

## On a Pfaffian System Containing Parameters

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### Introduction

The purpose of this paper is to study a Pfaffian system containing parameters. When we have no so-called turning point case, our result gives a fundamental solution.

We consider a completely integrable Pfaffian system of the form

$$(A) \quad dz = \left( \sum_{i=1}^n \frac{A^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) z$$

where we suppose that:

1)  $x = (x_1, \dots, x_n)$  is a complex variable in  $\mathbb{C}^n$ ,  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\nu)$  is a complex parameter in  $\mathbb{C}^\nu$  with  $\|\varepsilon\| = \max_{1 \leq j \leq \nu} |\varepsilon_j|$  sufficiently small.  $\mu(i)$  are  $\nu$ -vectors with nonnegative components  $\{\mu_j(i)\}$ .  $\varepsilon^{\mu(i)}$  stands for a scalar expression

$$(0.1) \quad \varepsilon^{\mu(i)} = \varepsilon_1^{\mu_1(i)} \dots \varepsilon_\nu^{\mu_\nu(i)}.$$

2)  $z$  is an  $m$ -vector with components  $\{z_j\}$ .

3) The  $A^i(x, \varepsilon)$ 's are  $m$  by  $m$  matrices whose components are holomorphic and bounded functions of  $(x, \varepsilon)$  for

$$(0.2) \quad (x, \varepsilon) \in U(r_0) \times \Sigma(\varepsilon_0, \varphi, \alpha)$$

with

$$(0.3) \quad U(r) = \left\{ x \mid \|x\| = \max_{1 \leq j \leq n} |x_j| < r \right\},$$

$$(0.4) \quad \Sigma(\varepsilon_0, \varphi, \alpha) = \{ \varepsilon \mid 0 < \|\varepsilon\| < \varepsilon_0, |\arg \varepsilon - \varphi| < \alpha \}.$$

Here,  $\arg \varepsilon$  denotes a  $\nu$ -vector with components  $\{\arg \varepsilon_j\}$ ,  $\varphi$  and  $\alpha$  are also  $\nu$ -vectors with components  $\{\varphi_j\}$  and  $\{\alpha_j\}$ , so that a vectorial inequality is to be understood to be component-wise inequalities.

4) The  $A^i(x, \varepsilon)$ 's have uniformly asymptotic expansions in powers of  $\varepsilon$

$$(0.5) \quad A_i(x, \varepsilon) \simeq \sum_{|k| \geq 0} A_k^i(x) \varepsilon^k \quad \text{as } \varepsilon \rightarrow 0$$

(cf. § 1, Chapter I). The coefficients  $A_k^i(x)$  are holomorphic and bounded matrix functions of  $x$  for  $x \in U(r)$ . Here  $|k|$  is the length of a multi-index  $k$  with components  $\{k_j\}$ , i.e.

$$(0.6) \quad |k| = k_1 + \cdots + k_v,$$

and  $\varepsilon^k = \varepsilon_1^{k_1} \cdots \varepsilon_v^{k_v}$ .

The integrability condition is given by

$$(0.7) \quad \varepsilon^{\mu(j)} \frac{\partial A^i(x, \varepsilon)}{\partial x_j} + A^i(x, \varepsilon) A^j(x, \varepsilon) = \varepsilon^{\mu(i)} \frac{\partial A^j(x, \varepsilon)}{\partial x_i} + A^j(x, \varepsilon) A^i(x, \varepsilon)$$

for any  $(i, j)$ . Since the both sides are considered as asymptotic expansions in powers of  $\varepsilon$ , these relations can be also expressed by using the  $A_k^i(x)$ 's appearing in the asymptotic expansions (0.5). When the  $A^i(x, \varepsilon)$ 's are expressed by a formal power series of the form (0.5) with holomorphic coefficients, the system (A) is called a *formally integrable system* if (0.7) holds as a formal power series of  $\varepsilon$ . In particular, we have, therefore,

$$(0.8) \quad [A_0^i(x), A_0^j(x)] \equiv A_0^i(x) A_0^j(x) - A_0^j(x) A_0^i(x) = 0$$

which means that the  $A_0^i(x)$ 's are mutually commutable.

We assume furthermore that

5) For any  $i$ , the eigenvalues  $\{\lambda_1^i, \dots, \lambda_m^i\}$  of the matrix  $A_0^i(0)$  satisfy the condition

$$(0.9) \quad \{\lambda_1^i, \dots, \lambda_p^i\} \cap \{\lambda_{p+1}^i, \dots, \lambda_m^i\} = \emptyset,$$

namely,

$$(0.10) \quad \lambda_\alpha^i \neq \lambda_\beta^i \quad \text{for any } (\alpha, \beta) \text{ such that } \alpha \leq p, \beta > p.$$

This assumption is called ASSUMPTION  $\mathfrak{A}$ .

Then, there exists a nonsingular constant matrix  $S$  such that the matrices  $S^{-1} A_0^i(0) S$  are at the same time block-diagonalized as follows

$$(0.11) \quad S^{-1} A_0^i(0) S = \text{diag} [A_0^{i,1}, A_0^{i,2}]$$

with the same partition independently of  $i$  and

$$(0.12) \quad A_0^{i,1} = \begin{pmatrix} \lambda_1^i & & * \\ & \ddots & \\ 0 & & \lambda_p^i \end{pmatrix}, \quad A_0^{i,2} = \begin{pmatrix} \lambda_{p+1}^i & & * \\ & \ddots & \\ 0 & & \lambda_m^i \end{pmatrix} \text{ (upper-triangular)}$$

for some  $p$  (Cf. [1] p. 333 Corollaire 2).

The theorem which we are going to establish can be stated as follows:

**Theorem A (Main Theorem).** *We suppose that all the assumptions imposed on system (A) are satisfied. Then, for any direction  $\arg \varepsilon = \theta$  in  $\Sigma(\varepsilon_0, \varphi, \alpha)$ , there exists a nonsingular transformation*

$$(0.13) \quad z = \mathfrak{U}(x, \varepsilon)v$$

defined in the domain of the form

$$(0.14) \quad (x, \varepsilon) \in U(r'_0) \times \Sigma(\varepsilon'_0, \theta, \alpha') \subset U(r_0) \times \Sigma(\varepsilon_0, \theta, \alpha)$$

such that system (A) is transformed into an integrable Pfaffian system of the form

$$(C) \quad dv = \left( \sum_{i=1}^n \frac{C^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) v$$

with

$$(0.15) \quad C^i(x, \varepsilon) = \text{diag} [C^{i,1}(x, \varepsilon), C^{i,2}(x, \varepsilon)].$$

Here the matrix  $\mathfrak{U}(x, \varepsilon)$  is holomorphic and bounded for (0.14), admits for (0.14) a uniformly asymptotic expansion in powers of  $\varepsilon$ , with holomorphic and bounded coefficients for  $x \in U(r'_0)$ , as  $\varepsilon$  tends to 0 through the sector  $\Sigma(\varepsilon'_0, \theta, \alpha')$ . And, the minor matrices  $C^{i,1}(x, \varepsilon)$  and  $C^{i,2}(x, \varepsilon)$  are holomorphic and bounded functions of  $(x, \varepsilon)$  in the domain (0.14) such that  $C^{i,1}(0, 0) = A_0^{i,1}$ ,  $C^{i,2}(0, 0) = A_0^{i,2}$ , and admit uniformly asymptotic expansions analogous to that for  $\mathfrak{U}(x, \varepsilon)$ .

**Theorem B (General solution).** *Instead of ASSUMPTION  $\mathfrak{A}$  we assume that, for any  $i$ , the matrix  $A_0^i(0)$  has mutually distinct eigenvalues. Then, for any direction  $\arg \varepsilon = \theta$  contained in  $\Sigma(\varepsilon_0, \varphi, \alpha)$ , we can find a small sector  $\Sigma(\varepsilon'_0, \theta, \alpha')$  such that, in a domain of the form (0.14), there exists a fundamental solution matrix of the form*

$$(0.16) \quad U(x, \varepsilon) = P(x, \varepsilon) \exp \left( \frac{Q(x, \varepsilon)}{\varepsilon^{\mu(0)}} \right).$$

Here,  $P(x, \varepsilon)$  is a nonsingular matrix, and  $Q(x, \varepsilon)$  is a diagonal matrix, whose components are holomorphic and bounded functions in (0.14), and admit there uniformly asymptotic expansions in powers of  $\varepsilon$  with holomorphic and bounded coefficients of  $x \in U(r'_0)$  as  $\varepsilon \rightarrow 0$ .  $\mu(0)$  is a  $\nu$ -vector with components  $\left\{ \max_{1 \leq i \leq n} \mu_j(i) \right\}$ .

Once Theorem A has been established, Theorem B will be almost trivial. So we would like to prove the Theorem A, only.

In Chapter I, the definition and the properties of asymptotic expansions to be used here will be discussed. And, a theorem concerning the simultaneous block-diagonalization of several commutable matrices will be proved. Chapter II is devoted to the proof of existence of a transformation for formal block-diagonalization. Namely, in a view point of purely formal sense for asymptotic power series, the existence of a matrix  $\mathfrak{U}(x, \varepsilon)$  used for Theorem A will be proved. In Chapter III, we discuss an asymptotic validity of a matrix  $\mathfrak{U}(x, \varepsilon)$  obtained in Chapter II.

The notations about vectors and multi-indices are as follows:

1) For two  $\nu$ -ple indices  $k$  and  $l$  with components  $\{k_j\}$  and  $\{l_j\}$ ,  $k \leq l$  means  $k_j \leq l_j$  for every  $i$ , and  $k < l$  means  $k \leq l$  and  $k_j < l_j$  for some  $j$ .

2) For two  $\nu$ -vector  $\alpha$  and  $\gamma$  with components  $\{\alpha_j\}$  and  $\{\gamma_j\}$ , a vectorial inequality  $\alpha < \gamma$  means component-wise inequalities,  $\alpha_1 < \gamma_1, \dots, \alpha_\nu < \gamma_\nu$ .

3)  $I_s$  denotes the  $s$  by  $s$  unit matrix.

The author wishes to express his thanks to Professor Tosihusa Kimura for helpful advices during the preparation of this paper.

## Chapter I. Preliminaries

### § 1. Asymptotic expansions in powers of several parameters

Let there be given a function  $A(x, \varepsilon)$  which is holomorphic and bounded for

$$(1.1) \quad (x, \varepsilon) \in U(r) \times \Sigma(\varepsilon_0, \varphi, \alpha)$$

where

$$(1.2) \quad U(r) = \{x \mid \|x\| < r\} \subset \mathbf{C}^n$$

and

$$(1.3) \quad \Sigma(\varepsilon_0, \varphi, \alpha) = \{\varepsilon \mid |\arg \varepsilon - \varphi| < \alpha, 0 < \|\varepsilon\| < \varepsilon_0\} \subset \mathbf{C}^\nu,$$

$r, \varepsilon_0$  are positive constants,  $\alpha$  is a  $\nu$ -vector with components  $\{\alpha_j\}$  positive numbers, and  $\varphi$  is a  $\nu$ -vector with components  $\{\varphi_j\}$  real numbers.

*We shall say that the function  $A(x, \varepsilon)$  admits uniformly for  $x \in U(r)$  an asymptotic expansion*

$$(1.4) \quad A(x, \varepsilon) \simeq \sum_{k \geq 0} A_k(x) \varepsilon^k$$

*as  $\varepsilon$  tends to the origin through the sector  $|\arg \varepsilon - \varphi| < \alpha$ , if, for any natural number  $N$ , there exists a positive constant  $C_N$  independent of  $x$  in  $U(r)$  such that*

$$(1.5) \quad \left| A(x, \varepsilon) - \sum_{|k| \leq N} A_k(x) \varepsilon^k \right| \leq C_N \|\varepsilon\|^{N+1}$$

for (1.1). Simply, we shall also say that  $A(x, \varepsilon)$  admits a uniformly asymptotic expansion (1.4) for  $(x, \varepsilon) \in U(r) \times \Sigma(\varepsilon_0, \varphi, \alpha)$  as  $\varepsilon \rightarrow 0$ .

Obviously, we have

**Proposition 1.1.** *If there be given two functions with uniformly asymptotic expansions*

$$(1.6) \quad A(x, \varepsilon) \simeq \sum_{k \geq 0} A_k(x) \varepsilon^k, \quad B(x, \varepsilon) \simeq \sum_{j \geq 0} B_j(x) \varepsilon^j,$$

for (1.1), then the functions  $A(x, \varepsilon) + B(x, \varepsilon)$  and  $A(x, \varepsilon)B(x, \varepsilon)$  admit uniformly asymptotic expansions of the form

$$(1.7) \quad A(x, \varepsilon) + B(x, \varepsilon) \simeq \sum_{k \geq 0} (A_k(x) + B_k(x)) \varepsilon^k,$$

and

$$(1.8) \quad A(x, \varepsilon)B(x, \varepsilon) \simeq \sum_{k \geq 0} C_k(x) \varepsilon^k$$

with

$$(1.9) \quad C_k(x) = \sum_{k' + k'' = k} A_{k'}(x) B_{k''}(x).$$

We want to prove the following.

**Proposition 1.2.** *The coefficients of the uniformly asymptotic expansion (1.4) are uniquely determined.*

*Proof.* It is sufficient to prove that if we have

$$0 \simeq \sum_{k \geq 0} A_k(x) \varepsilon^k \quad \text{for } (x, \varepsilon) \in U(r) \times \Sigma(\varepsilon_0, \varphi, \alpha).$$

Then we have necessarily  $A_k(x) \equiv 0$  for  $x \in U(r)$ .

Suppose that all the  $A_k(x)$ 's do not vanish identically. Then there would be an index  $k_0$  such that

$$(1.10) \quad \begin{aligned} A_k(x) &\not\equiv 0 & \text{for } |k| = |k_0|, \\ A_k(x) &\equiv 0 & \text{for } |k| < |k_0|. \end{aligned}$$

By the definition of uniformly asymptotic expansions, we can find a positive constant  $C$  such that

$$(1.11) \quad \left| \sum_{|k| \leq |k_0|} A_k(x) \varepsilon^k \right| = \left| \sum_{|k|=|k_0|} A_k(x) \varepsilon^k \right| \leq C \|\varepsilon\|^{|k_0|+1},$$

for (1.1). Observe that if  $\varepsilon$  belongs to  $\sum (\varepsilon_0, \varphi, \alpha)$ ,  $t\varepsilon$  does too as long as  $t > 0$  is sufficiently small. Hence, inequality (1.11) gives

$$\left| \sum_{|k|=|k_0|} A_k(x) (t\varepsilon)^k \right| \leq C \|t\varepsilon\|^{|k_0|+1},$$

or

$$\left| \sum_{|k|=|k_0|} A_k(x) \varepsilon^k \right| \leq t \cdot C \|\varepsilon\|^{|k_0|+1} = o(1).$$

Hence, we must have

$$(1.12) \quad \sum_{|k|=|k_0|} A_k(x) \varepsilon^k = 0$$

for (1.1). The expression (1.12) is a polynomial in  $\varepsilon$ , so every coefficients must be zero identically, which contradicts assumption (1.10).

Therefore  $A_k(x) \equiv 0$  for all  $k$ .

The following two propositions are almost trivial.

**Proposition 1.3.** *A bounded function  $A(x, \varepsilon)$  which admits a uniformly asymptotic expansion*

$$A(x, \varepsilon) \simeq \sum_{k \geq 0} A_k(x) \varepsilon^k$$

for (1.1) enables us to differentiate termwise with respect to  $x \in U(r)$ . Namely, the uniformly asymptotic expansion

$$(1.13) \quad \frac{\partial A(x, \varepsilon)}{\partial x_i} \simeq \sum_{k \geq 0} \frac{\partial A_k(x)}{\partial x_i} \varepsilon^k$$

holds for (1.11).

**Proposition 1.4.** *A bounded function  $A(x, \varepsilon)$  with a uniformly asymptotic expansion in the form*

$$A(x, \varepsilon) \simeq \sum_{k \geq 0} A_k(x) \varepsilon^k$$

permits us to integrate termwise with respect to  $x$ , so that the uniformly asymptotic expansion

$$(1.14) \quad \int_a^x A(\xi, \varepsilon) d\xi \simeq \sum_{k \geq 0} \left( \int_a^x A_k(\xi) d\xi \right) \varepsilon^k$$

holds, where the integration is to be taken along a path contained in  $U(r)$  joining the two points  $a$  and  $x$  in  $U(r)$ .

When a formal power series in  $\varepsilon$  with coefficients bounded for  $x \in U(r)$  is given, we need to construct an analytic function which admits the formal power series as a uniformly asymptotic expansion for an arbitrary sector with respect to  $\varepsilon$ .

The following proposition is a kind of a well-known Borel-Ritt theorem (Cf. [4] Theorem 9.3).

**Proposition 1.5.** *Let  $A_k(x)$  for every  $k$  be holomorphic and bounded function in  $x \in U(r)$ . Let there be given any sector  $\sum (\varepsilon_0, \varphi, \alpha)$  in the  $\varepsilon$ -space. Then there exists a holomorphic and bounded function  $\mathfrak{A}(x, \varepsilon)$  of  $(x, \varepsilon)$  for a domain of the form (1.1) which admits a uniformly asymptotic expansion*

$$(1.15) \quad \mathfrak{A}(x, \varepsilon) \simeq \sum_{k \geq 0} A_k(x) \varepsilon^k$$

for (1.1).

*Proof.* Without loss of generality, we can assume that

$$(1.16) \quad \varphi = (0, \dots, 0), \quad \varepsilon_0 < 1.$$

For a given  $\nu$ -ple  $\alpha$ , we can find a sufficiently small positive constant  $\beta$  so that

$$(1.17) \quad 0 < \beta < \frac{1}{2}$$

$$(1.18) \quad \|\arg \varepsilon^\beta\| < \|\beta[\alpha]\| < \frac{2\pi}{\nu}.$$

Let

$$(1.19) \quad b_k = \begin{cases} \left( \sup_x A_k(x) \right)^{-1} (x \in U(r)), & \text{if } A_k(x) \not\equiv 0 \\ 0, & \text{if } A_k(x) \equiv 0. \end{cases}$$

A described function  $\mathfrak{A}(x, \varepsilon)$  is expressed by the power series

$$(1.20) \quad \mathfrak{A}(x, \varepsilon) = \sum_{k \geq 0} A_k(x) \gamma_k(\varepsilon) \varepsilon^k$$

where

$$(1.21) \quad \begin{aligned} \gamma_0(\varepsilon) &= 1 \\ \gamma_k(\varepsilon) &= 1 - \exp\left(-\frac{b_k}{\varepsilon^{\beta(k)}}\right), \quad k > 0. \end{aligned}$$

The  $\beta(k)$ 's are  $\nu$ -ples  $(\beta_1(k), \dots, \beta_r(k))$  with

$$(1.22) \quad \beta_j(k) = \begin{cases} \beta & \text{if } k_j \geq 0 \\ 0 & \text{if } k_j = 0, \end{cases}$$

where  $k_j$  is the  $j$ -th component of  $k$ .

Observe that, if  $\operatorname{Re} z < 0$ ,

$$(1.23) \quad |1 - e^z| = \left| \int_0^z e^\zeta d\zeta \right| \leq \int_0^{|z|} |e^\zeta| |d\zeta| \leq |z|,$$

where the integration is to be taken along a segment between the two points 0 and  $z$ . By using (1.18) and (1.19), we have, therefore,

$$|A_k(x) \gamma_k(\varepsilon) \varepsilon^k| \leq \sup_{x \in U(r)} |A_k(x)| \cdot b_k \cdot |\varepsilon^{k - \beta(k)}| \leq |\varepsilon^{k - \beta(k)}|$$

for  $\varepsilon \in \Sigma(\varepsilon_0, 0, \alpha)$ . Hence,

$$|\mathfrak{A}(x, \varepsilon)| \leq \sum_{k \geq 0} |\varepsilon^{k - \beta(k)}|.$$

Since

$$\begin{aligned} |\varepsilon^{k - \beta(k)}| &= |\varepsilon_1|^{k_1 - \beta_1(k)} \cdots |\varepsilon_\nu|^{k_\nu - \beta_\nu(k)} \\ &\leq |\varepsilon_1|^{k_1 - \beta} \cdots |\varepsilon_\nu|^{k_\nu - \beta}, \end{aligned}$$

the power series (1.20) is uniformly convergent for  $U(r) \times \Sigma(\varepsilon_0, 0, \alpha)$ , so  $\mathfrak{A}(x, \varepsilon)$  is a holomorphic and bounded function of  $(x, \varepsilon)$ , there.

It remains to prove that

$$(1.24) \quad \left| \mathfrak{A}(x, \varepsilon) - \sum_{|k| \leq N} A_k(x) \varepsilon^k \right| \leq O(\|\varepsilon\|^{N+1}),$$

for any  $N \geq 1$ . Observe that

$$\begin{aligned} & \left| \mathfrak{A}(x, \varepsilon) - \sum_{|k| \leq N} A_k(x) \varepsilon^k \right| \\ & \leq \left| \sum_{0 < |k| \leq N} A_k(x) \exp\left(-\frac{b_k}{\varepsilon^{\beta(k)}}\right) \varepsilon^k \right| + \left| \sum_{|k| > N} A_k(x) \gamma_k(\varepsilon) \varepsilon^k \right|. \end{aligned}$$

Since

$$\|\arg \varepsilon^{\beta(k)}\| \leq \|\arg \varepsilon^\beta\| \leq |\arg \beta \alpha| < \frac{\pi}{2},$$

there exists a constant  $C_N^1$  such that

$$\left| \sum_{0 < |k| \leq N} A_k(x) \exp\left(-\frac{b_k}{\varepsilon^{\beta(k)}}\right) \varepsilon^k \right| \leq C_N^1 \|\varepsilon\|^{N+1}.$$

Of course,  $C_N^1$  is independent of  $x$ . On the other hand,

$$\begin{aligned} \left| \sum_{|k| > N} A_k(x) \gamma_k(\varepsilon) \varepsilon^k \right| &\leq \sum_{|k| > N} |\varepsilon^{k - \beta(k)}| \\ &\leq \sum_{|k| > N} |\varepsilon^{k - \beta \mathbf{1}}| \leq \sum_{|k| > N} \|\varepsilon\|^{|k| - \nu/2} \leq C_N^2 \|\varepsilon\|^{N - \nu/2}, \end{aligned}$$

where  $\mathbf{1} = (1, \dots, 1)$ . Thus we have

$$(1.25) \quad \left| \mathfrak{A}(x, \varepsilon) - \sum_{|k| \leq N} A_k(x) \varepsilon^k \right| \leq C_N \|\varepsilon\|^{N - \nu/2}$$

for (1.1) with  $C_N$  a certain positive constant. It is easy to see that inequality (1.25) implies our desired inequality.

## § 2. Simultaneous block-diagonalization of mutually commutable matrices

In this section, we have to prove a useful proposition which will play an important role in a block-diagonalization of a Pfaffian systems.

The following lemma is well known:

**Lemma 2.1.** *For any two constant square matrices  $A$  and  $B$  of order  $q$ , we consider a matrix equation with respect to unknown  $X$*

$$(2.1) \quad AX - XB = 0.$$

*Then a necessary and sufficient condition for equation (2.1) to have a non-zero solution  $X$  is that the matrices  $A$  and  $B$  have at least one common eigenvalue (Cf. Theorem 4.1 in [4]).*

Now we want to prove the following theorem.

**Theorem 2.1.** *Let there be given  $m$  by  $m$  matrices  $A^i(x)$  ( $i=1, \dots, M$ ) holomorphic at  $x=0 \in \mathbb{C}^n$ . Assume that these matrices are mutually commutable. Moreover assume that the eigenvalues  $\{\lambda_1^1, \dots, \lambda_m^1\}$  of the matrix  $A^1(0)$  satisfy*

$$(2.2) \quad \{\lambda_1^1, \dots, \lambda_p^1\} \cap \{\lambda_{p+1}^1, \dots, \lambda_m^1\} = \emptyset,$$

*for some positive integer  $p < m$ . Then, there exists an  $m$  by  $m$  matrix  $S(x)$ , which is holomorphic and nonsingular in a neighbourhood of the origin  $x=0$ , such that the  $A^i(x)$ 's are simultaneously block-diagonalized as follows:*

$$(2.3) \quad S^{-1}(x)A^i(x)S(x) = \text{diag} \{B^{i,1}(x), B^{i,2}(x)\}.$$

Here  $B^{i,1}(x)$  and  $B^{i,2}(x)$  are  $p$  by  $p$  and  $(m-p)$  by  $(m-p)$  matrices holomorphic at  $x=0$ . For any  $i$ , the matrices  $B^{i,1}(0)$  and  $B^{i,2}(0)$  are of upper-triangular form:

$$B^{i,1}(0) = \begin{bmatrix} \lambda_1^i & * & \cdots & * \\ & \cdot & & \vdots \\ & & \cdot & * \\ & & & \lambda_p^i \end{bmatrix}, \quad B^{i,2}(0) = \begin{bmatrix} \lambda_{p+1}^i & * & \cdots & * \\ & \cdot & & \vdots \\ & & \cdot & * \\ & & & \lambda_m^i \end{bmatrix}.$$

**Corollary.** In addition, if we assume that, instead of (2.2), the eigenvalues  $\{\lambda_1^i, \dots, \lambda_m^i\}$  are mutually distinct, then the matrices (2.3) are reduced to diagonal ones:

$$(2.4) \quad S^{-1}(x)A^i(x)S(x) = \text{diag} [\lambda_1^i(x), \dots, \lambda_m^i(x)]$$

for every  $i$ .

*Proof of Theorem 2.1.* We can prove the following proposition by the same way as that of Theorem 25.1 in [4].

**Proposition.** Under condition (2.2), there exists a nonsingular and holomorphic matrix  $S(x)$  so that

$$S^{-1}(x)A^1(x)S(x) = \begin{pmatrix} B^{1,1}(x) & 0 \\ 0 & B^{1,2}(x) \end{pmatrix} \begin{matrix} p \\ m-p \end{matrix}$$

where the eigenvalues of  $B^{1,1}(0)$  and  $B^{1,2}(0)$  are  $\lambda_1^1, \dots, \lambda_p^1$  and  $\lambda_{p+1}^1, \dots, \lambda_m^1$ , respectively.

Let us set

$$S^{-1}(x)A^i(x)S(x) = \begin{pmatrix} B^{i,1}(x) & B^{i,12}(x) \\ B^{i,21}(x) & B^{i,2}(x) \end{pmatrix} \begin{matrix} p \\ m-p \end{matrix} \quad (i=2, \dots, n).$$

Since

$$[S^{-1}(x)A^1(x)S(x), S^{-1}(x)A^i(x)S(x)] = 0,$$

it follows that

$$(2.5) \quad B^{1,1}(x)B^{i,12}(x) - B^{i,12}(x)B^{1,1}(x) = 0$$

$$(2.6) \quad B^{1,2}(x)B^{i,21}(x) - B^{i,21}(x)B^{1,2}(x) = 0.$$

We regard (2.5) as an equation for  $p(m-p)$  components of  $B^{i,12}(x)$ . Then by using Lemma 2.1, the Jacobian of (2.5) at  $x=0$  is nonsingular. Hence, by the implicit function theorem, we have

$$B^{i,12}(x)=0.$$

By the same way, we have  $B^{i,21}(x)=0$ . Therefore,  $A^i(x)$  ( $i=1, \dots, n$ ) are block-diagonalized simultaneously by  $S(x)$ .

## Chapter II. Formal block-diagonalization of a Pfaffian system

We consider a completely integrable system

$$(A) \quad dz = \left( \sum_{i=1}^n \frac{A^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) z$$

under the assumptions stated in the introduction. Since the  $A^i(x, \varepsilon)$ 's are supposed to have uniformly asymptotic expansions in powers of  $\varepsilon$ , system (A) can be also regarded as a formally integrable Pfaffian system.

### § 3. Formal block-diagonalization

We assume that ASSUMPTION  $\mathfrak{A}$  stated in the introduction is satisfied.

We want to prove the following theorem.

**Theorem 3.1** (*Formal block-triangularization*). Under ASSUMPTION  $\mathfrak{A}$ , there exists a matrix  $\mathfrak{S}(x, \varepsilon)$  which is of the form

$$(3.1) \quad \mathfrak{S}(x, \varepsilon) = S(x) \begin{pmatrix} I_p & S(x, \varepsilon) \\ 0 & I_{m-p} \end{pmatrix}$$

and is expressed by a formal power series

$$(3.2) \quad \mathfrak{S}(x, \varepsilon) = \sum_{k \geq 0} \mathfrak{S}_k(x) \varepsilon^k$$

with the properties that

- (i)  $S(x)$  is an  $m$  by  $m$  matrix which is holomorphic, bounded and nonsingular for  $x$  in a neighbourhood of the origin of  $\mathbf{C}^n$ , namely for  $x \in U(r_0)$ ,
- (ii) every  $\mathfrak{S}_k(x)$  is a holomorphic and bounded matrix for  $x \in U(r_0)$ ,
- (iii) the formal transformation

$$(3.3) \quad z = \mathfrak{S}(x, \varepsilon) u$$

changes formally system (A) into a formally integrable Pfaffian system

$$(B) \quad du = \left( \sum_{i=1}^n \frac{B^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) u,$$

- (iii, a) the  $B^i(x, \varepsilon)$ 's are expressed by formal power series

$$(3.4) \quad B^i(x, \varepsilon) = \sum_{k \geq 0} B_k^i(x) \varepsilon^k$$

where the  $B_k^i(x)$ 's are holomorphic and bounded for  $x \in U(r_0)$  and have the block-triangular form with the same partition as  $\mathfrak{S}(x, \varepsilon)$ ,

$$(3.5) \quad B_k^i(x) = \begin{pmatrix} B_k^{i,11}(x) & 0 \\ B_k^{i,21}(x) & B_k^{i,22}(x) \end{pmatrix} \quad (|k| \geq 1)$$

and

$$(3.6) \quad B_0^i(x) = \begin{pmatrix} B_0^{i,11}(x) & 0 \\ 0 & B_0^{i,22}(x) \end{pmatrix} \quad (k=0),$$

(iii, b) the eigenvalues of the matrices  $B_0^{i,11}(0)$  and  $B_0^{i,22}(0)$  coincide respectively with  $\{\lambda_1^i, \dots, \lambda_p^i\}$  and  $\{\lambda_{p+1}^i, \dots, \lambda_m^i\}$ .

Furthermore we have the following theorem.

**Theorem 3.2** (Formal block-diagonalization). *Consider a formally integrable Pfaffian system of the form (B) with ASSUMPTION  $\mathfrak{A}$ . Then we can construct a matrix  $\mathfrak{F}(x, \varepsilon)$  of the form*

$$(3.7) \quad \mathfrak{F}(x, \varepsilon) = \begin{pmatrix} I_p & 0 \\ T(x, \varepsilon) & I_{m-p} \end{pmatrix}$$

which is expressed by a formal power series

$$(3.8) \quad \mathfrak{F}(x, \varepsilon) = \sum_{k \geq 0} \mathfrak{F}_k(x) \varepsilon^k$$

with the properties that

- (i) the  $\mathfrak{F}_k(x)$ 's are holomorphic for  $x \in U(r_0)$ ,
- (ii) the formal transformation

$$(3.9) \quad u = \mathfrak{F}(x, \varepsilon)v$$

changes formally system (B) into another formal integrable Pfaffian system of the form

$$(C) \quad dv = \left( \sum_{i=1}^n \frac{C^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) v,$$

(iii, a) the  $C^i(x, \varepsilon)$ 's are expressed by formal series in  $\varepsilon$

$$(3.10) \quad C^i(x, \varepsilon) = \sum_{k \geq 0} C_k^i(x) \varepsilon^k,$$

where every coefficient matrix is holomorphic and bounded for  $x \in U(r_0)$  and of block-diagonal form with the same partition as  $\mathfrak{L}(x, \varepsilon)$ ,

$$(3.11) \quad C_k^i(x) = \begin{pmatrix} C_k^{i,11}(x) & 0 \\ 0 & C_k^{i,22}(x) \end{pmatrix},$$

(iii, b) the matrix  $C_0^{i,11}(0)$  and  $C_0^{i,22}(0)$  have the same eigenvalues as those of  $B_0^{i,11}(0)$  and  $B_0^{i,22}(0)$  respectively.

#### § 4. Proof of Theorem 3.1

Let

$$(4.1) \quad A^i(x, \varepsilon) = \sum_{k \geq 0} A_k^i(x) \varepsilon^k,$$

be the asymptotic expansion. Since system (A) is integrable, the relations

$$(4.2) \quad [A_0^i(x), A_0^j(x)] = 0$$

must be satisfied.

Then, by virtue of Theorem 2.1, there exists a matrix function  $S(x)$ , which is holomorphic, bounded and nonsingular for  $x$  in a certain neighbourhood of the origin  $U(r_1)$ , such that the matrices  $S(x)^{-1}A_0^i(x)S(x)$  are simultaneously block-diagonalized with the partition of the same size

$$(4.3) \quad S^{-1}(x)A_0^i(x)S(x) = \begin{pmatrix} A_0^{i,1}(x) & 0 \\ 0 & A_0^{i,2}(x) \end{pmatrix} \quad (i=1, \dots, n).$$

Moreover,  $A_0^{i,1}(0)$  and  $A_0^{i,2}(0)$  are of the upper-triangular form

$$(4.4) \quad A_0^{i,1}(0) = \begin{bmatrix} \lambda_1^i & & * & \cdots & * \\ & \ddots & & \ddots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & * \\ & & & & \lambda_p^i \end{bmatrix}, \quad A_0^{i,2}(0) = \begin{bmatrix} \lambda_{p+1}^i & & * & \cdots & * \\ & \ddots & & \ddots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & * \\ & & & & \lambda_n^i \end{bmatrix}.$$

Hence, by applying the transformation

$$(4.5) \quad z = S(x)z^1,$$

we can assume without loss of generality that, from the outset, the matrices  $A^i(x)$  have the form  $A_0^i(x) = \text{diag} [A_0^{i,1}(x), A_0^{i,2}(x)]$ , with (4.4).

The proof of Theorem 3.1 is based on the following fundamental lemma.

**Lemma 4.1.** *Let there be given an integrable system (A) with ASSUMPTION  $\mathfrak{A}$ .*

Moreover, assume that, for a positive integer  $N$ , the matrices  $A_k^i(x)$  for  $|k| \leq N-1$  appearing in (4.1) have the form

$$(4.6, N) \quad A_k^i(x) = \begin{pmatrix} A_k^{i,11}(x) & 0 \\ A_k^{i,21}(x) & A_k^{i,22}(x) \end{pmatrix},$$

with the same partition as (4.3), in particular

$$(4.6, 0) \quad A_0^i(x) = \text{diag} [A_0^{i,1}(x), A_0^{i,2}(x)],$$

with (4.4).

Then there exists a matrix function  $\mathfrak{S}_{(N)}(x, \varepsilon)$ , which is a polynomial in  $\varepsilon$  with holomorphic and bounded coefficients for  $x \in U(r_0)$ , of the form

$$(4.7) \quad \mathfrak{S}_{(N)}(x, \varepsilon) = I + \sum_{|l|=N} \mathfrak{S}_l(x) \varepsilon^l,$$

$$(4.8) \quad \mathfrak{S}_l(x) = \begin{pmatrix} 0 & S_l(x) \\ 0 & 0 \end{pmatrix}$$

with the same partition as before,  $r_0$  being a positive constant independent of  $N$ . If we apply the transformation

$$(4.9) \quad z = \mathfrak{S}_{(N)}(x, \varepsilon)u$$

to system (A), Pfaffian system (A) is changed into an integrable system

$$(E) \quad du = \left( \sum_{i=1}^n \frac{E^i(x, \varepsilon)}{e^{\mu(\varepsilon)}} dx_i \right) u$$

with the properties that:

(i) every  $E^i(x, \varepsilon)$  admits a uniformly asymptotic expansion

$$(4.10) \quad E^i(x, \varepsilon) = \sum_{k \geq 0} E_k^i(x) \varepsilon^k,$$

where the  $E_k^i(x)$ 's are holomorphic and bounded for  $x \in U(r_0)$  and, in particular,

$$(4.11) \quad E_k^i(x) = A_k^i(x) \quad |k| < N$$

$$(4.12) \quad E_k^i(x) = \begin{pmatrix} E_k^{i,11}(x) & 0 \\ E_k^{i,21}(x) & E_k^{i,22}(x) \end{pmatrix}, \quad |k| = N.$$

*Remark.* When  $N=1$ , condition (4.6,  $N$ ) is unnecessary.

Suppose that this lemma has been established. Theorem 3.1 can be proved in the following manner. We start from system (A) with ASSUMPTION  $\mathfrak{A}$  and (4.6, 0), which satisfies the assumptions in Lemma 4.1 for  $N=1$ . By using Lemma 4.1, we

can find a transformation  $z^1 = \mathfrak{S}_{(1)}(x, \varepsilon)z^2$  by which the transformed system ( $A_\varepsilon$ ) satisfies the assumptions in Lemma 4.1 for  $N=2$ . Thus, after  $N'$ -times successively repeated applications of Lemma 4.1, we have an integrable system ( $A_{N'+1}$ ) which fills all the assumptions in Lemma 4.1 for  $N=N'+1$ . It is obvious that the transformation from  $z$  to  $z^{N'+1}$  is given by

$$(4.13, N') \quad \mathfrak{S}^{N'}(x, \varepsilon) = \mathfrak{S}_{(1)}(x, \varepsilon) \cdots \mathfrak{S}_{(N')}(x, \varepsilon).$$

Then, we have a desired formal power series

$$(4.14) \quad \mathfrak{S}(x, \varepsilon) = \lim_{N' \rightarrow \infty} \mathfrak{S}^{N'}(x, \varepsilon) = \sum_{k \geq 0} \mathfrak{S}_k(x) \varepsilon^k.$$

The discussion about an analytic meaning of series (4.14) will be given in Chapter III.

*Proof of Lemma 4.1.* Since

$$\begin{aligned} E^i(x, \varepsilon) &= \left( I + \sum_{|l|=N} \mathfrak{S}_l(x) \varepsilon^l \right)^{-1} A^i(x, \varepsilon) \left( I + \sum_{|l|=N} \mathfrak{S}_l(x) \varepsilon^l \right) \\ &\quad - \varepsilon^{\mu(i)} \left( I + \sum_{|l|=N} \mathfrak{S}_l(x) \varepsilon^l \right)^{-1} \frac{\partial}{\partial x_i} \sum_{|l|=N} \mathfrak{S}_l(x) \varepsilon^l, \end{aligned}$$

we derive  $E_k^i(x) = A_k^i(x)$  for  $|k| \leq N-1$ , and

$$(4.15) \quad E_k^i(x) + \mathfrak{S}_k(x) E_0^i(x) = A_k^i(x) + A_0^i(x) \mathfrak{S}_k(x)$$

for  $|k|=N$ . In particular,

$$(4.16) \quad E_0^i(x) = A_0^i(x).$$

Let us set

$$A_k^i(x) = \begin{pmatrix} A_k^{i,11}(x) & A_k^{i,12}(x) \\ A_k^{i,21}(x) & A_k^{i,22}(x) \end{pmatrix}, \quad E_k^i(x) = \begin{pmatrix} E_k^{i,11}(x) & E_k^{i,12}(x) \\ E_k^{i,21}(x) & E_k^{i,22}(x) \end{pmatrix},$$

with the same partition as before. Then it follows from (4.6, 0), (4.8), (4.15) and (4.16) that

$$E_k^{i,12} = A_0^{i,1}(x) S_k(x) - S_k(x) A_0^{i,2}(x) + A_k^{i,12}(x) \quad \text{for } |k|=N.$$

Now we shall show that  $S_k(x)$  can be chosen so that

$$(4.17, i) \quad E_k^{i,12}(x) = 0 \quad (i=1, \dots, n)$$

for  $|k|=N$ . To do this, observe that

$$(4.18) \quad \frac{\partial}{\partial x_i} A_{k-\mu(i)}^i(x) - \frac{\partial}{\partial x_j} A_{k-\mu(j)}^i(x) = \sum_{k'+k''=k} [A_{k'}^i(x), A_{k''}^j(x)]$$

for  $i, j=1, \dots, n$ , (formally integrability condition of (A)) where the term  $(\partial/\partial x_i)A_{k-\mu(i)}^j(x)$  is gotten rid of if  $k-\mu(i) \geq 0$ . From (4.6, 0), (4.6, N) and (4.18), we derive

$$(4.19) \quad A_k^{i,12}(x)A_0^{j,2}(x) - A_0^{i,1}(x)A_k^{i,12}(x) + A_0^{i,1}(x)A_k^{j,12}(x) - A_k^{j,12}(x)A_0^{i,2}(x) = 0$$

for  $|k|=N, i, j=1, \dots, n$ ,

$$(4.20) \quad A_0^{i,1}(x)A_0^{j,1}(x) - A_0^{i,1}(x)A_0^{i,1}(x) = 0,$$

$$(4.21) \quad A_0^{i,2}(x)A_0^{j,2}(x) - A_0^{i,2}(x)A_0^{i,2}(x) = 0,$$

for  $i, j=1, \dots, n$ . Set

$$\mathfrak{A}^i(x) = \begin{pmatrix} A_0^{i,1}(x) & A_k^{i,12}(x) \\ 0 & A_0^{i,2}(x) \end{pmatrix}$$

for  $i=1, \dots, n, |k|=N$ . Then, from (4.19), (4.20) and (4.21), we obtain

$$(4.22) \quad [\mathfrak{A}^i(x), \mathfrak{A}^j(x)] = 0,$$

and

$$(4.23) \quad [\mathfrak{A}^i(0), \mathfrak{A}^j(0)] = 0,$$

for  $i, j=1, \dots, n$ . On the other hand, by virtue of (4.4), (4.23), and ASSUMPTION  $\mathfrak{A}$ , there exists a nonsingular matrix  $U = \begin{pmatrix} P & R \\ 0 & Q \end{pmatrix}$  such that  $U^{-1}\mathfrak{A}^i(0)U = \begin{pmatrix} \mathfrak{A}_1^i & 0 \\ 0 & \mathfrak{A}_2^i \end{pmatrix}$  with the same partition as before, where  $\mathfrak{A}_1^i$  and  $\mathfrak{A}_2^i$  are of upper-triangular form, and have the eigenvalues same as those of  $A_0^{i,1}(0)$  and  $A_0^{i,2}(0)$ , respectively. Applying the Theorem 2.1 to  $U^{-1}\mathfrak{A}^i(x)U$  ( $i=1, \dots, n$ ), we obtain  $T(x) = \begin{pmatrix} I_p & T^0(x) \\ 0 & I_{m-p} \end{pmatrix}$

such that  $T^{-1}(x)U^{-1}\mathfrak{A}^i(x)UT(x) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$ , i.e.

$$-(PT^0(x)Q^{-1} + RQ^{-1})A_0^{i,2}(x) + A_0^{i,1}(x)(PT^0(x)Q^{-1} + RQ^{-1}) + A_k^{i,12}(x) = 0,$$

for  $i=1, \dots, n, |k|=N$ . This implies that

$$S_k(x) = PT^0(x)Q^{-1} + RQ^{-1}$$

satisfies (3.17,  $i$ ) for all  $i=1, \dots, n$ , simultaneously. Furthermore, if we regard

$$(4.17, 1) \quad A_0^{i,1}(x)S_k(x) - S_k(x)A_0^{i,2}(x) + A_k^{i,12}(x) = 0$$

as linear equations for the  $p(m-p)$  components of  $S_k(x)$ , it is easy to see that this

has a unique solution  $S_k(x)$  holomorphic and bounded for  $x \in U(r_0)$ , where  $r_0$  is chosen so that  $A_0^{1,1}(x)$  and  $A_0^{1,2}(x)$  have no common eigenvalues if  $x \in \overline{U(r_0)}$ . Therefore  $S_k(x)$  is holomorphic and bounded for  $x \in U(r_0)$ ,  $r_0$  being a positive constant independent of  $N$ .

Finally, the formal integrability of  $(E)$  is verified by a simple computation. This completes the proof of the lemma.

**§ 5. Sketch of the proof for Theorem 3.2**

The desired matrix  $\mathfrak{F}(x, \varepsilon)$  can be given such that

$$\mathfrak{F}(x, \varepsilon) = \sum_{k \geq 0} \mathfrak{F}_k(x) \varepsilon^k = \prod_{N=1}^{\infty} \mathfrak{F}_{(N)}(x, \varepsilon).$$

Here, the matrices

$$\mathfrak{F}_{(N)}(x, \varepsilon) = I + \prod_{|i|=N}^{\infty} \mathfrak{F}_i(x) \varepsilon^i$$

with

$$\mathfrak{F}_i(x) = \begin{pmatrix} 0 & 0 \\ T_i(x) & 0 \end{pmatrix}$$

are determined inductively as follows.

Assume that the transformation

$$u = \mathfrak{F}_{(1)}(x, \varepsilon) \cdots \mathfrak{F}_{(N)}(x, \varepsilon) u^N$$

changes system  $(B)$  into

$$(B)_N \quad du^N = \left( \sum_{i=1}^n \frac{B^{i,N}(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) u^N \quad ((B)_0 = (B))$$

with

$$B^{i,N}(x, \varepsilon) = \sum_{k \geq 0} B_k^{i,N}(x) \varepsilon^k,$$

$$B_k^{i,N}(x) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \text{for } |k| \leq N.$$

Then, in the same way as for the case of  $\mathfrak{S}(x, \varepsilon)$ , one can determine  $\mathfrak{F}_{(N+1)}(x, \varepsilon)$  so that the transformation  $u^N = \mathfrak{F}_{(N+1)}(x, \varepsilon) u^{N+1}$  changes  $(B)_N$  into

$$(B)_{N+1} \quad du^{N+1} = \left( \sum_{i=1}^n \frac{B^{i,N+1}(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) u^{N+1}$$

with

$$B^{i,N+1}(x, \varepsilon) = \sum_{k \geq 0} B_k^{i,N+1}(x) \varepsilon^k,$$

$$B_k^{i,N+1}(x) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \text{for } |k| \leq N+1.$$

Thus we can construct a desired formal power series  $\mathfrak{L}(x, \varepsilon)$ .

### Chapter III. Block-diagonalization of a Pfaffian system

Under the assumptions stated in the introductions, we study a completely integrable system

$$(A) \quad dz = \left( \sum_{i=1}^n \frac{A^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) z,$$

and want to prove Theorem A. In Chapter II, under the same assumptions, we have already proved the existence of a formal transformation of the form

$$(T) \quad z = \mathfrak{S}(x, \varepsilon) \mathfrak{L}(x, \varepsilon) v$$

by which system (A) can be block-diagonalized as follows

$$(C) \quad dv = \left( \sum_{i=1}^n \frac{C^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) v.$$

Here the matrices  $C^i(x, \varepsilon)$  are formally expressed by

$$C^i(x, \varepsilon) = \sum_{k \geq 0} C_k^i(x) \varepsilon^k$$

where coefficient matrices  $C_k^i(x)$  are holomorphic and of block-diagonal form with the same partition as the  $A_0^i(0)$ . Since  $\mathfrak{S}(x, \varepsilon)$  and  $\mathfrak{L}(x, \varepsilon)$  are expressed by formal power series in  $\varepsilon$ , system (C) has only a formal meaning. We shall show that the transformation (T) has an analytic meaning.

#### § 6. Analytic block-diagonalization

Analytic validity of transformation (T) is clarified in the theorems below.

**Theorem 6.1.** *Let there be given a completely integrable system (A) which satisfies the assumptions stated in the introduction. Take any direction  $\arg \varepsilon = \theta$  contained in the sector  $\sum (\varepsilon_0, \varphi, \alpha) \subset \mathbb{C}^v$ . Then there exists a holomorphic matrix function  $\mathfrak{S}(x, \varepsilon)$  defined for*

$$(6.1) \quad (x, \varepsilon) \in U(r') \times \Sigma(\varepsilon'_0, \theta, \alpha')$$

with the properties that:

(i)  $\mathfrak{S}(x, \varepsilon)$  admits a uniformly asymptotic expansion to the formal series appearing in Theorem 3.1,

$$(6.2) \quad \mathfrak{S}(x, \varepsilon) \simeq \sum_{k \geq 0} \mathfrak{S}_k(x) \varepsilon^k$$

as  $\varepsilon$  tends to zero through the sector  $\Sigma(\varepsilon'_0, \theta, \alpha')$ ,

(ii) the transformation

$$(6.3) \quad z = \mathfrak{S}(x, \varepsilon)u$$

changes system (A) into a completely integrable system

$$(B) \quad du = \left( \sum_{i=1}^n \frac{B^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) u,$$

(ii, a) the matrices  $B^i(x, \varepsilon)$ 's are holomorphic for (6.1) and of lower block-triangular form

$$(6.4) \quad B^i(x, \varepsilon) = \begin{pmatrix} B^{i,11}(x, \varepsilon) & 0 \\ B^{i,21}(x, \varepsilon) & B^{i,22}(x, \varepsilon) \end{pmatrix},$$

(ii, b) we have uniformly asymptotic expansions

$$(6.5) \quad B^i(x, \varepsilon) \simeq \sum_{k \geq 0} B_k^i(x) \varepsilon^k \quad (\varepsilon \rightarrow 0)$$

for (6.1), where the  $B_k^i(x)$ 's are holomorphic and bounded matrix functions of  $x$  in  $U(r'_0)$ .

**Theorem 6.2.** Under the same assumptions as in Theorem 6.1, system (B) can be transformed into a completely integrable system

$$(C) \quad dv = \left( \sum_{i=1}^n \frac{C^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) v$$

by applying a transformation

$$(6.6) \quad u = \mathfrak{L}(x, \varepsilon)v$$

with the properties that:

(i)  $\mathfrak{L}(x, \varepsilon)$  is holomorphic for

$$(6.7) \quad (x, \varepsilon) \in U(r''_0) \times \Sigma(\varepsilon''_0, \theta, \alpha'')$$

and admits a uniformly asymptotic expansion to the formal series appearing in Theorem 3.2,

$$(6.8) \quad \mathfrak{F}(x, \varepsilon) \simeq \sum_{k \geq 0} \mathfrak{F}_k(x) \varepsilon^k \quad (\varepsilon \rightarrow 0)$$

for (6.7),

(ii) the  $C^i(x, \varepsilon)$ 's are block-diagonal matrices with the same partition

$$(6.9) \quad C^i(x, \varepsilon) = \text{diag} [C^{i,1}(x), C^{i,2}(x)]$$

defined for (6.7), and have uniformly asymptotic expansions

$$C^i(x, \varepsilon) \simeq \sum_{k \geq 0} C_k^i(x) \varepsilon^k \quad (\varepsilon \rightarrow 0)$$

for (6.7).

If Theorems 6.1 and 6.2 are once established, Theorem A can be derived easily. So we shall prove these two theorems in the remaining part of this paper.

### §7. Proof of Theorem 6.1

By applying, if necessary, a nonsingular and holomorphic transformation of the form

$$z = S(x)w$$

we can assume without loss of generality that every matrix  $A_0^i(x)$  has a block-diagonal one with the same partition for every index  $i$ . So we consider system (A) with block-diagonal matrices  $A_0^i(x)$ .

As was already shown in the proof of Theorem 3.1, a formal transformation of the form

$$z = \mathfrak{S}(x, \varepsilon)u \quad \mathfrak{S}(x, \varepsilon) = \begin{pmatrix} I_p & S(x, \varepsilon) \\ 0 & I_{m-p} \end{pmatrix}$$

changes system (A) into a system of the form

$$du = \left( \sum_{i=1}^n \frac{B^i(x, \varepsilon)}{\varepsilon^{\mu(i)}} dx_i \right) u,$$

$$B^i(x, \varepsilon) = \begin{pmatrix} B^{i,11}(x, \varepsilon) & 0 \\ B^{i,21}(x, \varepsilon) & B^{i,22}(x, \varepsilon) \end{pmatrix}.$$

By a direct calculation, we see that the equation for determining a matrix  $S(x, \varepsilon)$  is given by

$$(S) \quad \varepsilon^{\mu(i)} \frac{\partial S}{\partial x_i} = A^{i,12}(x, \varepsilon) + A^{i,11}(x, \varepsilon)S - SA^{i,22}(x, \varepsilon) - SA^{i,21}(x, \varepsilon)S$$

where  $A^{i,\alpha\beta}(x, \varepsilon)$  denotes the  $(\alpha, \beta)$ -block of the matrix  $A^i(x, \varepsilon)$  in our partition.

Since system (A) is completely integrable, we have identities

$$\varepsilon^{\mu(j)} \frac{\partial A^i}{\partial x_j} + A^i A^j = \varepsilon^{\mu(i)} \frac{\partial A^j}{\partial x_i} + A^j A^i.$$

This relation is also written by relations among the  $A^{i,\alpha\beta}(x, \varepsilon)$ 's. Then, a simple computation shows that system (S) is also completely integrable.

On the other hand, as was already proved, system (S) is formally solved. Namely, there exists a formal solution of the form

$$(7.1) \quad S = \sum_{k>0} S_k(x) \varepsilon^k$$

with  $S_k(x)$  holomorphic and bounded for  $x \in U(r_0)$ .

The following facts should be noticed.

- (S.i) System (S) is a completely integrable Pfaffian system.
- (S.ii) System (S) is nonlinear.
- (S.iii) System (S) possesses a formal solution such that

$$S(x, 0) = 0.$$

(S.iv) The leading term of the right-hand member of (S) is  $A_0^{i,1}(0)S - SA_0^{i,2}(0)$ . If we regard (S) as  $p(m-p)$ -vector equation with respect to the  $p(m-p)$  unknown components of  $S$ , by virtue of ASSUMPTION  $\mathfrak{A}$ , Lemma 2.1, and the completely integrability condition of (S), one can assume that, for every  $i$ , the Jacobian matrix with respect to  $S$  is nonsingular and of upper-triangular form.

Therefore, the problem of studying an analytic meaning of the formal solution (7.1) is reduced to the problem of solving the following nonlinear problem.

**Lemma 7.1.** *Let there be given a completely integrable nonlinear Pfaffian system*

$$(7.2) \quad dw = \sum_{i=1}^n \frac{1}{\varepsilon^{\mu(i)}} f^i(x, w, \varepsilon) dx_i$$

where we assume that

- (1)  $w$  is an  $M$ -vector with components  $\{w_j\}$ , the  $\mu(i)$ 's are multi-indices with positive components.
- (2) The  $f^i(x, w, \varepsilon)$ 's are  $M$ -vectors whose components are holomorphic and bounded functions for

$$(7.3) \quad (x, w, \varepsilon) \in U(r_0) \times \{w \mid \|w\| < w_0\} \times \sum (\varepsilon_0, \varphi, \alpha).$$

Moreover, we have asymptotic expansions in powers of

$$(7.4) \quad f^i(x, w, \varepsilon) \simeq \sum_{k \geq 0} f_k^i(x, w) \varepsilon^k$$

uniformly for  $x \in U(r_0)$ ,  $\|w\| < w_0$ , as  $\varepsilon$  tends to zero through the sector  $\Sigma(\varepsilon_0, \varphi, \alpha)$ . The coefficients  $f_k^i(x, w)$  are holomorphic and bounded vector functions for  $x \in U(r_0)$ ,  $\|w\| < w_0$ .

(3) For every  $i$ , the Jacobian matrix  $A^i$  defined by

$$A^i = \frac{\partial f^i}{\partial w}(0, 0, 0)$$

is nonsingular and of upper triangular form.

(4) There exists a formal solution of the form

$$(7.5) \quad w = \sum_{|k| \geq 1} c_k(x) \varepsilon^k$$

with  $c_k(x)$  holomorphic and bounded in  $U(r)$ .

Then, for any direction  $\arg \varepsilon = \theta$  in the sector  $\Sigma(\varepsilon_0, \varphi, \alpha)$ , system (7.2) possesses an actual solution  $w = \phi(x, \varepsilon)$  which is holomorphic and bounded for

$$(7.6) \quad (x, \varepsilon) \in U(r') \times \Sigma(\varepsilon'_0, \theta, \alpha')$$

and admits a uniformly asymptotic expansion to the formal solution (7.5) for (7.6) as  $\varepsilon$  tends to zero through the sector  $\Sigma(\varepsilon'_0, \theta, \alpha')$ .

By the help of this lemma, analytic validity of a formal block-triangularization, namely Theorem 6.1 has been now established.

*Proof of Lemma 7.1.* By virtue of Proposition 1.5, there exists a vector function  $c(x, \varepsilon)$  such that

$$c(x, \varepsilon) \simeq \sum_{|k| \geq 1} c_k(x) \varepsilon^k$$

uniformly for  $x \in U(r_0)$  as  $\varepsilon$  tends to zero in  $\Sigma(\varepsilon_0, \varphi, \alpha)$ . Then the transformation  $z = w - c(x, \varepsilon)$  changes system (7.2) into

$$(7.7) \quad dz = \sum_{i=1}^n \frac{1}{\varepsilon^{\mu(i)}} (A^i z + p^i(x, z, \varepsilon)) dx_i$$

with

$$p^i(x, z, \varepsilon) = a^i(x, \varepsilon) + G^i(x, \varepsilon)z + g^i(x, z, \varepsilon),$$

where  $G^i(x, \varepsilon)$  are  $M$  by  $M$  matrices such that

$$G^i(x, \varepsilon) = 0(\|x\| + \|\varepsilon\|),$$

and  $a^i(x, \varepsilon)$  and  $g^i(x, z, \varepsilon)$  are  $M$ -vectors such that

$$a^i(x, \varepsilon) \simeq 0 \quad \text{uniformly for } x \in U(r_0) \text{ in } \Sigma(\varepsilon_0, \varphi, \alpha)$$

$$g^i(x, z, \varepsilon) = 0(\|z\|^2).$$

Let us set

$$V(x) = \prod_{i=1}^n \exp[\varepsilon^{-\mu(i)} A_i x^i].$$

Then, by virtue of the integrability condition of (7.7), the solution of the integral equation

$$(7.8) \quad z = \sum_{i=1}^n \int_{\Gamma_i(x, a)} V(x) V^{-1}(\xi) \varepsilon^{-\mu(i)} p^i(\xi, z(\xi, \varepsilon), \varepsilon) d\xi_i$$

satisfies system (7.7). Here, the path of integration  $\Gamma_i(x, a)$  will be given later such that  $\Gamma_1(x, a) \cup \dots \cup \Gamma_n(x, a)$  ends in  $x$ . Therefore, it is sufficient to prove that the equation (7.8) possesses a solution  $z(x, \varepsilon)$  such that

i)  $z(x, \varepsilon)$  is holomorphic for  $U(r') \times \Sigma(\varepsilon'_0, \theta, \alpha')$ ,  $r', \varepsilon'_0$ , and  $\alpha' = (\alpha'_j)$  being sufficiently small positive constants,

ii)  $z(x, \varepsilon) \simeq 0$  uniformly for  $x \in U(r')$ , as  $\varepsilon$  tends to zero in  $\Sigma(\varepsilon'_0, \theta, \alpha')$ .

If we choose the paths of integration as follows, the proof of the above assertion is very similar to that of Theorem 26.1 of [4]. So the proof of this is omitted.

*Determination of the paths of integration.* Let us set

$$A^i = \text{diag}[\rho_1^i, \dots, \rho_M^i] + N^i,$$

where  $N^i$  is nilpotent and of upper-triangular form. Since  $A^i$  ( $i=1, \dots, n$ ) are nonsingular, there exist real constants  $\theta_0^i$  such that, for  $\varepsilon \in \{\varepsilon | \arg \varepsilon = \theta\}$ ,

$$\cos(\arg(\varepsilon^{-\mu(i)} \rho_l^i \exp(\sqrt{-1} \theta_0^i))) \neq 0 \quad (l=1, \dots, M).$$

Hence, we can choose sufficiently small positive constants  $\alpha'_1, \dots, \alpha'_n, \Delta^1, \dots, \Delta^n$ , and  $\beta$  such that

$$|\cos(\arg(e^{-\mu(i)} \rho_l^i \exp(\sqrt{-1} \theta_0^i)))| > \beta \cdot |\rho_l^i|^{-1} > 0$$

if  $\varepsilon \in \{\varepsilon | |\arg \varepsilon - \theta| < \alpha'\} \subset \{\varepsilon | |\arg \varepsilon - \varphi| < \alpha'\}$ ,  $\alpha' = (\alpha'_1, \dots, \alpha'_n)$ ,

$$|\theta^i - \theta_0^i| < \Delta^i \quad (i=1, \dots, n).$$

Then, we set

$$U^0 = \{x | |\arg(a \pm x) - \theta_0| < \Delta\}^{a, *}$$

$$\Sigma^0 = \{\varepsilon | |\arg \varepsilon - \theta| < \alpha'\},$$

$$\gamma_+(x_i) = \{\xi_i | \xi_i = tx_i + (1-t)a_i, 0 \leq t \leq 1\},$$

$$\gamma_-(x_i) = \{\xi_i | \xi_i = tx_i + (1-t)(-a_i), 0 \leq t \leq 1\},$$

\*)  $A^a$  denotes the closure of  $A$ .

where

$$\Theta_0 = (\Theta_0^1, \dots, \Theta_0^n), \quad \Delta = (\Delta^1, \dots, \Delta^n)$$

and  $a = (a_1, \dots, a_n)$  is a point contained in  $U(r_0)$  such that  $\arg a = \Theta_0$ . Furthermore, for  $(x, \varepsilon) \in U^0 \times \Sigma^0$ , we put

$$\begin{aligned} L_+(i) &= \{l | \operatorname{Re}(\varepsilon^{-\mu(i)} \rho_i^l(x_i - \xi_i)) < -\beta |\varepsilon^{-\mu(i)}(x_i - \xi_i)| \text{ for } \xi_i \in \gamma_+(x_i)\}, \\ L_-(i) &= \{l | \operatorname{Re}(\varepsilon^{-\mu(i)} \rho_i^l(x_i - \xi_i)) < -\beta |\varepsilon^{-\mu(i)}(x_i - \xi_i)| \text{ for } \xi_i \in \gamma_-(x_i)\}. \end{aligned}$$

Note that  $L_+(i) \cup L_-(i) = \{1, \dots, M\}$ .

Let  $\gamma_i^l(x, a)$  be a line segment from

$$(x_1, \dots, x_{i-1}, a(i, l), a(i+1, l), \dots, a(n, l))$$

to

$$(x_1, \dots, x_{i-1}, x_i, a(i+1, l), \dots, a(n, l)),$$

with

$$a(i, l) = \begin{cases} a_i & l \in L_+(i) \\ -a_i & l \in L_-(i). \end{cases}$$

Then, for each  $i$ , the path of integration  $\Gamma_i(x, a)$  is given as a set of  $M$  paths  $\gamma_i^l(x, a)$ , which is used in the integration of the  $l$ -th component of the integrand vector. Thus the paths of integration are determined.

### § 8. Outline of the proof of Theorem 6.2

In the same way as for the matrix  $\mathfrak{S}(x, \varepsilon)$ , it is shown that the transformation  $u = \mathfrak{X}(x, \varepsilon)v$  with  $\mathfrak{X}(x, \varepsilon) = \begin{pmatrix} I_p & 0 \\ T(x, \varepsilon) & I_{m-p} \end{pmatrix}$  changes system (B) into system (C), if  $T(x, \varepsilon)$  satisfies the nonlinear Pfaffian system

$$(8.1) \quad \varepsilon^{\mu(i)} \frac{\partial T}{\partial x_i} = B^{i,21}(x, \varepsilon) + B^{i,22}(x, \varepsilon)T - TB^{i,11}(x, \varepsilon).$$

This system satisfies the conditions which are similar to conditions (S.i), (S.ii), (S.iii) and (S.iv). Hence, applying Lemma 7.1 to system (8.1), we conclude that  $\mathfrak{X}(x, \varepsilon)$  have the properties stated in Theorem 6.2.

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(Ricevita la 18-an de januaro, 1978)  
(Reviziita la 23-an de oktobro, 1978)