Funkcialaj Ekvacioj, **22** (1979), 121–126

Sets of Fixed Points of Nonlinear Mappings in Function Spaces

By

Stanisław SZUFLA

(A. Mickiewicz University, Poland)

1. Let K be a bounded convex subset of a normed space, and let E be a Hausdorff topological vector space. Denote by $C=C(K, E)$ the space of all bounded continuous functions $K\to E$ with the topology of uniform convergence. In this paper we prove the connectedness of the set $\{x \in C: x=F(x)\}$ for a certain class of nonlinear mappings $F: C\to C$. The method of proof is based on a connectedness argument given by Hukuhara [2] for integral equations in the finite dimensional case.

Assume that $t_0 \in K$, $x_0 \in E$ and that Ω is the family of all open balanced neighbourhoods of 0 in E. Denote by Φ the set of all continuous functions $F: C\to C$ such that

1° $F(C)$ is an equiuniformly continuous subset of C, i.e., for any $U \in \Omega$ there exists $\varepsilon>0$ such that

$F(x)(t)-F(x)(s) \in U$ for each $x \in C$, $t, s \in K$ such that $\|t-s\|\leqslant\varepsilon$;

2° $F(x)(t_0)=x_0$ for every $x \in C$;

3° for every $\varepsilon>0$

$$x|K_\varepsilon=y|K_\varepsilon \Rightarrow F(x)|K_\varepsilon=F(y)|K_\varepsilon \qquad (x, y \in C),$$

where $K_\varepsilon=B(t_0, \varepsilon)\cap K$ and $B(t_0, \varepsilon)$ is the closed ball of center t_0 and radius ε.

Let I denote the identity mapping on C. First we shall prove a modification of some result of Vidossich given in [8].

Lemma 1. *For any* $F \in \Phi$ *there exists a sequence* (F_n), $n=1, 2, \cdots$, *such that* $I-F_n$ *is a homeomorphism* $C\to C$, *and* $\lim_{n\to\infty} F_n(x)=F(x)$ *uniformly in* $x \in C$.

Proof. For any positive integer n let

$$r_n(t)=\begin{cases} t_0 & \text{if } t \in K_{1/n} \\ (1-1/n\,\|t-t_0\|)t+t_0/n\,\|t-t_0\| & \text{if } t \in K\setminus K_{1/n}. \end{cases}$$

It can be easily proved that r_n is a continuous mapping of K into itself. By this, the equality

$$F_n(x)(t)=F(x)(r_n(t)) \qquad (x \in C,\ t \in K) \tag{1}$$

defines a continuous mapping $F_n(x)\colon K\to E$, and therefore F_n maps $C\to C$. Since $\|r_n(t)-t\|\leqslant 1/n$, from (1) and 1° we deduce that

$$\lim_{n\to\infty} F_n(x)=F(x) \qquad \text{uniformly in } x\in C.$$

Now we shall prove that $I—F_n$ is a homeomorphism $C\to C$. For any continuous function $x\colon K_\varepsilon\to E$ denote by $\tilde{x}$ the function defined by the formula

$$\tilde{x}(t)=x(p(t)) \qquad (t\in K),$$

where

$$p(t)=\begin{cases} t & \text{if } t\in K_\varepsilon \\ t_0+\varepsilon(t-t_0)/\|t-t_0\| & \text{if } t\in K\setminus K_\varepsilon. \end{cases}$$

Then $\tilde{x}$ is a continuous extension of x and $\tilde{x}(K)=x(K_\varepsilon)$. Since K is bounded, there is a positive integer m such that $K\subset B(t_0, m/n)$. Assume that $y\in C$. Putting

$$x_1(t)=y(t)+x_0 \qquad \text{for } t\in K_{1/n}$$

and

$$x_i(t)=y(t)+F_n(\tilde{x}_{i-1})(t) \qquad \text{for } t\in K_{i/n}\ (i=2,\cdots,m),$$

we see that $x_m\in C$ and $x_m-F_n(x_m)=y$. Conversely, if $x\in C$ and $x-F_n(x)=y$, then $x|K_{i/n}=x_i$ for $i=1,\cdots,m$, so that $x=x_m$. This proves that $I—F_n$ is a bijection $C\to C$. The continuity of F_n is a direct consequence of the continuity of F. Suppose that $(x_\alpha;\alpha\in A)$ is a net in C and $\lim_{\alpha\in A}(x_\alpha-F_n(x_\alpha))=x-F_n(x)$. From the definition of F_n and 2° it follows that

$$(2) \qquad \lim_{\alpha\in A} x_\alpha(t)=x(t) \qquad \text{uniformly in } t\in K_{1/n}.$$

Put $z_\alpha=(x_\alpha-x)|K_{1/n}$. From the definition of $\tilde{z}_\alpha$ and (2) it is clear that $\lim_{\alpha\in A}\tilde{z}_\alpha(t)=0$ uniformly in $t\in K$, so that $\lim_{\alpha\in A}(x+\tilde{z}_\alpha)=x$ in C. As F is continuous, this shows that $\lim_{\alpha\in A}F(x+\tilde{z}_\alpha)=F(x)$. Since $(x+\tilde{z}_\alpha)|K_{1/n}=x_\alpha|K_{1/n}$, by 3° we have $F(x+\tilde{z}_\alpha)|K_{1/n}=F(x_\alpha)|K_{1/n}$, and hence $\lim_{\alpha\in A}F(x_\alpha)(t)=F(x)(t)$ uniformly in $t\in K_{1/n}$. As $x_\alpha(t)=(x_\alpha(t)-F_n(x_\alpha)(t))+F(x_\alpha)(r_n(t))$ and $r_n(t)\in K_{1/n}$ for $t\in K_{2/n}$, this implies that

$$\lim_{\alpha\in A} x_\alpha(t)=x(t) \qquad \text{uniformly in } t\in K_{2/n}.$$

Since $r_n(t)\in K_{(i-1)/n}$ for $t\in K_{i/n}$, by repeating this argument we find $\lim_{\alpha\in A}x_\alpha(t)=x(t)$ uniformly on $K_{i/n}$ $(i=3,\cdots,m)$. Thus $\lim_{\alpha\in A}x_\alpha=x$ in C, which completes the proof.

Let $\tilde{\Omega}$ denote the family of all open balanced neighbourhoods of 0 in C. It is interesting that for any $F \in \Phi$ the well-known Hukuhara theorem [2] is also true.

Theorem 1. *Let $F \in \Phi$ and $U \in \tilde{\Omega}$. Then the set*

$$S_U = \{x \in C : x - F(x) \in U\}$$

is non-empty and connected.

Proof. Since $I-F_n$ in Lemma 1 is a homeomorphism and approximates, uniformly, $I-F$, it follows that the set S_U is non-empty. Assume that $x_1, x_2 \in S_U$. As U is balanced, the set $T=\{r(x_i - F(x_i)) : |r| \leqslant 1, i=1, 2\}$ is contained in U. Since T is compact, we can choose a V in $\tilde{\Omega}$ such that $T+V+V \subset U$. By Lemma 1 there exists a positive integer n such that $F(x)-F_n(x) \in V$ for every $x \in C$, where F_n is the mapping defined by (1). Put

$$a(r) = \begin{cases} -r(x_1 - F_n(x_1)) & \text{for } -1 \leqslant r \leqslant 0 \\ \ \ r(x_2 - F_n(x_2)) & \text{for } \ \ 0 \leqslant r \leqslant 1. \end{cases}$$

It is clear that $a(r) \in T+V$ for $r \in [-1, 1]$. Let $u_r = (I-F_n)^{-1}(a(r))$. Because $I-F_n$ is a homeomorphism $C \rightarrow C$ and $a(r)$ depends continuously on r, we see that $r \rightarrow u_r$ is a continuous mapping of $[-1, 1]$ into C. Moreover

$$u_r - F(u_r) = F_n(u_r) - F(u_r) + a(r) \in T+V+V \subset U \qquad \text{for } -1 \leqslant r \leqslant 1,$$
$$u_{-1} = (I-F_n)^{-1}(a(-1)) = (I-F_n)^{-1}(x_1 - F_n(x_1)) = x_1$$

and

$$u_1 = (I-F_n)^{-1}(a(1)) = (I-F_n)^{-1}(x_2 - F_n(x_2)) = x_2.$$

From this we conclude that for any $x_1, x_2 \in S_U$ there exists a continuous curve in S_U connecting x_1 and x_2, which proves that S_U is arcwise connected.

The following result is a generalized Kneser theorem for the equation $x=F(x)$ with $F \in \Phi$.

Theorem 2. *Assume that $F \in \Phi$ and the mapping $I-F$ is 0-closed, i.e.,*

(3) $0 \in \overline{(I-F)(V)} \Rightarrow 0 \in (I-F)(V)$ *for every closed subset V of C.*

Then the set $S=\{x \in C : x=F(x)\}$ is nonempty and connected.

Proof. For any $U \in \tilde{\Omega}$ there exists an element z_U in S_U, namely z_U which satisfies $z_U - F(z_U) \in U$. Therefore $0 \in \overline{(I-F)(C)}$ and in view of (3) there is a z such that $z-F(z)=0$. Thus $S \neq \emptyset$. Suppose that S is not connected. Then there are nonempty open sets G_1, G_2 such that $S \subset G_1 \cup G_2$, $S \cap G_1 \neq \emptyset$, $S \cap G_2 \neq \emptyset$ and $G_1 \cap G_2 =$

$\varnothing$; let $G=G_1\cup G_2$. Suppose that for every $U\in\tilde{\Omega}$ there exists $x_U\in S_U\setminus G$. Let $V=\overline{\{x_U\colon U\in\tilde{\Omega}\}}$. Since $x_U-F(x_U)\in U$, $0\in\overline{(I-F)(V)}$. Therefore by (3) there is $y\in V$ such that $y=F(y)$. Moreover, $V\subset C\setminus G$, as G is open and $x_U\in C\setminus G$ for every $U\in\tilde{\Omega}$. Hence $y\in S\setminus G$ in contradiction with $S\subset G$. Consequently, there exists $U\in\tilde{\Omega}$ such that $S_U\subset G$. On the other hand, $G_1\cap S_U\neq\varnothing\neq G_2\cap S_U$, because $S\subset S_U$. This proves that S_U is not connected, which contradicts Theorem 1. Hence S is connected.

Remark. In particular, the condition (3) is satisfied whenever the set $F(C)$ is relatively compact in C.

2. Now we consider the differential equation

$$x'=f(t,x)\,,\qquad x(0)=x_0, \tag{4}$$

where f is a bounded continuous function from $J\times E$ into E, $J=[0,a]$ is a compact interval in R, and E is a complete locally convex topological vector space. Let Ω denote the family of all open, balanced and convex neighbourhoods of 0 in E.

Lemma 2. *For any $u\in C$ and $U\in\Omega$ there exists a V in Ω such that*

$$f(t,x(t))-f(t,u(t))\in U\qquad \textit{for every } t\in J$$

whenever $x\in C$ and $x(t)-u(t)\in V$ for every $t\in J$.

Proof. From the continuity of f it follows that for any $s\in J$ there exist a neighbourhood T_s of s in R and $V_s\in\Omega$ such that

$$f(t,y)-f(s,u(s))\in\tfrac{1}{2}U\qquad\text{for } t\in T_s \text{ and } y\in u(s)+V_s. \tag{5}$$

On the other hand, by the continuity of u, for any $s\in J$ there exists a neighbourhood H_s of s, contained in T_s, such that

$$u(t)-u(s)\in\tfrac{1}{2}V_s\qquad\text{for } t\in H_s.$$

Since J is compact, there is a finite subset $\{s_1,\cdots,s_n\}$ of J such that $J\subset\bigcup_{i=1}^n H_{s_i}$. Let $V=\frac{1}{2}\bigcap_{i=1}^n V_{s_i}$. Assume that $x\in C$ and $x(t)-u(t)\in V$ for every $t\in J$. For any $t\in J$ we choose s_i such that $t\in H_{s_i}$. Because $u(t)\in u(s_i)+\frac{1}{2}V_{s_i}\subset u(s_i)+V_{s_i}$ and $x(t)=(x(t)-u(t))+u(t)\in u(s_i)+\frac{1}{2}V_{s_i}+V\subset u(s_i)+\frac{1}{2}V_{s_i}+\frac{1}{2}V_{s_i}=u(s_i)+V_{s_i}$, *by* (5), we get

$$\begin{aligned} f(t,x(t))-f(t,u(t))&=(f(t,x(t))-f(s_i,u(s_i)))\\ &\quad+(f(s_i,u(s_i))-f(t,u(t)))\in\tfrac{1}{2}U+\tfrac{1}{2}U=U. \end{aligned}$$

which ends the proof.

It is known [4] that (4) is equivalent to the integral equation

$$x(t)=x_0+\int_0^t f(s, x(s))ds \qquad (t \in J),$$

where $\int$ denotes the integral defined in [4].

Put

$$F(x)(t)=x_0+\int_0^t f(s, x(s))ds \qquad \text{for } x \in C \text{ and } t \in J. \tag{6}$$

From Lemma 2 we deduce that F is a continuous mapping of C into itself. Moreover, the set $F(C)$ is equiuniformly continuous. Indeed, as the set $f(J\times E)$ is bounded, for any $U \in \Omega$ there is a number $\varepsilon>0$ such that $\varepsilon\,\overline{\mathrm{conv}}\,f(J\times E)\subset U$, and therefore

$$F(x)(t)-F(x)(\tau)=\int_\tau^t f(s, x(s))ds \in (t-\tau)\,\overline{\mathrm{conv}}\,f(J\times E)\subset \frac{t-\tau}{\varepsilon}U\subset U$$

for every $x \in C$, t, $\tau \in J$ such that $0\leqslant t-\tau\leqslant\varepsilon$.

From this, by Theorem 2, we conclude that the following is true.

Theorem 3. *If f is such that the mapping $I-F$ satisfies* (3), *where F is defined by* (6), *then the set S of all solutions of* (4) *defined on J is nonempty and connected in $C(J, E)$. In particular, if the set $f(J\times E)$ is relatively compact in E, then S is a continuum in $C(J, E)$.*

Acknowledgement. The author expresses his appreciation to the referee for his suggestions to amend an earlier version of the paper.

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nuna adreso:
Os. Powstań Narodowych 59 m.6
61216 Poznań
Poland

(Ricevita la 10-an de februaro, 1978)
(Reviziita la 8-an de septembro, 1978)
(Reviziita la 25-an de septembro, 1978)