proof of Theorem I, and in view of our a priori ordering of Q we obtain $Q_{A}(z)=Q_{B}(z)$ and C must be diagonally blocked according to the blocks of $Q=Q_{A}=Q_{B}$. Then the right hand side of (3.8) is C, $U_{A}=U_{B}=U$, and (3.8) may be written as

$$F_{A}^{-1}(z)T(z)F_{B}(z)=z^{J_{A}}\hat{C}z^{-J_{B}}, \tag{3.9}$$

where $\hat{C}=UCU^{-1}$. The right hand side is the direct sum of super-blocks, which we now consider separately. Recall in each super-block

$$J_{A}=J_{A}'+U'$$

where

$$J_{A}'=\operatorname{diag}\{J_{s}(A),\cdots,J_{s}(A)\}$$

and

$$U'=\operatorname{diag}\left\{0_{s},\frac{1}{p}I_{s},\cdots,\frac{p-1}{p}I_{s}\right\}.$$

Since the eigenvalues of $J_s(A)$ come from a universal system of representatives mod $1/p$, the eigenvalues of J_A come from a corresponding system of representatives mod 1, and the same holds for J_B. Because the left hand side of (3.9) is a formal meromorphic series, the right hand side is single-valued, and from Lemma 1 we conclude that

$$J_A\hat{C}=\hat{C}J_B \tag{3.10}$$

in each super-block.

Since $J_s(A)+((j-1)/p)I_s$ and $J_s(B)+((k-1)/p)I_s$ have, for $j\neq k$, no common eigenvalues, then $\hat{C}$ must be diagonally blocked (into s-blocks). Letting $\hat{C}=\operatorname{diag}\{\hat{C}_1,\cdots,\hat{C}_p\}$ and $C=\operatorname{diag}\{C_1,\cdots,C_p\}$ in $\hat{C}U=UC$, we see that $\hat{C}=C$ and $C_1=\cdots=C_p$. Also, because $C=\hat{C}$ commutes with U', then from (3.10) we have in each block of the super-block

$$J_s(A)C_j=C_jJ_s(B), \tag{3.11}$$

which implies that $J_s(A)$ is similar to $J_s(B)$. But from our normalization we conclude that $J_s(A)=J_s(B)(=J_s)$, hence $J_A=J_B(=J)$ and C commutes with J. Finally, from (3.9) we obtain

$$F_A^{-1}(z)T(z)F_B(z)=C \tag{3.12}$$

in each super-block and consequently for the total matrices, where C has the structure described in Theorem II.

Conversely, if $G_A(z)=G_B(z)(=G(z))$ and C is a constant nonsingular matrix which commutes with G, then

$$T(z)=F_A(z)CF_B^{-1}(z)$$

is a formal meromorphic transformation such that $H_AC=TH_B$, hence $x=Ty$ transforms $[A]$ into $[B]$.

d. *A formal meromorphic canonical form.*

Let $G(z)=G_m(z)=z^JU\exp[Q(z)]$ and recall from Section 2 that $L=U^{-1}JU$ is a formal monodromy matrix for $G(z)$, hence there exists a single-valued matrix of elementary functions, $E_f(z)$, such that $G(z)=E_f(z)z^JU$. To calculate $E_f(z)$ we remark that within a super-block we have

$$Q(z)=\hat{q}(z^{1/p}e^{2\pi iU'}),$$

where $\hat{q}$ is a polynomial without constant term such that $\hat{q}(z^{1/p})=q(z)$. It follows that within a super-block

$$\begin{aligned} E_f(z) &= z^J U \exp\,[Q(z)]U^{-1}z^{-J} \\ &= \exp\,[\hat{q}(z^{1/p}z^J U e^{2\pi i U'} U^{-1} z^{-J})] \\ &= \exp\,[\hat{q}(z^{1/p} z^{U'} R^{-1} z^{-U'})] \end{aligned}$$

(note that $z^{J'}$ and R commute). If we define

$$W = \begin{bmatrix} O_s & I_s & O_s & \cdots & O_s \\ & & \ddots & & \vdots \\ & & & \ddots & I_s \\ zI_s & \cdots & \cdots & \cdots & O_s \end{bmatrix}$$

then $z^{1/p} z^{U'} R^{-1} z^{-U'} = W$, hence we have

$$E_f(z) = \exp\,[\hat{q}(W)] \tag{3.13}$$

for every super-block. *So each super-block of $E_f(z)$ is the exponential of a scalar polynomial in the matrix variable W.*

If we calculate $G'(z)G^{-1}(z)$, we have for each super-block $zG'(z)G^{-1}(z) = J + z^J U \hat{q}'(z^{1/p} e^{2\pi i U'})(1/p) z^{1/p} e^{2\pi i U'} U^{-1} z^{-J}$, hence

$$zG'(z)G^{-1}(z) = J + \hat{q}'(W)W/p. \tag{3.14}$$

For every integer $k = mp + l$, $0 < l \leq p$, we have

$$W^k = z^m W^l,$$

and we obtain W^l by calculating R^{-l} and multiplying the non-zero blocks not above the diagonal by z; for instance

$$W^2 = \begin{bmatrix} O_s & O_s & I_s & O_s & \cdots & O_s \\ & \ddots & & \ddots & & \vdots \\ & & \ddots & & \ddots & O_s \\ & & & \ddots & & I_s \\ zI_s & & & & \ddots & O_s \\ O_s & zI_s & & & & O_s \end{bmatrix}, \qquad W^p = zI.$$

We see that for $k \geq 0$, W^k is a matrix polynomial in z of degree $m+1$. Since the order of the solutions of $[A]$ is bounded by r, the Poincaré rank of $[A]$, we see that the highest power of z occurring in $Q(z)$ is less than or equal to r. Therefore, $\deg \hat{q} \leq pr$, and it follows that $zG'(z)G^{-1}(z)$ is a matrix polynomial $P(z)$ in z of degree at most r. Hence A is formally meromorphically equivalent to $[P(z)z^{-1}]$ *which has Poincaré rank not exceeding the rank of* $[A]$. This means that a Birkhoff reduction (see [16]) can be made using formal meromorphic transformations. Whether it can be done with a convergent meromorphic transformation still remains an open question.

e. *Formal root-meromorphic invariants.*

We conclude this section with a statement about the invariants corresponding to formal root-meromorphic transformations, that is, $T(z)$ which are formal meromorphic transformations in $z^{1/g}$, where g is a positive integer. Note that $H(z)$ from Theorem I can be written as

$$H(z) = \Phi(z) z^{J'} \exp [Q(z)], \qquad \text{where } \Phi(z) = F(z) z^{U'} U$$

is a formal root-meromorphic transformation and in the columns of Φ corresponding to each super-block of $Q(z)$ we can take $g=p$. It follows that the eigenvalues of J' can be changed by arbitrary rational numbers if we allow arbitrary roots in $F(z)$. We now choose, a priori (with the help of the Axiom of Choice) a fixed system of representatives modulo the rational numbers and change the eigenvalue of J_s to come from this system of representatives. We also make an a priori choice for a fixed ordering of the Jordan blocks in this new J_s and denote it by $J_s^{(r)}$ and $J^{(r)} = \operatorname{diag}\{J_s^{(r)}, \cdots, J_s^{(r)}\}$. We use the same notation for the total matrix. Hence a formal fundamental solution matrix can be represented as

$$H_A(z) = \Phi_A(z) G_A(z), \qquad \text{where } G_A(z) = z^{J_A^{(r)}} \exp (Q_A(z)).$$

Then we have the following analogue of Theorem II in the formal root-meromorphic case:

Theorem II′. *Two formal meromorphic differential equations* $[A]$ *and* $[B]$ *are formally root-meromorphically equivalent if and only if*

$$Q_A(z) = Q_B(z) \quad \text{and} \quad J_A^{(r)} = J_B^{(r)}.$$

Furthermore, in case of equivalence, all formal root-meromorphic transformations are given by

$$T(z) = \Phi_A(z) C \Phi_B^{-1}(z),$$

where C *is a constant, non-singular matrix which commutes with* $Q(=Q_A=Q_B)$ *and* $J^{(r)}(=J_A^{(r)}=J_B^{(r)})$, *that is,* C *is blocked diagonally like* $Q(z)$ *and in each super-block it commutes with the corresponding* $J_s^{(r)}$.

The proof of Theorem II′ is similar to the proof of Theorem II but simpler, and we omit it here. We remark that for the purpose of comparing formal meromorphic and root-meromorphic invariants, it is natural after having selected $J_s^{(r)}$ to select J_s so that the block structure is compatible and $J_s = J_s^{(r)} + D_s$, where D_s is a diagonal matrix with rational entries. The remarks following the statement of Theorem II have exact analogues in the formal root-meromorphic case, which we also omit here.

§ 4. Formal analytic and Birkhoff invariants

Analytic transformations have the form $x=T(z)y$, where $T(z)=\sum_0^\infty T_k z^{-k}$ converges in a neighborhood of ∞ and $\det T_0 \neq 0$. *Birkhoff transformations* correspond to those analytic $T(z)$ for which $T_0=I$. If the series is just assumed to be a formal series, we call the transformations *formally analytic*, respectively *formally Birkhoff*. The corresponding differential equations are called *analytically* (*formally analytically*) or *Birkhoff* (*formally Birkhoff*) *equivalent* and the associated invariants are called by the same names.

It is convenient to introduce the notation F_m, F_a, F_b to denote, respectively, formal meromorphic, analytic, Birkhoff transformations. We now denote the meromorphically normalized formal solutions (used in Theorem II) by

$$H_m(z)=F_m(z)G_m(z)$$

where $G_m(z)=z^J U \exp[Q(z)]$ is the formal meromorphic invariant and F_m is determined from the differential equation only up to right hand constant non-singular factors C, where C commutes with G_m. Such matrices C are called *G_m-admissible*. In this section we see what can be adjoined to G_m to yield a complete system of formal analytic and formal Birkhoff invariants. The remaining invariants are in F_m, and the following lemma is helpful in recognizing them.

a. *The Hermite Normal Form.*

Lemma 2. *Each formal meromorphic transformation $F_m(z)$ can be uniquely factored as*

$$F_a(z)P(z)z^K, \tag{4.1}$$

where $F_a(z)$ is a formal analytic transformation, $P(z)$ is a lower triangular matrix polynomial (in z) such that $\operatorname{diag} P(z)=I$ *and $P(0)=I$, and K is a diagonal matrix with integer entries.*

Proof. According to the Gaussian algorithm (see [11; pp. 32–33]), $F_m(z)$ can be brought into the form $P(z)z^K$ with P and K as above, by means of a finite sequence of elementary row operations (each of which is a formal analytic transformation). To see the uniqueness, suppose

$$F_a(z)P(z)z^K=\tilde{F}_a(z)\tilde{P}(z)z^{\tilde{K}},$$

i.e.,

$$P(z)z^{K-\tilde{K}}\tilde{P}^{-1}(z)=F_a^{-1}(z)\tilde{F}_a(z). \tag{4.2}$$

Since the left hand side of (4.2) is lower triangular with diagonal $z^{K-\tilde{K}}$ and the right hand side is formally analytic we have $K=\tilde{K}$, then $P(z)\tilde{P}^{-1}(z)$ is formally analytic iff $P(z)\tilde{P}^{-1}(z)=I$, and finally $F_a(z)=\tilde{F}_a(z)$.

b. *Analytic normalization.*

According to Lemma 2, F_m can be uniquely factored into an analytic transformation F_a times $P(z)z^K$. But as we remarked, the differential equation determines F_m only up to constant right hand factors which are G_m-admissible. Such a $F_m(z)C$ can again be factored uniquely as $\tilde{F}_a$ times $\tilde{P}(z)z^{\tilde{K}}$. We say that $P(z)z^K$ *is equivalent to* $\tilde{P}(z)z^{\tilde{K}}$ *relative to* G_m if there exists a G_m-admissible matrix C and a formal analytic transformation $T(z)$ such that

$$P(z)z^K C = T(z)\tilde{P}(z)z^{\tilde{K}}. \tag{4.3}$$

It follows from (4.3) that such a $T(z)$ must be, in fact, a polynomial in z^{-1} with nonsingular constant term.

Since the G_m-admissible matrices C as well as the analytic transformations form a group, then (4.3) defines an equivalence relation. From each equivalence class we select a representative. Given $F_m(z)=F_a(z)P(z)z^K$, a member of the equivalence class determined by $P(z)z^K$ can be reached by factoring $F_m(z)C=\tilde{F}_a(z)\tilde{P}(z)z^{\tilde{K}}$ for some G_m-admissible C. Therefore we can find such a C so that the *selected* representative appears in the factorization. Hence there exists a formal fundamental solution matrix

$$H_a(z)=H_m(z)C=F_a(z)G_a(z), \tag{4.4}$$

where $F_a(z)$ is a formal analytic transformation, $G_a(z)=P(z)z^K G_m(z)$, and $P(z)z^K$ is the selected representative from its equivalence class. Such a formal solution is said to be *analytically normalized.*

c. *Formal analytic invariants.*

The formal analytic invariants of a differential equation can now be determined from an analytically normalized formal solution.

Theorem III. *Two formal meromorphic differential equations* $[A]$ *and* $[\tilde{A}]$ *having analytically normalized formal fundamental solution matrices* F_aG_a, $\tilde{F}_a\tilde{G}_a$, *resp., are formally analytically equivalent iff*

$$G_a=\tilde{G}_a,\ \text{i.e.,}\ G_m=\tilde{G}_m,\ P=\tilde{P},\ \text{and}\ K=\tilde{K}. \tag{4.5}$$

In case of equivalence, all formal analytic transformations are given by

$$T(z)=F_a(z)C(z)\tilde{F}_a^{-1}(z), \tag{4.6}$$

where $C(z)$ *is any analytic transformation satisfying*

$$P(z)z^K C = C(z)P(z)z^K \tag{4.7}$$

for some G_m*-admissible matrix* C.

Proof: If $[A]$ is formally analytically equivalent to $[\tilde{A}]$, there exists a formal analytic transformation $T(z)$ and a non-singular constant matrix C such that

$$T\tilde{H}_a = H_a C, \text{ i.e., } T\tilde{F}_a\tilde{P}z^{\tilde{K}}\tilde{G}_m = F_a P z^K G_m C. \tag{4.8}$$

According to Theorem II, $G_m = \tilde{G}_m$ and $G_m C = C G_m$ so (4.8) can be written as

$$Pz^K C = (F_a^{-1}T\tilde{F}_a)\tilde{P}z^{\tilde{K}}, \tag{4.9}$$

which means that Pz^K is equivalent to $\tilde{P}z^{\tilde{K}}$. But because H_a and $\tilde{H}_a$ are analytically normalized, then $Pz^K = \tilde{P}z^{\tilde{K}}$, i.e., $P = \tilde{P}$ and $K = \tilde{K}$. From (4.9) we obtain

$$Pz^K C = C(z)Pz^K, \qquad C(z) = F_a^{-1}T\tilde{F}_a, \tag{4.10}$$

where $C(z)$ is a (formal) analytic transformation, and this proves the necessity of (4.5), (4.6), and (4.7). To prove the sufficiency, assume $G_a = \tilde{G}_a$ and let C be a G_m-admissible matrix such that (4.7) holds with a formal analytic $C(z)$ (actually a polynomial in z^{-1} with non-singular costant term). Then

$$T(z) = F_a(z)C(z)\tilde{F}_a^{-1}(z)$$

is a formal analytic transformation and satisfies $T\tilde{H}_a = F_a(z)C(z)G_a(z) = F_a(z)G_a(z)C = H_a C$, hence $[A]$ and $[\tilde{A}]$ are formally analytically equivalent by means of T. This completes the proof of Theorem III.

d. *Remarks on formal analytic invariants*

(1) If we set $A = \tilde{A}$ in the theorem, we see that G_a is uniquely determined by A and is a complete system of formal analytic invariants. This invariant is free within our specifications and modulo the last equivalence relation on $P(z)z^K$. The matrix G_a determines the *formal analytic type* of the solutions and $[G'_a G_a^{-1}]$ may be called the *formal analytic canonical form* of the differential equation $[A]$.

(2) The G_m-admissible matrices C which satisfy (4.10) for some $C(z)$, a polynomial in z^{-1} with non-zero constant determinant, form a sub-group of the G_m-admissible matrices, which we call the G_a*-admissible matrices.* The formal analytic series $F_a(z)$ in H_a is determined by the differential equation only up to such right hand factors $C(z)$.

e. *Formal Birkhoff normalization.*

If we write

$$F_a(z)=\tilde{F}_0+\tilde{F}_1z^{-1}+\cdots \quad\text{and}\quad C(z)=C_0+C_1z^{-1}+\cdots,$$

then note that according to the above discussion, $\tilde{F}_0$ (in H_a) may be replaced by $\tilde{F}_0C_0$, where $C(z)$ corresponds to a G_a-admissible C (4.7). The set of such matrices C_0 form a group which is completely determined by the invariant G_a. We say that F_0 and $\tilde{F}_0$ are *equivalent relative to* G_a if there exists a C_0 from this group such that $\tilde{F}_0C_0=F_0$. This defines an equivalence relation and from each equivalence class we make an a priori selection of a fixed representative denoted by F_0. Since $\tilde{F}_0C_0$ runs through the complete equivalence class as C runs through the G_a-admissible matrices, there exists a formal fundamental solution matrix

$$\begin{aligned}H_b(z)&=H_a(z)C=F_a(z)C(z)G_a(z)=(\tilde{F}_0C_0+\cdots)G_a\\&=(F_0+\cdots)G_a(z).\end{aligned}$$

Hence $H_b(z)$ may be written as

$$H_b(z)=F_b(z)G_b(z), \qquad \text{where } G_b(z)=F_0G_a(z) \tag{4.11}$$

and $F_b(z)$ is a formal Birkhoff transformation. We say that such a formal fundamental solution matrix (4.11) is *Birkhoff-normalized.*

f. *Formal Birkhoff invariants.*

Theorem IV. *Two formal meromorphic differential equations* $[A]$ *and* $[\tilde{A}]$ *having Birkhoff-normalized formal fundamental solution matrices* F_bG_b, $\tilde{F}_b\tilde{G}_b$ *resp., are formally Birkhoff equivalent iff*

$$G_b=\tilde{G}_b,\ \textit{that is},\ G_a=\tilde{G}_a\ \textit{and}\ F_0=\tilde{F}_0. \tag{4.12}$$

Moreover, in case of equivalence, all formal Birkhoff transformations are given by

$$T(z)=F_b(z)C_b(z)\tilde{F}_b^{-1}(z), \tag{4.13}$$

where $C_b(z)=F_0C(z)F_0^{-1}$ *and* $C(z)$ *corresponds to any* G_a*-admissible matrix* C *with* $C_0=I$.

Proof: If $[A]$ is formally Birkhoff equivalent to $[\tilde{A}]$, there exists a formal Birkhoff transformation T and a non-singular constant matrix C such that

$$T\tilde{H}_b=H_bC,\ \text{i.e.,}\ T\tilde{F}_b\tilde{F}_0\tilde{G}_a=F_bF_0G_aC. \tag{4.14}$$

From Theorem III we obtain $G_a=\tilde{G}_a$ and $G_a(z)C=C(z)G_a(z)$, where C is G_a-admissible. Hence (4.14) becomes

$$T(z)\tilde{F}_b(z)\tilde{F}_0=F_b(z)F_0C(z). \tag{4.15}$$

From the constant term of (4.15) we obtain $\tilde{F}_0=F_0C_0$, hence $\tilde{F}_0$ is equivalent to F_0

relative to $G_a = \tilde{G}_a$. By our selection we therefore obtain $F_0 = \tilde{F}_0$ and $C_0 = I$. Hence C is G_a-admissible, $C_0 = I$, and $T(z) = F_b(z)C_b(z)\tilde{F}_b^{-1}(z)$.

Conversely, if $G_b = \tilde{G}_b$ and $C_b(z)$ is any formal Birkhoff transformation such that $G_b(z)C = C_b(z)G_b(z)$ for some G_a-admissible C, then T given by (4.13) is a formal Birkhoff transformation such that

$$T(z)\tilde{H}(z) = F_b(z)C_b(z)G_b(z) = F_b(z)G_b(z)C = H(z)C,$$

hence $[A]$ and $[\tilde{A}]$ are formally Birkhoff equivalent by means of T.

g. *Remarks on formal Birkhoff invariants.*

(1) If we set $A = \tilde{A}$ in Theorem IV, we see that G_a and F_0, that is, G_b, are uniquely determined by $[A]$ and form a complete system of formal Birkhoff invariants. This invariant G_b is free within our specifications and modulo the last two equivalences. The differential equation $[G_b' G_b^{-1}]$ may be called the formal Birkhoff canonical form of the differential equation $[A]$, and G_b determines the *formal Birkhoff type* of the solutions.

(2) The set of G_a-admissible matrices C such that $C_0 = I$ are called the G_b-admissible matrices. Note that they form a subgroup of the G_a-admissible matrices which is completely determined by the invariant G_b. Also, the matrices $C_b(z)$ where $G_b(z)C = C_b(z)G_b(z)$ and C is G_b-admissible likewise form a group. If this group is not the trivial $\{I\}$ group, the formal Birkhoff series $F_b(z)$ could be further normalized, but we choose not to do this here.

§ 5. The case r = 0, a singularity of the first kind

When $r = 0$, the meromorphic differential equation $[A]$ is said to have a singularity of the *first kind* at ∞. This implies that $Q(z) \equiv 0$, hence $p = 1$ and $s = n$; furthermore, the formal solutions converge automatically, so that there is no difference between formal and proper invariants. According to our theory, the Birkhoff invariant is given by

$$G_b(z) = F_0 P(z) z^K z^J.$$

We only discuss the case of a convergent power series

$$A(z) = A_0 z^{-1} + A_1 z^{-2} + \cdots, \tag{5.1}$$

where A_0 is in (lower triangular) Jordan form and normalized as follows: We assume that the single Jordan blocks with equal eigenvalues are grouped together as *blocks*, that the blocks with congruent (mod 1) eigenvalues are grouped together as *big blocks*, and that the blocks inside the big blocks are arranged so that the real parts of the eigenvalues occur in decreasing order (these eigenvalues differ by non-zero integers).

Now we select a priori a fixed system of representatives for the complex numbers mod 1 and form the matrix M_0 by changing the eigenvalues of A_0 mod 1 so that they come from this system of representatives. Clearly

$$K_0 = A_0 - M_0 \tag{5.2}$$

is uniquely determined by A_0 and is a diagonal matrix with integral entries which are constant along blocks and strictly decreasing from one block to the next block inside a big block. According to Gantmacher [5; vol. II, pp. 148–164], the differential equation $[A]$ has a solution of the form

$$X(z) = F_b(z) z^{K_0} z^M, \tag{5.3}$$

where $F_b(z)$ is a (convergent) Birkhoff transformation and M is diagonally blocked according to the big blocks of A_0 and lower triangularly blocked inside the big blocks according to the blocks of A_0. Such a block structure is called *admissible with respect to* A_0, and we denote the matrix consisting of the diagonal blocks (corresponding to the blocks of A_0) by $\operatorname{diag}_{A_0} M$. Comparing the coefficients of z^{-1} in the logarithmic derivative of

$$F_b^{-1} X = z^{K_0} z^M,$$

we see that A_0 is the constant coefficient in

$$K_0 + z^{K_0} M z^{-K_0},$$

hence

$$M_0 = \operatorname{diag}_{A_0} M. \tag{5.4}$$

Next we investigate the freedom in the choice of M. Suppose that C is a constant matrix with A_0-admissible block structure satisfying

$$\operatorname{diag}_{A_0} C = I, \tag{5.5}$$

hence non-singular. Then

$$\tilde{M} = C^{-1} M C \tag{5.6}$$

is also A_0-admissible. Since

$$C(z) = z^{K_0} C z^{-K_0} \tag{5.7}$$

is a Birkhoff transformation, we see that

$$X(z)C=F_b(z)C(z)z^{K_0}z^{\tilde{M}}=\tilde{F}_b(z)z^{K_0}z^{\tilde{M}}$$

is another solution of the type considered. Observe that these matrices C form a group and that the corresponding matrices $\tilde{M}$ vary over a similarity class denoted by $\langle M\rangle_{A_0}$. It is convenient to select (a priori) a fixed representative in this similarity class which we denote by M from now on.

We now give a different, but particularly natural system of Birkhoff invariants using the selected M.

Proposition. *Two differential equations* $[A]$ *and* $[\tilde{A}]$ *of the normalized form* (5.1) *are Birkhoff equivalent iff*

$$A_0=\tilde{A}_0 \quad \textit{and} \quad M=\tilde{M}.$$

In case of equivalence, all Birkhoff transformations are given by

$$T(z)=F_b(z)C(z)\tilde{F}_b^{-1}(z),$$

where C *varies over the* A_0*-admissible matrices which satisfy* (5.5) *and commute with* $M=\tilde{M}$ *and* $C(z)$ *is given by* (5.7).

Proof: Let $X(z)=F_b(z)z^{K_0}z^M$, $\tilde{X}(z)=\tilde{F}_b(z)z^{\tilde{K}_0}z^{\tilde{M}}$ be our selected solutions and let T be a Birkhoff transformation from $[A]$ to $[\tilde{A}]$. Then there exists a constant non-singular matrix C satisfying $T\tilde{X}=XC$. Trivially we see that $A_0=\tilde{A}_0$, $K_0=\tilde{K}_0$, and becuase $z^MCz^{-\tilde{M}}$ must be single-valued (see Lemma 1 and Remark), we obtain $MC=C\tilde{M}$ since the eigenvalues of M and $\tilde{M}$ come from the same system of representatives in view of (5.4). This implies, in particular, that C is diagonally blocked according to the big blocks of $A_0=\tilde{A}_0$. Next, we observe that

$$C(z)=z^{K_0}Cz^{-K_0}=F_b^{-1}(z)T(z)\tilde{F}_b(z)$$

is a Birkhoff transformation, hence C must be lower triangularly blocked within the big blocks and $\operatorname{diag}_{A_0} C=I$. Therefore $\langle M\rangle_{A_0}=\langle\tilde{M}\rangle_{A_0}$ and by our selection it follows that $M=\tilde{M}$; but then C commutes with M.

Conversely, if $A_0=\tilde{A}_0$, $M=\tilde{M}$, and C, T are arbitrarily selected as specified in the Proposition, then T is a Birkhoff transformation satisfying

$$T\tilde{X}=F_b(z)C(z)z^{K_0}z^M=F_b(z)z^{K_0}z^MC=XC.$$

Hence the differential equations are equivalent by means of T.

Remark 1. The proposition shows that A_0, M or A_0, $\langle M\rangle_{A_0}$ is a complete system of Birkhoff invariants. An equivalent system is K_0, M and the corresponding canonical form for the differential equation is given by the logarithmic derivative of $z^{K_0}z^M$. Within the above-mentioned specifications, the invariants K_0, M are free. It is natural to select M in such a way that a maximal number of entries vanishes. If we partition the big blocks according to the *single* Jordan blocks of A_0, then we can arrange that the rectangular blocks below the diagonal blocks have non-zero entries in at most the last column or first row, whichever is shorter. But this condition may not determine the representative uniquely yet. The problem of finding a natural representative remains open.

Remark 2. To consider analytic equivalence, it is convenient to further normalize A_0 by putting the single Jordan blocks of a block of A_0 into a fixed (a priori) order. At the same time we permit matrices C, where condition (5.5) is replaced by the requirement that $\operatorname{diag}_{A_0} C$ should be non-singular and commute with A_0. The similarity class changes accordingly as well as the selection of the representative M. It is convenient to consider the more general type of solution

$$X(z)=F_a(z)z^{K_0}z^M, \qquad F_a(z)=F_0+F_1z^{-1}+\cdots,$$

where F_0 is non-singular and commutes with A_0, hence it is blocked diagonally according to the blocks of A_0. With these changes the discussion and the Proposition remain true for analytic equivalence. The proof changes only slightly.

Remark 3. The invariants K_0, M can, of course, be used to calculate G_b. We may assume that $[A]$ has a solution of the form $X=z^{K_0}z^M$, where K_0 and M are as in the Proposition. Clearly, there is a constant non-singular matrix C such that $XC=F_bG_b$. Remember that the eigenvalues of J are selected mod 1; here we can use the same system of representatives as for A_0. Also, the arrangement of the single Jordan blocks in J is fixed a priori. Moreover, we can assume that the blocks with equal eigenvalues are grouped together as *big blocks* and that the order of the big blocks is determined by the (representatives of the) eigenvalues; the same principle of arranging the big blocks should be used for A_0 and M_0. Since z^MCz^{-J} is single-valued, it follows that $J=C^{-1}MC$. Therefore, *J is a Jordan form of M.* Remember that the big blocks of M have one eigenvalue each, which are incongruent. It follows that the *big blocks of J are Jordan forms of the big blocks of M separately and*, in view of our arrangements, C is *diagonally blocked according to the big blocks.* Since the arrangement of the single Jordan blocks of J inside the big blocks is fixed, J is uniquely determined by M according to the remarks above.

Now let C be any constant non-singular matrix satisfying

$$J=C^{-1}MC. \tag{5.8}$$

Then $z^{K_0}z^MC=z^{K_0}Cz^J$ and using the unique factorization (see Section 4, Lemma 2) we have

$$z^{K_0}C=F_a(z)P(z)z^K=F_b(z)F_0P(z)z^K \tag{5.9}$$

from which we obtain $XC=F_b(z)G_b(z)$ provided we make these choices for the representatives $P(z)z^K$ and F_0. Other choices simply correspond to other choices of C in (5.8). Checking the proof of Lemma 2 for the case of $z^{K_0}C$, we find that every big block is treated separately and that inside a big block (using the same notation)

$$z^{K_0}C=F_a(z)z^K C', \tag{5.10}$$

where *K is a permutation of* K_0 and C' is a lower triangular matrix with ones on the diagonal and zeros in positions (i, j) below the diagonal where $k_i \leq k_j$ so that

$$z^K C' z^{-K}=P(z) \tag{5.11}$$

is a polynomial in z with $P(0)=I$. Since it is not clear how the triangular structure of M reflects itself in C and C', the structure of $G_b(z)$ is not entirely determined; nevertheless, the procedure above can be used to calculate $G_b(z)$ and gives some information about K and $P(z)$; in particular, *G_b is diagonally blocked according to the big blocks of J.*

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