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Well-Posedness and Stability of Linear Volterra Integrodifferential Equations in Abstract Spaces

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§ 1. Introduction

We study an abstract Volterra integrodifferential equation of the form

$$(E) \qquad x'(t)=cx(t)+\int_0^t K(t-\tau)x(\tau)d\tau+F(t), \qquad x(0)=x_0,$$

in a Banach or Hilbert space X. Here $'$ denotes the strong derivative with respect to the variable $t\in R^+=[0, \infty)$, $F: R^+\to X$ and c is a real scalar. For each $t\in R^+$, $K(t)$ is a linear operator with domain dense in X. In particular, we will consider the case with $K(t)=B(t)A+G(t)$ where A is the infinitesimal generator of a strongly continuous cosine family. Here $B(t)$ and $G(t)$ are real-valued functions of the form

$$B(t)=\beta+\int_0^t b(\tau)d\tau, \qquad G(t)=\gamma+\int_0^t g(\tau)d\tau$$

with b, g in $C^1(R^+)\cap L^1(R^+)$, and $\beta>0$ and γ real constants. If $K(t), F(t)$ and the solution $x(t)$ are sufficiently smooth, then the differentiated form of (E) is

$$(E_1) \qquad x''(t)=cx'(t)+Lx(t)+\int_0^t K_0(t-\tau)x(\tau)d\tau+f(t), \quad x(0)=x_0, \quad x'(0)=v_0,$$

where $L=K(0)$, $K_0(t)x=K'(t)x$ for each x in the domain of $K(t)$, $f(t)=F'(t)$ and $v_0=cx_0+F(0)$.

In Sections 2 and 3 we obtain conditions which imply that solutions $x(t)$ of (E) depend continuously on the pair (x_0, F). Here we also deal with the question of the existence of a unique solution to equation (E) on R^+. This work is motivated by the well-posedness results in Miller [13] and Grimmer and Miller [6].

Sections 4 and 5 deal with questions of stability and asymptotic behavior as $t\to\infty$ of solutions of equation (E). In this part of the paper we restrict our atten-

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tion to the case where the abstract space X is a Hilbert space.

An equation of the form (E) arises in a model for heat flow in materials with memory proposed by Gurtin and Pipkin [7]. This model exhibits certain behavior of a hyperbolic nature. An alternate parabolic model for heat flow proposed by Coleman and Gurtin [1] has been studied by many authors, see Miller [14] and its bibliography. Studies of well-posedness and stability similar to those here, but for abstract integrodifferential equations having parabolic form, appear in papers by Friedman and Shinbrot [5], and, more recently, Miller [13] and Miller and Wheeler [15].

Dafermos [2] has used Lyapunov techniques to investigate stability properties of equations in Hilbert space which have the same form as those studied in this paper. Also, in a series of papers (see [8], [9] and their bibliographies) Hannsgen has analyzed uniform stability properties of scalar integrodifferential equations with a parameter, and used these results to obtain stability results for equations in Hilbert space having the form of the equations studied here. In the work of Dafermos and Hannsgen it is assumed that the scalar kernels occurring in the equations studied satisfy positivity, monotonicity and certain convexity assumptions. We make no such assumptions here; instead, we assume that the corresponding kernels are integrable on $t \geq 0$. As a consequence of this we must use different techniques, and the stability results we obtain are of a different nature.

§ 2. Continuous dependence

This section deals with the question of continuous dependence of solutions $x(t)$ of equation (E) on the pair (x_0, F).

We recall the notation $L=K(0)$, $K_0(t)=K'(t)$ of the Introduction. We will assume throughout this section that the forcing function $F(t)$ belongs to $C^1(R^+; X)$, and that

(A1) The operator L is a closed linear operator with domain $D(L)$ dense in X while for each $t \in R^+$, the domain of the linear operator $K(t)$ contains $D(L)$. Moreover, for each $x \in D(L)$, $K(t)x \in C^1(R^+; X)$, and there exist continuous positive functions κ and κ_0 with $\kappa \in L^\infty(R^+)$ and $\kappa_0 \in L^1(R^+)$ so that

$$\|K(t)x\| \leq \kappa(t)\,\|x\|_L, \qquad \|K_0(t)x\| \leq \kappa_0(t)\,\|x\|_L \tag{2.1}$$

for all $t \in R^+$ and all $x \in D(L)$.

Here $\|x\|_L \equiv \|x\| + \|Lx\|$ denotes the graph norm.

Let BU denote the Banach space $BU=\{f \in C(R^+; X): f$ is bounded and uniformly continuous$\}$ with norm $\|f\|=\sup\{\|f(t)\|: t \in R^+\}$. Define the Banach space $Z=X\times X\times BU\times BU$ with the norm $\|(x, u, f, F)\|=\|x\|+\|u\|+\|f\|+\|F\|$. Define the linear map C on the domain

$$D(C)=\{(x, u, f, F) \in Z: x \text{ and } F(0) \in D(L),\ F \in C^2,\ F' \in BU,\ F''+K(\cdot)[cx+F(0)]+K_0(\cdot)x \in BU\}$$

by

(2.2) $$C(x, u, f, F) = ([L+c^2I]x+cF(0)+F'(0),\ LF(0),\ F',\ F''+K(\cdot)[cx+F(0)]+K_0(\cdot)x).$$

Here I denotes the identity operator.

We begin by showing

Lemma 2.1. *The operator C is closed.*

Proof. Let $z_n=(x_n, u_n, f_n, F_n) \in D(C)$, $z_n \to z_0 = (x_0, u_0, f_0, F_0) \in Z$ and $Cz_n \to (y, w, g, G)$ in Z. Since L is closed and $LF_n(0) \to w$, we see that $F_0(0) \in D(L)$ and $LF_0(0)=w$. Also, since L is closed, we find that $x_0 \in D(L)$ and $Lx_0=y-c^2x_0-cF_0(0)-g(0)$.

Next we observe that

(2.3) $$F_n''(t) \to G(t)-K(t)[cx_0+F_0(0)]-K_0(t)x_0, \qquad n \to \infty,$$

uniformly for t on compact intervals in R^+. To verify (2.3) we use the bounds (2.1) to write

$$\begin{aligned} &\|F_n''(t)-G(t)+K(t)[cx_0+F_0(0)]+K_0(t)x_0\| \\ &\quad \le \|F_n''-G+K(\cdot)[cx_n+F_n(0)]+K_0(\cdot)x_n\| \\ &\qquad +\kappa(t)\,\|c(x_n-x_0)+F_n(0)-F_0(0)\|_L+\kappa_0(t)\,\|x_n-x_0\|_L, \end{aligned}$$

and (2.3) follows by letting $n \to \infty$. Thus, we apply the fundamental theorem, and use (2.3) to take limits under the integral to obtain

$$g(t)=g(0)+\int_0^t \{G(\tau)-K(\tau)[cx_0+F_0(0)]-K_0(\tau)x_0\}d\tau.$$

A second use of the fundamental theorem and passage to the limit yields

$$F_0(t)=F_0(0)+tg(0)+\int_0^t\int_0^s \{G(\tau)-K(\tau)[cx_0+F_0(0)]-K_0(\tau)x_0\}d\tau ds.$$

We can differentiate this last line to see that $F_0 \in C^2$, $F_0'=g \in BU$ and $F_0''+K(\cdot)[cx_0+F_0(0)]+K_0(\cdot)x_0=G \in BU$. Thus, $z_0 \in D(C)$ and $Cz_0=(y, w, g, G)$; hence, C is closed. Q.E.D.

Hext we show

Lemma 2.2. *$D(C)$ is dense in Z.*

Proof. Let $z=(x, u, f, F) \in Z$ and $\varepsilon>0$. We must find $z_0=(x_0, u, f, F_0) \in D(C)$ so that $\|z-z_0\|<\varepsilon$. Since $\overline{D(L)}=X$, pick $x_0 \in D(L)$ so that $\|x-x_0\|<\varepsilon/2$. To find an F_0, we begin by using the denseness of $D(L)$ to pick $\alpha \in D(L)$ satisfying $\|F(0)-\alpha\|<\varepsilon/6$. Next, let $\xi(t)$ be a nonnegative function so that $\xi \in C^\infty(-\infty, \infty)$, $\xi(0)=1$, $\xi(t)=0$ for $|t|\geq 1$ and $\xi'(-1)=\xi'(0)=0$. Set $\phi(t)=[cx_0+\alpha]te^t+\xi(t)x_0$, and define

$$g(t)=\int_t^\infty K(\tau)\phi(t-\tau)d\tau.$$

It follows using the first inequality in (2.1) and the definition of ϕ, that $g \in BU$. Moreover,

$$g'(t)=-K(t)x_0+\int_t^\infty K(\tau)\phi'(t-\tau)d\tau,$$

$$g''(t)=-K_0(t)x_0-K(t)[cx_0+\alpha]+\int_t^\infty K(\tau)\phi''(t-\tau)d\tau.$$

Hence, g' and $g''+K(\cdot)[cx_0+\alpha]+K_0(\cdot)x_0$ lie in BU. Let ψ be a C^2-function with $\psi^{(j)} \in BU$ for $j=0, 1, 2$, and $\|F-g-\psi\|<\varepsilon/6$. (For example, let $m(t)$ be a non-negative, scalar, C^∞-function with support in $[-1, 1]$ such that

$$\int_{-1}^1 m(\tau)d\tau=1.$$

Extend the domain of definition of $F-g$ to $(-\infty, \infty)$ by setting $F(t)-g(t)=F(0)-g(0)$ when $t\leq 0$, and mollify $F-g$ to obtain

$$\psi_\delta(t)=\delta^{-1}\int_{-\infty}^\infty m((t-\tau)/\delta)[F(\tau)-g(\tau)]d\tau$$

for each $\delta>0$. Then $\psi_\delta^{(j)} \in BU$ for $j=0, 1, 2$, and $\|F-g-\psi_\delta\|<\varepsilon/6$ for δ sufficiently small.)

Define $F_0(t)=\alpha+\psi(t)-\psi(0)+g(t)-g(0)$. Then $F_0 \in BU$, $F_0(0)=\alpha \in D(L)$, $F_0' \in BU$ and $F_0''+K(\cdot)[cx_0+F_0(0)]+K_0(\cdot)x_0 \in BU$. Also,

$$\begin{aligned}\|F-F_0\| &\leq \|F-\psi-g\|+\|\alpha-\psi(0)-g(0)\| \\ &<\varepsilon/6+\|\alpha-F(0)\|+\|F(0)-\psi(0)-g(0)\| \\ &<\varepsilon/2.\end{aligned}$$

Thus, $z_0=(x_0, u, f, F_0) \in D(C)$ and $\|z-z_0\|<\varepsilon$. Q.E.D.

Define

$$\rho(\lambda)=[L+K_0^*(\lambda)+cK^*(\lambda)+c^2I-\lambda^2I]^{-1}$$

at all points λ with Re $\lambda > 0$ for which the inverse exists as a bounded linear map on X. Here * denotes the Laplace transform so that, for example,

$$K_0^*(\lambda)x = \int_0^\infty K_0(\tau)x \exp(-\lambda\tau)d\tau$$

for all $x \in D(L)$ and all points λ with Re $\lambda \geq 0$.

Lemma 2.3. *Suppose $\rho(\lambda)$ exists for some λ with* Re $\lambda > 0$. *Then, given $z \in Z$, there exists $z_1 \in D(C)$ so that*

$$(C - \lambda^2 I)z_1 = z. \tag{2.4}$$

Moreover, if Re λ *is sufficiently large, z_1 can be chosen so that*

$$\|z_1\| \leq M(\lambda)\,\|z\| \tag{2.5}$$

where $M(\lambda)$ does not depend on z.

Proof. Suppose that λ, Re $\lambda > 0$, is so that $\rho(\lambda)$ exists. Given $z = (x, u, f, F) \in Z$, we must find $z_1 = (x_1, u_1, f_1, F_1)$ in $D(C)$ so that (2.4) holds. Set $F_1(0) = 0$, $x_1 = \rho(\lambda)[F^*(\lambda) + x]$, and $F_0'(0) = \lambda^2 x_1 - Lx_1 - c^2 x_1 + x$. Also, define $u_1 = -u/\lambda^2$ and $f_1 = (F_1' - f)/\lambda^2$ where

$$\begin{aligned} 2\lambda F_1(t) = {} & e^{\lambda t}\left\{F_1'(0) + \int_0^t e^{-\lambda\tau}G(\tau)d\tau\right\} \\ & - e^{-\lambda t}\left\{F_1'(0) + \int_0^t e^{\lambda\tau}G(\tau)d\tau\right\}, \end{aligned}$$

with

$$G(t) = F(t) - K(t)cx_1 - K_0(t)x_1.$$

Our choice of x_1 and $F_1'(0)$ yields $F_1'(0) + G^*(\lambda) = 0$; hence, we can rewrite $F_1(t)$ as

$$\begin{aligned} 2\lambda F_1(t) = {} & -e^{\lambda t}\int_t^\infty e^{-\lambda\tau}G(\tau)d\tau \\ & - e^{-\lambda t}\left\{F_1'(0) + \int_0^t e^{\lambda\tau}G(\tau)d\tau\right\}. \end{aligned} \tag{2.6}$$

It is easy to verify that this $z_1 \in D(C)$ and that equation (2.4) holds.

It remains to show that if $\sigma = \text{Re}\,\lambda$ is large enough, then there exists a constant $M(\lambda)$ which does not depend on z so that (2.5) holds for this choice of z_1. We begin by finding a constant $M_1(\lambda)$ which is independent of z so that

$$\|F_1'(0)\| \leq M_1(\lambda)\,\|z\|. \tag{2.7}$$

To obtain (2.7), recall that $F_1'(0)=-G^*(\lambda)$. Then, using the definition of $G(t)$, the bounds (2.1), and the definition $F_1'(0)=\lambda^2x_1-Lx_1-c^2x_1+x$, we obtain

$$\begin{aligned}\|F_1'(0)\| &\le \|F\|\,\sigma^{-1}+[|c|\,\kappa^*(\sigma)+\kappa_0^*(\sigma)]\,\|x_1\|_L\\ &\le \|F\|\,\sigma^{-1}+[|c|\,\kappa^*(\sigma)+\kappa_0^*(\sigma)][(1+|\lambda|^2+|c|^2)\,\|x_1\|+\|x\|+\|F_1'(0)\|].\end{aligned}$$

For σ sufficiently large, $|c|\,\kappa^*(\sigma)+\kappa_0^*(\sigma)<1/2$; hence, using the definition of x_1, (2.7) follows with

$$M_1(\lambda)=2\sigma^{-1}+(1+|\lambda|^2+|c|^2)\,\|\rho(\lambda)\|\,(1+\sigma^{-1})+1.$$

Next, starting with equation (2.6), and using (2.1) and (2.7), we deduce that for $t\ge 0$,

$$\begin{aligned}\|F_1(t)\| &\le \Big\{e^{\sigma\tau}\int_t^\infty e^{-\sigma\tau}[\|F\|+(\kappa(\tau)\,|c|+\kappa_0(\tau))\,\|x_1\|_L]d\tau\\ &\quad +e^{-\sigma t}\Big(\|F_1'(0)\|+\int_0^t e^{\sigma\tau}[\|F\|+(\kappa(\tau)\,|c|+\kappa_0(\tau))\,\|x_1\|_L]d\tau\Big)\Big\}\Big/2\,|\lambda|\\ &\le |\lambda|^{-1}\{\sigma^{-1}\,\|F\|+(\|\kappa\|_\infty\,|c|\,\sigma^{-1}+\|\kappa_0\|_1)\,\|x_1\|_L+\|F_1'(0)\|/2\}\\ &\le |\lambda|^{-1}\{\sigma^{-1}+(\|\kappa\|_\infty\,|c|\,\sigma^{-1}+\|\kappa_0\|_1)[(1+|\lambda|^2+|c|^2)\,\|\rho(\lambda)\|\,(1+\sigma^{-1})+1+M_1(\lambda)]\\ &\qquad +M_1(\lambda)/2\}\,\|z\|\\ &=M_2(\lambda)\,\|z\|.\end{aligned}$$

Also, if we differentiate both sides of equation (2.6), we find after some manipulation that

$$F_1'(t)=-\lambda F_1(t)-e^{\lambda t}\int_t^\infty e^{-\lambda\tau}G(\tau)d\tau.$$

The second term can be bounded using the methods of the previous paragraph, and we obtain

$$\begin{aligned}\|F_1'(t)\| &\le \{|\lambda|\,M_2(\lambda)+\sigma^{-1}+(\|\kappa\|_\infty\,|c|\,\sigma^{-1}+\|\kappa_0\|_1)[(1+|\lambda|^2+|c|^2)\,\|\rho(\lambda)\|\,(1+\sigma^{-1})\\ &\qquad +M_1(\lambda)+1]\}\,\|z\|\\ &=M_3(\lambda)\,\|z\| \qquad \text{for } t\ge 0.\end{aligned}$$

Finally, using the definitions of x_1, u_1, f_1 and F_1 as well as the last two estimates, we have

$$\begin{aligned}\|z_1\| &= \|x_1\|+\|u_1\|+\|f_1\|+\|F_1\|\\ &\le \|\rho(\lambda)\|\,(1+\sigma^{-1})\,\|z\|+(\|u\|+\|F_1'\|+\|f\|)\,|\lambda|^{-2}+\|F_1\|\\ &\le \{\|\rho(\lambda)\|\,(1+\sigma^{-1})+(1+M_3(\lambda))\,|\lambda|^{-2}+M_2(\lambda)\}\,\|z\|\\ &=M(\lambda)\,\|z\|,\end{aligned}$$

where $M(\lambda)$ does not depend on z. Q.E.D.

By a *solution* of equation (E) we mean a function $x: R^+ \to D(L)$ so that $x(0)=x_0$, $x(t) \in C^1(R^+; X)$, $Lx(t) \in C(R^+; X)$ and (E) is true for all $t \geq 0$. By a *generalized solution* we mean the limit uniformly on compact subsets of $0 \leq t < \infty$ of a sequence of solutions.

We make the additional assumption.

$(A2)$ For any point $(x_0, u, f, F) \in D(C)$, there exists a unique solution $x(t, x_0, F)$ of equation (E) on R^+.

Let π_1 denote the projection onto the first coordinate of Z. For $(x_0, u, f, F) \in D(C)$ define

$$U(t)(x_0, u, f, F) = \left(x(t), u, f, F_t + \int_0^t K_{t-\tau} x(\tau) d\tau\right) \tag{2.8}$$

where $x(t)=x(t, x_0, F)$, and $F_t(s)=F(t+s)$ denotes translation. Thus, for each $t \geq 0$

$$\left(F_t + \int_0^t K_{t-\tau} x(\tau) d\tau\right)(s) = F(t+s) + \int_0^t K(t+s-\tau) x(\tau) d\tau$$

for all $s \geq 0$.

Lemma 2.4. *Let $z \in D(C^2)$ and $x(t)=\pi_1 U(t) z$. Then $x''(t)=\pi_1 U(t) Cz$ for $t \geq 0$.*

Proof. Fix $z=(x_0, u, f, F) \in D(C^2)$ and let $x(t)=\pi_1 U(t) z$. Let $X(t)$ be the solution of (E) with $X(0)=[L+c^2 I]x_0 + cF(0) + F'(0)$ and forcing function $\phi = F'' + K(\cdot)[cx_0 + F(0)] + K_0(\cdot)x$; thus, $X(t)=\pi_1 U(t) Cz$.

Let $*$ denote convolution. An application of the fundamental theorem yields

$$X(t)=[L+c^2 I]x_0 + cF(0) + F'(0) + 1 * cX(t) + 1 * K * X(t) + 1 * \phi(t).$$

Define $y' = cx_0 + F(0) + 1 * X$, $y(0)=x_0$. Then $y''(t)=X(t)$. Also,

$$\begin{aligned}
y'(t) &= cx_0 + F(0) + t\{[L+c^2 I]x_0 + cF(0) + F'(0)\} \\
&\quad + 1*1*cX(t) + 1*1*K*X(t) + 1*1*\phi(t) \\
&= cx_0 + F(0) + t\{[L+c^2 I]x_0 + cF(0) + F'(0)\} \\
&\quad + K*y(t) - 1*Kx_0 - 1*1*Kcx_0 - 1*1*KF(0) + 1*1*cX(t) \\
&\quad + 1*1*\{F''(t) + K(t)cx_0 + K(t)F(0) + K_0(t)x_0\} \\
&= cx_0 + tc^2 x_0 + tcF(0) + 1*1*cX(t) + K*y(t) + F(t) \\
&= cy(t) + K*y(t) + F(t).
\end{aligned}$$

By hypothesis (A2), (E) has a unique solution with initial value x_0 and forcing function F; hence $y(t)=x(t)$, and $x''(t)=X(t)$. Q.E.D.

Define the normed linear space

$$W=\{(x, u, f, F) \in Z: x \in D(L), F \in C^1, F' \in BU\}$$

with norm

$$\|(x, u, f, F)\|_W=\|(x, u, f, F)\|+\|[L+c^2I]x+cF(0)+F'(0)\|+\|F'\|.$$

For fixed $T>0$ define

$$C(W)=\{(y(t), u, f, F(t, \cdot)): [0, T]\to W: \text{continuous}\}$$

with norm $\|z\|_{C(W)}=\sup\{\|z(t)\|_W: 0\leq t\leq T\}$.

Lemma 2.5. *$C(W)$ is a Banach space.*

Proof. The completeness of $C(W)$ is an immediate consequence of the completeness of W. The proof of the completeness of W is a routine exercise which uses the definition of the W-norm and the fact that the operator L is closed; hence, we will omit the details. Q.E.D.

We are now ready to state and prove our result on continuous dependence.

Theorem 2.1. *Suppose $(A1)$ and $(A2)$ hold, and let $\rho(\lambda)$ exist for some λ with* $\mathrm{Re}\,\lambda>0$ *so large that inequality (2.5) of Lemma 2.3 is valid. Then the solution $x(t, x_0, F)$ of (E) depends continuously on (x_0, F) in the sense that for any $T>0$, there is an $M=M(T)>0$, such that for $0\leq t\leq T$,*

$$\|x(t, x_0, F)\|\leq M(\|x_0\|+\sup\{\|F(\tau)\|: 0\leq\tau<\infty\}) \tag{2.9}$$

for all $(x_0, u, f, F) \in D(C)$.

Remark. In the next section we exhibit a class of kernels K satisfying $(A1)$ for which the existence-uniqueness hypothesis $(A2)$ is valid, and for which $\rho(\lambda)$ exists for all sufficiently large positive λ.

Proof. Fix $T>0$. It is not difficult to check that $U: D(C)\to C(W)$ where, for $0\leq t\leq T$, $U(t)(x_0, u, f, F)$ is defined by (2.8) when $(x_0, u, f, F) \in D(C)$. Moreover, U is a closed operator from $D(C)$ with its graph norm $\|z\|_C=\|z\|+\|Cz\|$ into $C(W)$.

To verify that U is closed, first observe that since C is closed, $D(C)$ with its graph norm is complete. Thus, let the sequence $\{z_n=(x_n, u_n, f_n, F_n)\}$ and $z_0=(x_0, u_0, f_0, F_0)$ be so that $\|z_n-z_0\|_C\to 0$. Also, suppose that

$$Uz_n=(x_n(t), u_n, f_n, \mathscr{F}_n(t, \cdot))\to(y(t), v, g, \mathscr{G}(t, \cdot))$$

in $C(W)$. Here, $\mathscr{F}_n(t, \cdot)$ is defined by

$$\mathscr{F}_n(t, s) = F_n(t+s) + \int_0^t K(t+s-\tau)x_n(\tau)d\tau, \qquad s \geq 0.$$

Then, $\|x_n(t) - y(t)\| \to 0$,

$$\|\mathscr{F}_n(t, \cdot) - \mathscr{G}(t, \cdot)\| \to 0, \qquad \left\| \frac{\partial}{\partial s}\mathscr{F}_n(t, \cdot) - \frac{\partial}{\partial s}\mathscr{G}(t, \cdot) \right\| \to 0,$$

$$\left\| [L + c^2 I][x_n(t) - y(t)] + c\mathscr{F}_n(t, 0) - c\mathscr{G}(t, 0) + \frac{\partial}{\partial s}[\mathscr{F}_n(t, 0) - \mathscr{G}(t, 0)] \right\| \to 0,$$

where each convergence is uniform for t in $[0, T]$. It follows that $\|Lx_n(t) - Ly(t)\| \to 0$ uniformly for $0 \leq t \leq T$. Thus, we may integrate equation (E) for $x_n'(t)$ from 0 to t, apply the fundamental theorem, use the first bound in inequality (2.1) and let $n \to \infty$ to obtain

$$y(t) = x_0 + c\int_0^t y(s)ds + \int_0^t \int_0^s K(s-\tau)y(\tau)d\tau ds + \int_0^t F_0(s)ds.$$

Hence, y satisfies (E) with initial value x_0 and forcing function F_0. By the uniqueness hypothesis in $(A2)$ $y(t) = x_0(t)$ for $0 \leq t \leq T$. Finally, to see that $\mathscr{G}(t, \cdot) = \mathscr{F}_0(t, \cdot)$, observe that for $0 \leq t \leq T$ and $s \in R^+$,

$$\begin{aligned} \|\mathscr{G}(t, s) - \mathscr{F}_0(t, s)\| &\leq \|\mathscr{F}_n(t, s) - \mathscr{F}_0(t, s)\| + \|\mathscr{F}_n(t, s) - \mathscr{G}(t, s)\| \\ &= \|\mathscr{F}_n(t, s) - \mathscr{F}_0(t, s)\| + o(1) \end{aligned}$$

as $n \to \infty$. Then, using the definition of $\mathscr{F}_n$ and $\mathscr{F}_0$, and the first bound of (2.1), let $n \to \infty$ to obtain $\mathscr{G}(t, s) = \mathscr{F}_0(t, s)$. This completes the proof that U is closed.

To continue with the proof of Theorem 2.1, let λ, $\operatorname{Re}\lambda > 0$, be such that $\rho(\lambda)$ exists and inequality (2.5) holds. Fix $z = (x_0, u, f, F) \in D(C)$. Choose $z_1 = (x_1, u_1, f_1, F_1) \in D(C^2)$ satisfying (2.4) and (2.5), and write $x_1(t) = \pi_1 U(t)z_1$. Using (2.4) and Lemma 2.4, we find that for $0 \leq t \leq T$,

$$\begin{aligned} \|x(t)\| &= \|\pi_1 U(t)z\| = \|\pi_1 U(t)(C - \lambda^2 I)z_1\| \\ &\leq \|x_1''(t)\| + \|\pi_1 U(t)\lambda^2 z_1\| \\ &\leq (1 + |\lambda|^2)\{\|x_1''(t)\| + \|\pi_1 U(t)z_1\|\}. \end{aligned} \tag{2.10}$$

From the definition of the norm, $\|\cdot\|_W$, we have

$$\|\pi_1 U(t)z_1\| \leq \|U(t)z_1\|_W, \qquad 0 \leq t \leq T.$$

Also, differentiating equation (E) for $x_1(t)$ and then substituting equation (E) for $x_1'(t)$ into the resulting expression yields

$$x_1''(t)=[L+c^2I]x_1(t)+c\left(F_1(t)+\int_0^t K(t-\tau)x_1(\tau)d\tau\right)$$
$$+\int_0^t K_0(t-\tau)x_1(\tau)d\tau+F_1'(t).$$

Hence, using the definition of $\|\cdot\|_W$, we have

$$\|U(t)z_1\|_W\geq\|x_1''(t)\|, \qquad 0\leq t\leq T.$$

Thus, after substituting the last two inequalities into inequality (2.10), and using the definition of $\|\cdot\|_{C(W)}$, we find that

$$\|x(t)\|\leq 2(1+|\lambda|^2)\|Uz_1\|_{C(W)}, \qquad 0\leq t\leq T.$$

Since U is a closed linear map from $D(C)$ with its graph norm to $C(W)$, U is bounded [18, p. 181]. Combining this with (2.4) and (2.5), we obtain

$$\begin{aligned}\|x(t)\|&\leq 2(1+|\lambda|^2)\|U\|[\|z_1\|+\|Cz_1\|]\\&\leq 2(1+|\lambda|^2)\|U\|[\|z\|+(1+|\lambda|^2)\|z_1\|]\\&\leq 2(1+|\lambda|^2)\|U\|[1+M(\lambda)(1+|\lambda|^2)]\|z\|\\&=M\|z\|\end{aligned}$$

where $M=M(T)$.

This inequality is true for all choices of u and f in $z=(x_0, u, f, F)$; in particular, it is true with $z=(x_0, 0, 0, F)$. But, for this choice of z, the last inequality is simply (2.9).
Q.E.D.

§ 3. Well-Posed problems

We say that (E) is *uniformly well-posed* if for each pair (x_0, F) with $(x_0, 0, 0, F)\in D(C)$, there exists a unique solution $x(t, x_0, F)$ of equation (E) on R^+, and if for any $T>0$, $\|x(t, x_0, F)\|\to 0$ uniformly on $[0, T]$ as $\|(x_0, 0, 0, F)\|\to 0$. In this section, we give an example of a class of problems which are uniformly well-posed. We begin by obtaining conditions that guarantee that the existence and uniqueness hypothesis $(A2)$ holds.

Consider the initial value problem

$$x'(t)=cx(t)+\int_0^t [B(t-\tau)Ax(\tau)+G(t-\tau)x(\tau)]d\tau+F(t), \qquad x(0)=x_0, \tag{3.1}$$

where c is a real scalar, $B(t)$ and $G(t)$ are real-valued functions in $C^2(R^+)$ with $B(0)>0$, and $F\in C^2(R^+; X)$. The closed, densely defined linear operator A will be assumed to satisfy

$(A3)$ A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R=(-\infty, \infty)$.

The fundamental work on cosine families is that of H. O. Fattorini [3, 4] and M. Sova [17]. For more recent results concerning cosine families and abstract nonlinear second order differential equations, we refer the reader to the paper by C. C. Travis and G. F. Webb [19].

Since $x(t)$ satisfies (3.1) if and only if $y(t)=\exp(-ct)x(t)$ satisfies

$$y'(t)=\int_0^t \exp(-c(t-\tau))[B(t-\tau)Ay(\tau)+G(t-\tau)y(\tau)]d\tau+\exp(-ct)F(t),$$
$$y(0)=x_0,$$

we may assume without loss of generality that $c=0$ in equation (3.1). Moreover, since $B(0)>0$, there is no loss of generality in assuming that $B(0)=1$. Finally, it is clear that solving equation (3.1) with $c=0$ and $B(0)=1$ is equivalent to finding $x(t) \in C^2(R^+; X)$ with $x(t) \in D(A)$ and $Ax(t) \in C(R^+; X)$ so that

$$x''(t)=[A+\gamma I]x(t)+\int_0^t [b(t-\tau)Ax(\tau)+g(t-\tau)x(\tau)]d\tau+f(t), \tag{3.2}$$

$x(0)=x_0$, $x'(0)=v_0$, where $b=B'$, $g=G'$, $f=F'$, $\gamma=G(0)$ and $v_0=F(0)$.

In order to state and prove our existence and uniqueness result for equation (3.2), it is necessary to introduce the following notation and lemma from the theory of cosine families. For proof we refer the reader to [19].

Let A be the infinitesimal generator of a strongly continuous cosine family $C(t)$. The operator $S(t)$, defined by $S(t)x=\int_0^t C(\tau)x d\tau$, $x \in X$, $t \in R$, is called the corresponding sine family. Also, define the sets $D(A)$ and E by

$$D(A)=\{x \in X: C(t)x \in C^2(R; X)\}, \qquad E=\{x \in X: C(t)x \in C^1(R; X)\}.$$

With this notation, Travis and Webb [19] prove

Lemma 3.1. *Assume that $(A3)$ holds and that $f \in C^1(R^+; X)$. Then for each $x_0 \in D(A)$ and $v_0 \in E$, there is a unique function $u(t) \in C^2(R^+; X)$ with $u(t) \in D(A)$ which solves the initial value problem*

$$u''(t)=Au(t)+f(t), \quad u(0)=x_0, \quad u'(0)=v_0, \tag{3.3}$$

for $t \in R^+$. Furthermore, this $u(t)$ can be written as

$$u(t)=C(t)x_0+S(t)v_0+\int_0^t S(t-\tau)f(\tau)d\tau, \qquad t\geq 0. \tag{3.4}$$

Using this lemma we prove

Theorem 3.1. *Let (A3) hold, $f \in C^1(R^+; X)$, and both b and g belong to $C^1(R^+)$. Then for each $x_0 \in D(A)$ and $v_0 \in E$, the initial value problem* (3.2) *has a unique solution $x(t)$ on R^+.*

Proof. Our proof is similar to the proof of Lemma 7.2 in [13].

Fix $T>0$. Let $0<t_0\le T$ and define $d(t_0)=\{w: w$ maps $[0, t_0]$ into $D(A)$ with $w(t)$, $w'(t)$ and $Aw(t)$ continuous$\}$. Since A is closed, it is easy to see that $d(t_0)$ with norm $|||\,w\,|||=\sup\{\|w(t)\|_A+\|w'(t)\|: 0\le t\le t_0\}$ is a Banach space. For $w \in d(t_0)$ define

$$\Psi_0 w(t)=\gamma w(t)+\int_0^t [b(t-\tau)Aw(\tau)+g(t-\tau)w(\tau)]d\tau$$

on $0\le t\le t_0$. Then $\Psi_0 w$ is strongly continuously differentiable with

$$\begin{aligned}(\Psi_0 w)'(t)=&\gamma w'(t)+b(0)Aw(t)+g(0)w(t)\\ &+\int_0^t [b'(t-\tau)Aw(\tau)+g'(t-\tau)w(\tau)]d\tau.\end{aligned}$$

From Lemma 3.1 it follows that for any $w \in d(t_0)$,

$$u''(t)=Au(t)+\{\Psi_0 w(t)+f(t)\}, \qquad u(0)=x_0,\ u'(0)=v_0,$$

has a unique solution $u=\Psi w$ which is again in $d(t_0)$.

Since $\Psi_0 w$ is linear, we see, using formula (3.4), that we can decide whether Ψw is a contraction map by computing the norm when $x_0=v_0=0$ and $f(t)\equiv 0$. If $w \in d(t_0)$ and $|||\,w\,|||\le 1$, then

$$\|\Psi_0 w(t)\|\le|\gamma|+\int_0^t (|b(\tau)|+|g(\tau)|)d\tau.$$

Thus, using formula (3.4) with $x_0=v_0=0$ and forcing function $\Psi_0 w(t)$, we obtain

$$\begin{aligned}\|\Psi w(t)\|&\le\int_0^t \|S(t-\tau)\|\,\|\Psi_0 w(\tau)\|\,d\tau\\ &\le M\int_0^t\left\{|\gamma|+\int_0^s (|b(\tau)|+|g(\tau)|)d\tau\right\}ds\\ &=M\left\{|\gamma|t+\int_0^t (t-s)(|b(s)|+|g(s)|)ds\right\},\end{aligned}$$

where M is a bound for $\|C(t)\|$ and $\|S(t)\|$ on $[0, T]$ and $0\le t\le t_0$. If we differentiate the expression for $\Psi w(t)$, we obtain

$$(\Psi w)'(t)=\int_0^t C(t-\tau)\Psi_0 w(\tau)d\tau.$$

Then, as above, we find that

$$\|(\Psi w)'(t)\| \leq M\left\{|\gamma|\, t + \int_0^t (t-s)(|b(s)|+|g(s)|)ds\right\}$$

for $0 \leq t \leq t_0$. Finally, if we integrate the expression for $\Psi w(t)$ by parts, and observe that $\Psi_0 w(0) = \gamma w(0)$, we obtain

$$\Psi w(t) = \int_0^t (t-\tau)C(\tau)\gamma w(0)d\tau + \int_0^t \int_0^{t-s} (t-s-u)C(u)(\Psi_0 w)'(s)duds.$$

Since A is closed, $w(0) \in D(A)$, and A commutes with $C(t)$ on $D(A)$, we have

$$A\int_0^t (t-\tau)C(\tau)\gamma w(0)d\tau = \int_0^t (t-\tau)C(\tau)\gamma Aw(0)d\tau.$$

Also, it is well-known from the theory of cosine families [17, p. 14] that for any $x \in X$, $\int_0^t (t-\tau)C(\tau)xd\tau \in D(A)$ and

$$A\int_0^t (t-\tau)C(\tau)xd\tau = C(t)x - x.$$

Thus, again observing that A is closed, we may apply A to the above expression for $\Psi w(t)$ and get

$$A\Psi w(t) = \int_0^t (t-\tau)C(\tau)\gamma Aw(0)d\tau + \int_0^t [C(t-\tau)-I](\Psi_0 w)'(\tau)d\tau.$$

Then, using the expression for $(\Psi_0 w)$, and recalling the definition of $|||\, w\, |||$ and the fact that $|||\, w\, ||| \leq 1$, we obtain

$$\begin{aligned}\|A\Psi w(t)\| &\leq \int_0^t (t-\tau)\, \|C(\tau)\|\, |\gamma|\, \|Aw(0)\|\, d\tau \\ &\quad + \int_0^t \|C(t-\tau)-I\|\, \|(\Psi_0 w)'(\tau)\|\, d\tau \\ &\leq M|\gamma|\, t + (M+1)\left\{(|\gamma|+|b(0)|+|g(0)|)t + \int_0^t (t-s)(|b'(s)|+|g'(s)|)ds\right\}.\end{aligned}$$

It follows that

$$|||\Psi w|||\leq(M+1)\Big\{(4|\gamma|+|b(0)|+|g(0)|)t_0 + \int_0^{t_0}(t_0-s)(2|b(s)|+2|g(s)|+|b'(s)|+|g'(s)|)ds\Big\}<1$$

for t_0 sufficiently small. The contraction mapping theorem implies the existence and uniqueness of a solution of (3.2) on $[0, t_0]$.

Translate (3.2) by t_0 to see that $y(t)=x(t+t_0)$ must satisfy

$$y''(t)=[A+\gamma I]y(t)+\int_0^t(b(t-\tau)Ay(\tau)+g(t-\tau)y(\tau))d\tau + \Big\{f(t+t_0)+\int_0^{t_0}(b(t+t_0-\tau)Ax(\tau)+g(t+t_0-\tau)x(\tau))d\tau\Big\},$$

$y(0)=x(t_0)$, $y'(0)=x'(t_0)$. Clearly the term in brackets, that is, the new forcing function, is strongly continuously differentiable, and $y(0)=x(t_0)\in D(A)$. To see that $y'(0)=x'(t_0)$ belongs to E, we note that $\Psi x=x$, and hence, using the representation (3.4),

$$x(t)=C(t)x_0+S(t)v_0+\int_0^t S(t-\tau)\{\Psi_0 x(\tau)+f(\tau)\}d\tau$$

for $0\leq t\leq t_0$. From the theory of cosine functions [19], we can differentiate the last expression and write

$$x'(t_0)=S(t_0)Ax_0+C(t_0)v_0+\int_0^{t_0}S(t_0-\tau)\{(\Psi_0 x)'(\tau)+f'(\tau)\}d\tau.$$

It is well-known [19] that $S(t_0)Ax_0\in E$. Since $C(r)C(t_0)=2^{-1}(C(r+t_0)+C(r-t_0))$ and $v_0\in E$, we have that

$$\frac{d}{dr}C(r)C(t_0)v_0=2^{-1}A(S(r+t_0)v_0+S(r-t_0)v_0),$$

and, hence, $C(t_0)v_0\in E$. Finally, if $g(t)$ is strongly continuous on $0\leq t\leq t_0$,

$$\begin{aligned}C(r)\int_0^{t_0}S(t_0-\tau)g(\tau)d\tau &=\int_0^{t_0}\int_0^{t_0-\tau}C(r)C(s)g(\tau)dsd\tau\\ &=2^{-1}\int_0^{t_0}\int_0^{t_0-\tau}(C(r+s)+C(r-s))g(\tau)dsd\tau.\end{aligned}$$

Using this last expression, it is easy to show that

$$h^{-1}(C(r+h)-C(r))\int_0^{t_0} S(t_0-\tau)g(\tau)d\tau$$
$$=\int_0^{t_0}(2h)^{-1}\int_0^h (C(r+t_0-\tau+u)-C(r-t_0+\tau+u))g(\tau)dud\tau.$$

It follows using the strong continuity of $g(\tau)$, the continuity of $C(\tau)x$ for each $x \in X$, and the dominated convergence theorem, that

$$\frac{d}{dr}C(r)\int_0^{t_0} S(t_0-\tau)g(\tau)d\tau$$
$$=2^{-1}\int_0^{t_0}(C(r+t_0-\tau)-C(r-t_0+\tau))g(\tau)d\tau.$$

Thus, since $(\Psi_0 x)'+f'$ is continuous, the last term in the expression for $x'(t_0)$ belongs to E. Hence, $x'(t_0)$ belongs to E.

Therefore, the same argument can be repeated to obtain a solution to (3.2) on $[t_0, 2t_0]$, $[2t_0, 3t_0]$, $\cdots$ until $[Nt_0, T]$ where $(N+1)t_0>T$. Since T is an arbitrary positive number, this proves existence and uniqueness on R^+. Q.E.D.

In order to give our example of a class of problems for which (E) is uniformly well-posed, we make the assumption

($A4$) The scalar functions B and G are of the form

$$B(t)=\beta+\int_0^t b(\tau)d\tau, \qquad G(t)=\gamma+\int_0^t g(\tau)d\tau$$

with b and g in $C^1(R^+)\cap L^1(R^+)$ and $\beta>0$ and γ real constants.

We have

Theorem 3.2. *Assume that* ($A3$) *and* ($A4$) *hold. Then the initial value problem* (3.1) *is uniformly well-posed.*

Proof. Since $\beta>0$, we may assume without loss generality that $\beta=1$. The existence of a unique solution $x(t, x_0, F)$ to (3.1) for each $(x_0, 0, 0, F) \in D(C)$ follows from Theorem 3.1 and the remarks preceding equation (3.2).

Define $K(t)$ by $K(t)=B(t)A+G(t)I$. Since ($A3$) and ($A4$) hold, $L=K(0)=A+\gamma I$ is closed and densely defined, $K(t)x$ is strongly continuously differentiable for each $x \in D(L)$, and there exist positive functions $\kappa(t) \in L^\infty(R^+)$ and $\kappa_0(t) \in L^1(R^+)$ such that the inequalities (2.1) hold. Thus, $K(t)$ satisfies the assumption ($A1$).

Finally, $\rho(\lambda)$ of Theorem 2.1 becomes

$$\rho(\lambda)=\{[A+\gamma I]+b^*(\lambda)A+g^*(\lambda)+c(B^*(\lambda)A+G^*(\lambda))+c^2I-\lambda^2I\}^{-1}.$$

Since $B^*(\lambda)$, $G^*(\lambda)$, $b^*(\lambda)$ and $g^*(\lambda)$ all tend to zero as Re $\lambda\to\infty$, $\rho(\lambda)$ exists if and only if

$$(\lambda^2-\gamma-g^*(\lambda)-cG^*(\lambda)-c^2)/(1+b^*(\lambda)+cB^*(\lambda))$$

is in the resolvent set of A when Re λ is sufficiently large. Since A is the infinitesimal generator of a strongly continuous cosine family, it is known [3, p. 90] that the resolvent set of A contains λ^2 whenever Re λ is sufficiently large. Thus, since the last expression is asymptotic to λ^2 as Re $\lambda\to\infty$, $\rho(\lambda)$ exists for all sufficiently large positive λ. Therefore, the hypotheses of Theorem 2.1 are satisfied, and problem (3.1) is well-posed. Q.E.D.

We remark that if ($A3$) and ($A4$) hold with b' and g' both of exponential order, if x_0 and $v_0=F(0)$ both lie in $D(A)$, and $F\in C^2(R^+;X)$ is such that $\|F''(t)\|$ is of exponential order, then the solution $x(t)$ of (3.1) is such that $\|x(t)\|$, $\|x'(t)\|$ and $\|Ax(t)\|$ are all of exponential order. This argument proceeds as follows. As above we may assume that $c=0$ and $\beta=1$. Hence, if we differentiate equation (3.1), we see that $x''(t)$ satisfies (3.2). Equation (3.2) together with (3.4) imply that $x(t)$ satisfies the equivalent equation

$$\begin{aligned}x(t)&=C(t)x_0+S(t)v_0\\ &\quad+\int_0^t S(t-\tau)\Big\{\gamma x(\tau)+\int_0^\tau[b(\tau-s)Ax(s)+g(\tau-s)x(s)]ds+F'(\tau)\Big\}d\tau.\end{aligned}\tag{3.5}$$

The argument used in the proof of Theorem 3.1 may be used again to see that $Ax(t)$ satisfies

$$\begin{aligned}Ax(t)&=C(t)Ax_0+S(t)Av_0+[C(t)-I][\gamma x_0+F'(0)]\\ &\quad+\int_0^t[C(t-\tau)-I]\Big\{\gamma x'(\tau)+b(0)Ax(\tau)+g(0)x(\tau)\\ &\quad+\int_0^\tau[b'(\tau-s)Ax(s)+g'(\tau-s)x(s)]ds+F''(\tau)\Big\}d\tau.\end{aligned}\tag{3.6}$$

If we set $w(t)=x'(t)$ and $z(t)=Ax(t)$, then equations (3.1), (3.5) and (3.6) are three coupled equations for x, w and z which contain only bounded linear operators. Since $\|C(t)\|$ and $\|S(t)\|$ are of exponential order [19], the exponential order of $\|x(t)\|$, $\|w(t)\|$ and $\|z(t)\|$ now follows using standard comparison arguments from integral equations (see, e.g. [12, pp. 121–122]).

The preceding remark justifies the Laplace transform arguments of the next section.

§ 4. Stability considerations

We now consider some stability properties of solutions of equations of the form (3.1) when the scalar kernels B and G satisfy assumption ($A4$). Throughout the remainder of this paper, we assume that the space X is a Hilbert space. We also make the additional assumption:

($A5$) A is a self-adjoint, negative definite, closed, densely defined linear operator. Moreover, the resolvent operator $R(\mu; A)=(A-\mu I)^{-1}$ is compact when it exists.

It is well-known (see Taylor [18, p. 343]) that if ($A5$) holds, then there exist eigenvalues $\{\mu_n\}$ satisfying $0>\mu_1\geq\mu_2\geq\cdots\to-\infty$, and a complete orthonormal set of corresponding eigenvectors $\{\phi_n\}$ associated with A. Moreover, a complex number μ is in the spectrum of A (i.e., $\mu\in\sigma(A)$) if and only if $\mu=\mu_n$ for some n, and in this case $A\phi_n=\mu_n\phi_n$.

We remark that ($A5$) implies ($A3$). Indeed, if $\{E_\lambda\}$ is the spectral resolution of the identity for A, then it is easy to check that

$$C(t)x=\int_{-\infty}^{0}\cos(\sqrt{-\lambda}t)dE_\lambda x, \qquad x\in X,$$

is a strongly continuous cosine family with infinitesimal generator A.

Since in assumption ($A4$) $\beta>0$, we may assume, as in Section 3, that $\beta=1$. In this case the differentiated form of equation (3.1) is

$$x''(t)=cx'(t)+[A+\gamma I]x(t)+\int_0^t[b(t-\tau)Ax(\tau)+g(t-\tau)x(\tau)]d\tau+f(t), \tag{4.1}$$

$x(0)=x_0$ $x'(0)=v_0$, where $f=F'$ and $v_0=cx_0+F(0)$. We choose to state our stability results in terms of equation (4.1).

Let $L^p(R^+; X)$ $(1\leq p\leq\infty)$ denote the space of measurable functions $f(t)$ from R^+ into X with $\|f(t)\|\in L^p(R^+)$.

Definition 4.1. Assume that both $b(t)$ and $g(t)$ belong to $C^1(R^+)\cap L^1(R^+)$ and that ($A5$) holds. The initial value problem (4.1) is called *L^2-stable* if for any $x_0\in X$, $v_0\in X$ and $f(t)\in L^1(R^+; X)$, the unique generalized solution $x(t, x_0, v_0, f)$ of (4.1) belongs to $L^2(R^+; X)$.

Our stability results will be stated in terms of Laplace transform conditions. We first give a condition which must necessarily hold if (4.1) is L^2-stable.

Theorem 4.1. *Assume that equation* (4.1) *is L^2-stable. Then it is necessary that* $1+b^*(\lambda)\neq0$ *when* $\operatorname{Re}\lambda>0$.

Proof. The method of proof is essentially the same as that for Theorem 5 of [14]. Namely, suppose $1+b^*(\lambda_0)=0$ for some λ_0 with $\operatorname{Re}\lambda_0>0$. Then, either $\lambda_0^2-c\lambda_0-\gamma-g^*(\lambda_0)=0$, or $(\lambda^2-c\lambda-\gamma-g^*(\lambda))/(1+b^*(\lambda))$ has a pole at λ_0. Since $\mu_n\to-\infty$ as $n\to\infty$, it follows that for all sufficiently large n one can find solutions $\lambda=\lambda_n$ of the equation

$$\lambda_n^2 - c\lambda_n - \gamma - g^*(\lambda_n) - \mu_n(1 + b^*(\lambda_n)) = 0$$

with Re $\lambda_n > 0$. It is an easy matter to verify that the function $x_n(t) = \exp(\lambda_n t)\phi_n$ is a solution of (4.1) with $x_0 = \phi_n$, $v_0 = \lambda_n \phi_n$ and

$$f(t) = \int_t^\infty \exp(\lambda_n(t-\tau))[\mu_n b(\tau) + g(\tau)] d\tau \phi_n.$$

Using the fact that Re $\lambda_n > 0$, it is not difficult to show that $\|f(t)\| \in L^1(R^+)$. Finally, since Re $\lambda_n > 0$, $\|x_n(t)\| \to \infty$, and it follows that (4.1) is not L^2-stable. Q.E.D.

Our hypotheses sufficient to guarantee the L^2-stability of problem (4.1) will include the following Laplace transform conditions:

$(T1)$ $\qquad 1 + b^*(\lambda) \neq 0 \qquad$ for Re $\lambda \geq 0$.

$(T2)$ $\quad \lambda^2 - c\lambda - \gamma - g^*(\lambda) - \mu_n(1 + b^*(\lambda)) \neq 0 \qquad$ for $n = 1, 2 \cdots$ and Re $\lambda \geq 0$.

As Theorem 4.1 and its proof show, a necessary condition for equation (4.1) to be L^2-stable is that the transform assumptions $(T1)$ and $(T2)$ hold for λ in the open half-plane Re $\lambda > 0$. As a practical matter, if L^2-stability is to be guaranteed, we should generally have that $(T1)$ and $(T2)$ hold on Re $\lambda = 0$ as well as in the open half-plane. Otherwise, if, say, $1 + b^*(\lambda) = 0$ had a solution on Re $\lambda = 0$, then a small perturbation could shift this root into the open right half-plane. However, the case where $1 + b^*(\lambda) \neq 0$ for Re $\lambda \geq 0$, $\lambda \neq 0$, but $1 + b^*(0) = 0$ will be considered separately in Section 5.

We now derive two lemmas needed in the proof of our theorem giving conditions which are sufficient to guarantee that equation (4.1) be L^2-stable.

Let $B(X)$ denote the space of bounded linear operators on X. Also, let $H^2(0; X)$ denote the Hardy class consisting of all functions $h(\lambda)$ from Re $\lambda > 0$ into X which are analytic on Re $\lambda > 0$ and which satisfy:

(i) $\sup \left\{ \int_{-\infty}^{\infty} |h(\sigma + i\tau)|^2 \, d\tau : \sigma > 0 \right\} < \infty$;

(ii) $h(i\tau) \equiv \lim_{\sigma \to 0+} h(\sigma + i\tau)$ exists almost everywhere and belongs to $L^2(R; X)$.

Lemma 4.1. *Let $p(t)$ and $q(t)$ be real-valued functions in $L^1(R^+)$, let $\alpha < 0$ and γ be real constants, and assume that the operator A satisfies hypothesis $(A5)$. Assume that*

(4.2) $\quad \lambda^2 - \alpha\lambda - \lambda p^*(\lambda) - \gamma - q^*(\lambda) - \mu_n \neq 0 \qquad$ *for* $n = 1, 2, \cdots$ *and* Re $\lambda \geq 0$.

Then the $B(X)$-valued function

$$T^*(\lambda) = [\lambda^2 - \alpha\lambda - \lambda p^*(\lambda) - \gamma - q^*(\lambda) - A]^{-1}$$

is defined for Re $\lambda \geq 0$ *and satisfies*

$$\sup\left\{\int_{-\infty}^{\infty} \|T^*(\sigma+i\tau)\|^2\, d\tau\colon \sigma\geq 0\right\}<\infty.$$

Proof. Recall that since the operator A satisfies ($A5$), $\|R(\lambda; A)\| \leq [\operatorname{dist}(\lambda, \sigma(A))]^{-1}$ (see, e.g., [18, p. 343]). Thus, Lemma 4.1 will follow provided we can show that there exists a positive constant M such that

$$\operatorname{dist}(\lambda^2-\alpha\lambda-\lambda p^*(\lambda)-\gamma-q^*(\lambda), \sigma(A))\geq M(|\tau|+1)$$

whenever $\lambda=\sigma+i\tau$ with $\sigma\geq 0$ and $-\infty<\tau<\infty$.

Suppose that the last inequality is not satisfied. Then there exists $\lambda_n=\sigma_n+i\tau_n$ such that

$$\operatorname{dist}(\lambda_n^2-\alpha\lambda_n-\lambda_n p^*(\lambda_n)-\gamma-q^*(\lambda_n), \sigma(A))\leq(|\tau_n|+1)/n. \tag{4.3}$$

If there exists a subsequence $n=n_k$ such that $\lambda_n\to\lambda_0$ as $n\to\infty$ and such that (4.3) holds, then the continuity of $p^*(\lambda)$ and $q^*(\lambda)$ implies that (4.2) fails for λ_0 and some eigenvalue $\mu_m \in \sigma(A)$, a contradiction. Therefore, we may suppose that $\lambda_n\to\infty$ in Re $\lambda\geq 0$ as $n\to\infty$.

We now consider two cases. First, suppose that there exists a subsequence $n=n_k$ such that $|\tau_n|\leq\sigma_n/2$. Write $p^*(\lambda_n)=p_{1n}+ip_{2n}$, $q^*(\lambda_n)=q_{1n}+iq_{2n}$, and recall that both $p^*(\lambda)$ and $q^*(\lambda)$ tend to zero as $\lambda\to\infty$ in Re $\lambda\geq 0$. Thus, as $n\to\infty$,

$$\begin{aligned}
&\operatorname{Re}(\lambda_n^2-\alpha\lambda_n-\lambda_n p^*(\lambda_n)-\gamma-q^*(\lambda_n))\\
&\quad=\sigma_n^2-\tau_n^2-\alpha\sigma_n-\sigma_n p_{1n}+\tau_n p_{2n}-\gamma-q_{1n}\\
&\quad\geq 3\sigma_n^2/4-\alpha\sigma_n-o(\sigma_n)\geq\sigma_n/2\geq|\tau_n|.
\end{aligned}$$

Since $\sigma(A)\subseteq(-\infty,0)$, the above calculation contradicts (4.3).

On the other hand, suppose that there exists a subsequence $n=n_k$ such that $|\tau_n|\geq\sigma_n/2$. Recalling that $\alpha<0$ and $\sigma_n\geq 0$, we see that in this case

$$\begin{aligned}
&|\operatorname{Im}(\lambda_n^2-\alpha\lambda_n-\lambda_n p^*(\lambda_n)-\gamma-q^*(\lambda_n))|\\
&\quad\geq(2\sigma_n-\alpha)|\tau_n|-\sigma_n|p_{2n}|-|\tau_n||p_{1n}|-|q_{2n}|\\
&\quad\geq-\alpha|\tau_n|-o(|\tau_n|)\geq-\alpha|\tau_n|/2 \qquad \text{as } n\to\infty.
\end{aligned}$$

This time (4.3) is contradicted since $\sigma(A)$ is real. Q.E.D.

Lemma 4.2. *Assume that the hypotheses of Lemma* 4.1 *hold. Define the* $B(X)$-*valued function* $R^*(\lambda)=\lambda T^*(\lambda)$ *for* Re $\lambda\geq 0$ *where* T^* *is as in Lemma* 4.1. *Then for each* $x\in X$

$$\sup\left\{\int_{-\infty}^{\infty}\|R^*(\sigma+i\tau)x\|^2\, d\tau\colon\sigma\geq 0\right\}<\infty.$$

Proof. For $n=1, 2, \cdots$, define the scalar functions

$$R_n^*(\lambda)=\lambda[\lambda^2-\alpha\lambda-\lambda p^*(\lambda)-\gamma-q^*(\lambda)-\mu_n]^{-1}, \qquad \operatorname{Re}\lambda\geq 0.$$

Since p and q belong to $L^1(R^+)$ and (4.2) holds, each $R_n^*(\lambda)$ is continuous on $\operatorname{Re}\lambda\geq 0$ and analytic on $\operatorname{Re}\lambda>0$. Moreover, each $R_n^*(\lambda)=O(|\lambda|^{-1})$ as $\lambda\to\infty$ in $\operatorname{Re}\lambda\geq 0$; hence, R_n^* belongs to $H^2(0; C)$ and by a theorem of Paley and Wiener [16, p. 8] each R_n^* is the Laplace transform of a real-valued function $R_n(t)\in L^2(R^+)$. Thus, for $n=1, 2, \cdots$

$$\sup\left\{\int_{-\infty}^{\infty}|R_n^*(\sigma+i\tau)|^2\,d\tau : \sigma\geq 0\right\}=\int_{-\infty}^{\infty}|R_n^*(i\tau)|^2\,d\tau. \tag{4.4}$$

We now show that

$$\sup\left\{\int_{-\infty}^{\infty}|R_n^*(i\tau)|^2\,d\tau : n=1, 2, \cdots\right\}<\infty. \tag{4.5}$$

Since each $R_n(t)\in L^2(R^+)$, it is clear that (4.5) holds provided that we show that the L^2-norms of the $R_n(t)$ are uniformly bounded for all sufficiently large n. Write $p^*=p_1+ip_2$, $q^*=q_1+iq_2$, and set $\eta_n=\sqrt{-\mu_n}$, $D=\|p\|_1$, $d=|\gamma|+\|q\|_1$. We have

$$|R_n^*(i\tau)|^2=\tau^2\{(\tau^2-\tau p_2(i\tau)+\gamma+q_1(i\tau)+\mu_n)^2+(\alpha\tau+\tau p_1(i\tau)+q_2(i\tau))^2\}^{-1},$$

and recalling that $p(t)$ and $q(t)$ are real-valued it is easy to check that

$$\int_{-\infty}^{\infty}|R_n^*(i\tau)|^2\,d\tau=2\int_0^{\infty}|R_n^*(i\tau)|^2\,d\tau.$$

For n sufficiently large, decompose this integral as

$$\begin{aligned}\int_0^{\infty}|R_n^*(i\tau)|^2\,d\tau&=\left(\int_0^{\eta_n-D-1}+\int_{\eta_n-D-1}^{\eta_n+D+1}+\int_{\eta_n+D+1}^{2\eta_n}+\int_{2\eta_n}^{\infty}\right)|R_n^*(i\tau)|^2\,d\tau\\&=I_1+I_2+I_3+I_4.\end{aligned}$$

Since $p^*(i\tau)$ and $q^*(i\tau)$ tend to zero as $\tau\to\infty$, we have

$$\begin{aligned}I_2&\leq\int_{\eta_n-D-1}^{\eta_n+D+1}\tau^2(\alpha\tau+\tau p_1(i\tau)+q_2(i\tau))^{-2}d\tau\\&\leq 2(D+1)\alpha^{-2}+o(1) \quad \text{as} \quad n\to\infty.\end{aligned}$$

If $\tau\geq\eta_n+D+1$, then

$$\begin{aligned}\tau^2-\tau p_2(i\tau)+\gamma+q_1(i\tau)+\mu_n&\geq\tau^2-D\tau-d+\mu_n\\&\geq 2\eta_n+D\eta_n+1-d>0\end{aligned}$$

for all large n. Thus,

$$I_3+I_4\leq\left(\int_{\eta_n+D+1}^{2\eta_n}+\int_{2\eta_n}^{\infty}\right)\tau^2(\tau^2-\tau p_2(i\tau)+\gamma+q_1(i\tau)+\mu_n)^{-2}d\tau$$
$$\leq\left(\int_{\eta_n+D+1}^{2\eta_n}+\int_{2\eta_n}^{\infty}\right)\tau^2(\tau^2-D\tau-d+\mu_n)^{-2}d\tau.$$

It is easily seen that

$$\begin{aligned}I_4&\leq\int_{2\eta_n}^{\infty}\tau^2(\tau^2-D\tau-d+\mu_n)^{-2}d\tau\\&=\eta_n^{-1}\int_2^{\infty}u^2(u^2-Du\eta_n^{-1}-d\eta_n^{-2}-1)^{-2}du\\&\leq\eta_n^{-1}\int_2^{\infty}u^2(3u^2/4-1)^{-2}du\\&\leq4\eta_n^{-1}\int_2^{\infty}u^{-2}du=2\eta_n^{-1},\qquad n\text{ large.}\end{aligned}$$

In order to estimate the remaining two integrals, we first observe that an elementary calculation yields the integral formula

$$\int\tau^2(\tau^2-\theta^2)^{-2}d\tau=\frac{1}{2}\tau(\theta^2-\tau^2)^{-1}+\frac{1}{4\theta}\log|(\tau-\theta)/(\tau+\theta)|. \tag{4.6}$$

Thus,

$$\begin{aligned}I_3&\leq\int_{\eta_n+D+1}^{2\eta_n}\tau^2(\tau^2-D\tau-d+\mu_n)^{-2}d\tau\\&\leq\int_{\eta_n+D+1}^{2\eta_n}\tau^2(\tau^2-2D\eta_n-d+\mu_n)^{-2}d\tau\to\frac{1}{4}\end{aligned}$$

as $n\to\infty$. Here we have used formula (4.6) with $\theta_n=\sqrt{2D\eta_n+d-\mu_n}$ to evaluate the last integral, and recalled that $\eta_n^2=-\mu_n$ when we let $n\to\infty$ in the resulting expression.

Finally, observe that for $\tau\leq\eta_n-D-1$ and n large we have

$$\begin{aligned}\tau^2-\tau p_2(i\tau)+\gamma+q_1(i\tau)+\mu_n&\leq\tau^2+D\tau+d+\mu_n\\&\leq-2\eta_n-D\eta_n+D+d+1<0.\end{aligned}$$

Thus, using formula (4.6) with $\theta_n=\sqrt{-D\eta_n+D^2+D-d-\mu_n}$, we find that

$$\begin{aligned}I_1&\leq\int_0^{\eta_n-D-1}\tau^2(\tau^2+D\tau+d+\mu_n)^{-2}d\tau\\&\leq\int_0^{\eta_n-D-1}\tau^2(\tau^2+D\eta_n-D^2-D+d+\mu_n)^{-2}d\tau\to(D+2)/2\end{aligned}$$

as $n\to\infty$. Thus (4.5) holds.

To conclude the proof of the lemma, fix $x \in X$ and observe that by the spectral decomposition formula

$$R^*(\lambda)x = \sum R_n^*(\lambda)\langle x, \phi_n\rangle\phi_n.$$

Here $\langle\cdot, \cdot\rangle$ denotes the inner product in X and the sum is from $n=1$ to $n=\infty$. By (4.4) and (4.5) we can find a constant M such that

$$\begin{aligned}\int_{-\infty}^{\infty} \|R^*(\sigma+i\tau)x\|^2\, d\tau &= \sum \int_{-\infty}^{\infty} |R_n^*(\sigma+i\tau)|^2\, d\tau\, |\langle x, \phi_n\rangle|^2 \\ &\leq \sum \int_{-\infty}^{\infty} |R_n^*(i\tau)|^2\, d\tau\, |\langle x, \phi_n\rangle|^2 \\ &\leq \sum M |\langle x, \phi_n\rangle|^2 = M\|x\|^2,\end{aligned}$$

and the proof is complete. Q.E.D.

Theorem 4.2. *Let A, b and g satisfy the conditions of Definition* 4.1, *and assume that* $(T1)$ *and* $(T2)$ *hold. In addition, let $b'(t) \in L^1(R^+)$ and $b(0)+c<0$. Then equation* (4.1) *is L^2-stable.*

Proof. Define the scalar resolvent function $r(t)$ by

$$r(t) = b(t) - b*r(t), \qquad t \geq 0, \tag{4.7}$$

where as in Section 2

$$b*r(t) = \int_0^t b(t-\tau)r(\tau)d\tau$$

denotes convolution. It is well-known (see [12, Chapter 4]) that $r(t) \in C(R^+)$. Since $(T1)$ holds, the classical result of Paley and Wiener [16, paragraph 18] yields $r(t) \in L^1(R^+)$. By differentiating equation (4.7), we see that we also have $r'(t) \in C(R^+) \cap L^1(R^+)$.

Fix x_0 and v_0 in $D(A)$, $f(t) \in C^1(R^+; X) \cap L^1(R^+; X)$, and let $x(t) = x(t, x_0, v_0 f)$ denote the unique solution of equation (4.1) on $t \geq 0$. An integration by parts yields

$$r*x''(t) = r'*x'(t) + b(0)x'(t) - r(t)v_0,$$

where we have observed that $r(0) = b(0)$. Thus, if we convolve both sides of equation (4.1) with $r(t)$, and use the last equality together with (4.7), we find that $x(t)$ satisfies

$$\begin{aligned}&x''(t) = \alpha x'(t) + p*x'(t) + [A+\gamma I]x(t) + q*x(t) + h(t), \qquad t \geq 0, \\ &x(0) = x_0, \qquad x'(0) = v_0.\end{aligned} \tag{4.8}$$

Here $\alpha = b(0) + c < 0$,

$$p(t)=r'(t)-cr(t), \qquad q(t)=g(t)-r*g(t)-\gamma r(t),$$

and

$$h(t)=f(t)-r*f(t)-r(t)v_0.$$

Clearly, the real-valued functions $p(t)$ and $q(t)$ lie in $C(R^+)\cap L^1(R^+)$, and $h(t) \in C^1(R^+; X)\cap L^1(R^+; X)$.

An elementary calculation yields

$$\lambda^2-\alpha\lambda-\lambda p^*(\lambda)-\gamma-q^*(\lambda)=(\lambda^2-c\lambda-\gamma-g^*(\lambda))/(1+b^*(\lambda)),$$

and using $(T2)$ it follows that (4.2) holds. Therefore, we can define the X-valued function $z^*(\lambda)$ on $\mathrm{Re}\,\lambda\geq 0$ by

$$z^*(\lambda)=R^*(\lambda)x_0+T^*(\lambda)[-\alpha x_0-p^*(\lambda)x_0+v_0+h^*(\lambda)]$$

where $R^*(\lambda)$ and $T^*(\lambda)$ are the $B(X)$-valued functions defined in Lemmas 4.1 and 4.2 respectively. Clearly, $z^*(\lambda)$ is continuous on $\mathrm{Re}\,\lambda\geq 0$ and analytic on $\mathrm{Re}\,\lambda>0$. Since $|p^*(\lambda)|\leq\|p\|_1$ and $\|h^*(\lambda)\|\leq\|h\|_1$, we have

$$\|z^*(\lambda)\|\leq\|R^*(\lambda)x_0\|+\|T^*(\lambda)\|([|\alpha|+\|p\|_1]\,\|x_0\|+\|v_0\|+\|h\|_1),$$

and by Lemmas 4.1 and 4.2 it follows that

$$\sup\left\{\int_{-\infty}^{\infty}\|z^*(\sigma+i\tau)\|^2\,d\tau : \sigma\geq 0\right\}<\infty.$$

Thus, $z^*(\lambda)$ lies in the Hardy space $H^2(0; X)$. Since X is a Hilbert space, the theorem of Paley and Wiener used in the proof of Lemma 4.2 is true for functions in $H^2(0; X)$ [5, p. 164]; hence, there exists a function $z(t)\in L^2(R^+; X)$ whose Laplace transform is $z^*(\lambda)$ for all $\mathrm{Re}\,\lambda\geq 0$.

Next, observe that by the remark at the end of Section 3, $\|x(t)\|$, $\|x'(t)\|$ and $\|Ax(t)\|$ are all of exponential order whenever both x_0 and v_0 lie in $D(A)$ and $f(t)\in C^1(R^+; X)\cap L^1(R^+; X)$ with $\|f'(t)\|$ of exponential order. Thus, there exists $\sigma_0\geq 0$ so that we may take Laplace transforms in equation (4.8) when $\mathrm{Re}\,\lambda\geq\sigma_0$. After an elementary calculation using our definitions of T^*, R^* and z^*, we obtain $x^*(\lambda)=z^*(\lambda)$ for $\mathrm{Re}\,\lambda\geq\sigma_0$. Uniqueness of Laplace transforms implies that $x(t)=z(t)$ for almost all $t\geq 0$; hence, $x(t)\in L^2(R^+; X)$ provided that the initial data and forcing function are sufficiently smooth. But by Theorem 3.2 we have that solutions of (4.1) depend continuously on parameters. Thus, an elementary density argument may be used to show that the generalized solution $x(t)$ of (4.1) belongs to $L^2(R^+; X)$ for all x_0, v_0 in X and $f(t)\in L^1(R^+; X)$. Q.E.D.

We remark that equation (4.1) must be L^2-unstable if $\alpha=b(0)+c>0$ in Theorem 4.2. To see this rewrite equation (4.1) in the form (4.8) and observe that when $\alpha>0$, $\lambda^2-\alpha\lambda-\gamma-\mu_n$ has roots at

$$\lambda_n=(\alpha\pm\sqrt{\alpha^2+4(\gamma+\mu_n)})/2;$$

hence, Re $\lambda_n=\alpha/2>0$ for all large n. On a circle of radius $\alpha/2$ about λ_n, the quadratic form $\lambda^2-\alpha\lambda-\gamma-\mu_n$ has modulus approximately equal to $\alpha|\lambda_n|$. Since $|\lambda p^*(\lambda)+q^*(\lambda)|=o(|\lambda|)$ as $\lambda\to\infty$ in Re $\lambda\geq 0$, Rouche's theorem implies that for all sufficiently large n (4.2) fails with λ in the open half-plane Re $\lambda>0$. Thus, ($T2$) is not satisfied for some λ in Re $\lambda>0$, and the proof of Theorem 4.1 shows that equation (4.1) is unstable. The case where $\alpha=0$ appears to be difficult to analyze.

§ 5. Homogeneous equations

In this section we obtain some additional stability results for the homogeneous equation

$$x''(t)=cx'(t)+[A+\gamma I]x(t)+\int_0^t [b(t-\tau)Ax(\tau)+g(t-\tau)x(\tau)]d\tau, \tag{5.1}$$

$x(0)=x_0$, $x'(0)=v_0$. Here the operator A and the scalar functions b and g satisfy the assumptions of the Section 4. We remark that the integrated version of equation (5.1) is equation (3.1) with forcing function $F(t)$ identically constant.

Theorem 5.1. *Let the hypotheses of Theorem* 4.2 *hold. Then for each* $x_0\in D(A)$ *and* $v_0\in E$, *the solution* $x(t)=x(t, x_0, v_0)$ *of equation* (5.1) *and* $x'(t)$ *are both in* $L^2(R; X)$.

Proof. Since $x(t)\in L^2(R^+; X)$ by Theorem 4.2, we need only show that $x'(t)\in L^2(R^+; X)$. To see that this is true, recall the expression for $x^*(\lambda)$ in the proof of Theorem 4.2, and write

$$\begin{aligned}(x')^*(\lambda)&=\lambda x^*(\lambda)-x_0\\ &=[\lambda R^*(\lambda)-I]x_0+R^*(\lambda)[-\alpha x_0-p^*(\lambda)x_0+v_0-r^*(\lambda)v_0]\end{aligned}$$

for Re $\lambda\geq 0$. Since $x_0\in D(A)$, an elementary calculation yields

$$\begin{aligned}[\lambda R^*(\lambda)-I]x_0=&[\alpha+p^*(\lambda)]R^*(\lambda)x_0\\ &+[q^*(\lambda)+\gamma]T^*(\lambda)x_0+T(\lambda)Ax_0.\end{aligned}$$

Since $p(t)$, $q(t)$ and $r(t)$ lie in $L^1(R^+)$, Lemmas 4.1 and 4.2 may be used to show that $(x')^*(\lambda)$ belongs to $H^2(0; X)$; hence, $x'(t)\in L^2(R^+; X)$ by the theorem of Paley and Wiener and uniqueness of Laplace transforms. Q.E.D.

We conclude this section by examining the stability of equation (5.1) in the case where $b^*(\lambda)\neq -1$ for Re $\lambda\geq 0$, $\lambda\neq 0$, but $b^*(0)=-1$. Observe that $1+b^*(\lambda)=$

$\lambda B^*(\lambda)$ must have a zero at $\lambda=0$ whenever the kernel $B(t)$ in the integrated form of equation (5.1) belongs to $L^1(R^+)$.

Theorem 5.2. *Let (A5) hold, and assume that b and g both belong to* $C^1(R^+)\cap L^1(R^+)$, *that* $b'(t)$, $t^2b(t)$ *and* $t^2g(t)$ *all belong to* $L^1(R^+)$, *and that* $b(0)+c<0$. *Assume that the transform conditions (T1) and (T2) hold for all* λ *satisfying* $\mathrm{Re}\,\lambda\geq 0$, $\lambda\neq 0$. *At* $\lambda=0$ *assume that the three conditions*

$$b^*(0)=-1, \qquad (b^*)'(0)\neq 0, \tag{5.2}$$

$$g^*(0)=-\gamma, \tag{5.3}$$

$$(c+(g^*)'(0))/(b^*)'(0)\neq -\mu_n \qquad (n=1,2,\cdots) \tag{5.4}$$

are satisfied. Then for each $x_0\in D(A)$ *and* $v_0\in E$, *the solution* $x(t,x_0,v_0)$ *of equation* (5.1) *has the form* $x(t)=\bar{x}+x_1(t)$ *where* $\bar{x}\in X$ *and* $x_1(t)\in L^2(R^+;X)$.

Proof. Let $r(t)$ be the scalar resolvent function defined in (4.7). Since the only zero of $1+b^*(\lambda)$ in $\mathrm{Re}\,\lambda\geq 0$ is a simple zero at $\lambda=0$, and since $t^2b(t)$ belongs to $L^1(R^+)$, a result of Jordan and Wheeler [11] yields that r has the form $r(t)=r_0+r_1(t)$ with r_0 a constant and $r_1(t)$ in $L^1(R^+)$. By substituting this expression for r into (4.7) and differentiating, we can easily show that $r_1'(t)\in L^1(R^+)$ as well.

If we use the above expression for $r(t)$ in equation (4.8) with $f(t)\equiv 0$, we find that $x(t)$ satisfies

$$x''(t)=\alpha x'(t)+p_1*x'(t)+[A+(\gamma-cr_0)I]x(t)+q*x(t)+h_1(t), \tag{5.5}$$

$x(0)=x_0$, $x'(0)=v_0$, where α and $q(t)$ are defined as in the proof of Theorem 4.2, and

$$p_1(t)=r_1'(t)-cr_1(t), \qquad h_1(t)=cr_0x_0-r_0v_0-r_1(t)v_0.$$

Here $p_1(t)\in C(R^+)\cap L^1(R^+)$. Also, since $tg(t)\in L^1(R^+)$ and (5.3) holds, it is easy to see that $\gamma+\int_0^t g(\tau)d\tau$ belongs to $L^1(R^+)$, and it follows that $q(t)\in C(R^+)\cap L^1(R^+)$.

An elementary calculation yields

$$\lambda^2-\alpha\lambda-\lambda p_1^*(\lambda)-\gamma+cr_0-q^*(\lambda)=(\lambda^2-c\lambda-\gamma-g^*(\lambda))/(1+b^*(\lambda))$$

for $\mathrm{Re}\,\lambda\geq 0$, $\lambda\neq 0$. Also, letting λ tend to zero in $\mathrm{Re}\,\lambda\geq 0$ and recalling that $1+b^*(\lambda)$ has a simple zero at $\lambda=0$, we obtain

$$-\gamma+cr_0-q^*(0)=-(c+(g^*)'(0))/(b^*)'(0).$$

Thus, using the transform hypotheses (T2) when $\mathrm{Re}\,\lambda\geq 0$, $\lambda\neq 0$, and (5.4) at $\lambda=0$, we see that the $B(X)$-valued functions

$$T_1^*(\lambda)\equiv[\lambda^2-\alpha\lambda-\lambda p_1^*(\lambda)-\gamma+cr_0-q^*(\lambda)-A]^{-1}$$

and $R_1^*(\lambda)\equiv\lambda T_1^*(\lambda)$ are defined for Re $\lambda\geq 0$, and that Lemmas 4.1 and 4.2 hold for $T_1^*(\lambda)$ and $R_1^*(\lambda)$, respectively.

Define the X-valued function $z_1^*(\lambda)$ on Re $\lambda\geq 0$, $\lambda\neq 0$, by

$$\begin{aligned}z_1^*(\lambda)\equiv R_1^*(\lambda)x_0+T_1^*(\lambda)[-\alpha x_0-p_1^*(\lambda)x_0+v_0-r_1^*(\lambda)v_0]\\ +\lambda^{-1}[T_1^*(\lambda)-T_1^*(0)][cr_0x_0-r_0v_0].\end{aligned}$$

From the resolvent equation [18, p. 257] we find that

$$\begin{aligned}&\lambda^{-1}[T_1^*(\lambda)-T_1^*(0)]\\ &=-\left\{\frac{\lambda^2-c\lambda-\gamma-g^*(\lambda)}{\lambda(1+b^*(\lambda))}+\frac{c+(g^*)'(0)}{\lambda(b^*)'(0)}\right\}T_1^*(0)T_1^*(\lambda)\end{aligned}$$

for Re $\lambda\geq 0$, $\lambda\neq 0$. Since $t^2b(t)$ and $t^2g(t)$ lie in $L^1(R^+)$, $b^*(\lambda)$ and $g^*(\lambda)$ are C^2 in Re $\lambda\geq 0$, and it is easy to show that the term in brackets in the last equation has a finite limit as λ tends to zero in Re $\lambda\geq 0$. Therefore, we may define $z_1^*(0)$ so as to make $z_1^*(\lambda)$ continuous at $\lambda=0$, and using Lemmas 4.1 and 4.2, we find that $z_1^*(\lambda)$ lies in the Hardy space $H^2(0;X)$. Thus, there exists a function $z_1(t)\in L^2(R^+;X)$ whose Laplace transform is $z_1^*(\lambda)$ for all Re $\lambda\geq 0$.

As in the proof of Theorem 4.2, we may take Laplace transforms in equation (5.5), and we find that

$$x^*(\lambda)=z_1^*(\lambda)+\lambda^{-1}T_1^*(0)[cr_0x_0-r_0v_0]$$

when Re $\lambda\geq\sigma_0$ for some $\sigma_0>0$. Then uniqueness of Laplace transforms implies that $x(t)=\bar{x}+x_1(t)$ where $\bar{x}=T_1^*(0)[cr_0x_0-r_0v_0]$ and $x_1(t)=z_1(t)$ for almost all $t\geq 0$.
Q.E.D.

We remark that extensions of Theorem 5.2 to the case where $1+b^*(\lambda)$ has several (possibly multiple) zeros both on Re $\lambda=0$ and in Re $\lambda>0$ would be of use in analyzing the structure of solutions of equations (5.1). Results of this nature for the nonhomogeneous equation (4.1) would also be of interest.

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