Stability Problem in Functional Differential Equations with Infinite Delay

By

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We shall discuss the stability problem for functional differential equations with infinite delay.

Our first aim is to establish a Razumikhin type Liapunov theorem as a sufficient condition for the stability with respect to the norm in $R^n$, which will be done in Section 1. In Section 2, we shall discuss relationships between the stability with respect to the norm in $R^n$ and the stability with respect to the norm in the phase space, and we shall apply the results to obtaining the stability of perturbed systems. Here, we should note that for the case of finite delay there is no difference between these two concepts of the stability, but if we choose the space $C_0$ of bounded continuous functions on $(-\infty, 0]$ with the uniform norm, non trivial solution $x(t)$ never tends to zero with respect to this norm, that is,

$$|x_t|_{C_0} = \sup_{s \leq 0} |x(t+s)| / 0 \quad \text{as} \quad t \to \infty,$$

while

$$\sup_{s \leq 0} |x(t+s)| e^{\tau t} \to 0 \quad \text{as} \quad t \to \infty$$

for a $\tau > 0$ if $x(t) \to 0$ as $t \to \infty$.

This fact also causes a trouble to discuss the stability of perturbed systems. For example, the zero solution of

$$\dot{x}(t) = -x(t) + X(t, x_t), \quad X(t, \phi) = \phi(-t)$$

is not asymptotically stable, though $|X(t, \phi)| = o(|\phi|_{C_0})$.

1. Razumikhin type theorem.

For a functional differential equations with infinite delay

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there are several ways to specify the phase space. In this paper, we shall choose the space \( B \) discussed by Hale [1] (also, see [2]) consisting of \( R^n \)-valued functions defined on \( (\infty, 0] \), which is endowed with a semi-norm \( \cdot \) and the conditions: If \( x(t) \) is defined on \( (\infty, A) \), continuous on \([\sigma, A), \sigma < A \), and \( x_\tau \in B \), then for \( t \in [\sigma, A) \)

(i) \( x_\tau \in B \),

(ii) \( x_\tau \) is continuous in \( t \) with respect to \( \cdot \),

(iii) there are a constant \( M_0 > 0 \) and continuous functions \( K(s) \) and \( M(s) \) such that

\[
\begin{align*}
(1) & & M_0 |x(t)| &\leq |x_\tau|, \\
(2) & & |x_\tau| &\leq K(t-\sigma) \sup_{s\leq t} |x(s)| + M(t-\sigma) |x_\tau|,
\end{align*}
\]

where \( x_\tau \) denotes the segment defined by \( x_\tau(s) = x(t + s) \) for \( s \in (\infty, 0] \) and \( \cdot \) denotes any norm in \( R^n \).

Clearly, for a real \( \gamma \) the space \( C_\gamma \) of continuous functions \( \phi \) having the limit

\[
\lim_{t \to -\infty} e^{\gamma t} \phi(s)
\]

satisfies all the conditions as a space \( B \) with the norm

\[
|\phi|_{C_\gamma} = \sup_{s \leq 0} e^{\gamma s} |\phi(s)|.
\]

Another example is given by the space \( C([-h, 0]), h \geq 0 \), of functions \( \phi \) continuous on \([-h, 0]\) with the semi-norm

\[
|\phi|_{C([-h, 0])} = \sup_{-h \leq s \leq 0} |\phi(s)|
\]

(cf. [3]).

In [2] we made hypotheses on the space \( B \), which stands for \( \tilde{B} \) in [2], in a slightly different way. However, it is not difficult to see that those hypotheses play their roles through almost only the relation (2), which was referred as the fundamental inequality in [2], and we can reproduce the theorems in [2] under the corresponding hypotheses on \( K \) and \( M \). For example, we have the following lemmas, where and hereafter we assume that \( f(t, \phi) \) in (E) is defined and continuous in the topology induced by \( \cdot \) on \([0, \infty) \times B_\tilde{H}, B_\tilde{H} = \{ \phi \in B; \ |\phi|_B < H \}\), and that the equation (E) has the zero solution.

**Lemma 1.** If the zero solution of (E) is unique, then there exists a continuous function \( L(t, \tau, \phi) \) such that \( L(t, \tau, 0) = 0 \) and

\[
| x_\tau(\tau, \phi) |_B \leq L(t, \tau, |\phi|_B) \quad \text{for } t \geq \tau
\]

as long as \( x_\tau(\tau, \phi) \in B_\tilde{H} \), where \( x_\tau(\tau, \phi) \) denotes the segment of a solution \( x(t, \tau, \phi) \) of
(E) through $(\tau, \phi)$. Moreover, if $f(t, \phi)$ satisfies
\[ |f(t, \phi)| \leq L_1 |\phi|_B \]
for a constant $L_1$, then $L(t, \tau, r)$ in (L) can be chosen in the form $L(t, \tau, r) = L(t - \tau) r$, that is,
\[ |x_t(\tau, \phi)|_B \leq L(t - \tau) |\phi|_B. \]

The first part of this lemma is nothing but Theorem 2.5 in [2], and the second part can be proved in the same way as in the proof of Theorem 2.2 in [2].

**Lemma 2.** If $f(t, \phi)$ in (E) is completely continuous on $[0, \infty) \times B_R$, then for any $\varepsilon \in (0, H)$ and any solution $x(t)$ of (E) satisfying $(\tau, x_\tau) \in [0, \infty) \times B$, either $x(t)$ exists for all $t \geq \tau$ or there exists a $t_1 > \tau$ for which $x(t)$ exists on $[\tau, t_1]$ and $|x_\tau|_B = \varepsilon$.

It is easy to see that our hypotheses on the space $B$ causes no trouble in the proof of [2; Theorem 2.4].

The following definitions will be made (see [2]).

**Definition.** The zero solution of (E) is said to be **stable** in $R^n$ if for any $\varepsilon > 0$ and any $\tau \geq 0$ there exists a $\delta = \delta(\varepsilon, \tau) > 0$ such that
\[ |x_\tau|_B < \delta \quad \text{implies} \quad |x(t)| < \varepsilon \quad \text{for all} \quad t \geq \tau; \]
**asymptotically stable in** $R^n$ if in addition to the stability for any $\tau \geq 0$ there exist a $\delta_0 = \delta_0(\tau) > 0$ and a function $T = T(\tau, \varepsilon)$ of $\varepsilon > 0$ such that
\[ |x_\tau|_B < \delta_0 \quad \text{and} \quad t \geq \tau + T \quad \text{imply} \quad |x(t)| < \varepsilon, \]
where $x(t)$ denotes any solution of (E). If $\delta, \delta_0$ and $T$ are independent of $\tau$, then the stabilities are called **uniform**, while the **exponential stability** in $R^n$ corresponds to the case where there are positive constants $\alpha, \delta_0$ and $\gamma$ such that
\[ |x(t)| \leq \gamma e^{-\alpha(t-\tau)} |x_\tau|_B \quad \text{if} \quad |x_\tau|_B < \delta_0 \quad \text{and} \quad t \geq \tau. \]

Moreover, if the $R^n$-norm $|x(t)|$ in the relations (3), (4), (5) can be replaced by the $B$-(semi) norm $|x_\tau|_B$, then the concept of the stabilities in $B$ will be obtained.

**Remark.** In the above, we should read that each of the relations (3), (4), (5) holds as long as $x(t)$ exists. However, under the complete continuity of $f(t, \phi)$ Lemma 2 guarantees that the relation (3), for example, can be read so that if $|x_\tau|_B < \delta$, then $x(t)$ exists for all $t \geq \tau$ and satisfies $|x(t)| < \varepsilon$ there.

The following Theorem is a simple version of the Liapunov-type theorem (cf. [4]).
Theorem 1. Suppose that there exists a continuous real-valued function $V(t, \phi)$ defined on $[0, \infty) \times B_{\mathbb{H}}, H \geq H_0 > 0$, which satisfies the conditions

(A) $a(|\phi(0)|) \leq V(t, \phi),$
(B) $V(t, \phi) \leq b(t, |\phi|_B),$
(C) $\dot{V}(t, \phi) \leq -c(t, V(t, \phi)),$

where $a(r), b(t, r), c(t, r)$ are non-negative, continuous, non-decreasing in $r \geq 0$, $a(r) > 0$ for $r > 0$ and $b(t, 0) = 0$.

Then, the zero solution of $(E)$ is stable in $\mathbb{R}^n$. Moreover, it is asymptotically stable in $\mathbb{R}^n$ if, for any $r > 0$ and $t \geq 0$,

(D) $\int_t^{t+r} c(s, r) ds \to \infty$ as $T \to \infty,$

and it is uniformly asymptotically stable in $\mathbb{R}^n$ if

(UD) the divergence in (D) is uniformly in $t$

and if $b(t, r)$ in (B) can be chosen independent of $t$, that is,

(UB) $V(t, \phi) \leq b(|\phi|_B).$

Here,

$V(t, \phi) = \sup \lim_{h \to 0^+} \frac{1}{h} \{ V(t + h, x_{t+h}) - V(t, \phi) \},$

where the "sup" is taken over the solutions $x(u)$ of $(E)$ through $(t, \phi)$.

It will be observed that the segments of the solution may belong to a more restrictive class in $B$ as the time elapses. This fact may allow us to choose a simpler Liapunov function.

Example 1. The zero solution of

$x(t) = -x(t),$

considered as a functional differential equations on the space $C([-1, 0]),$ is surely exponentially stable in $C([-1, 0]);$

$|x_t(\tau, \phi)|_{C([-1, s])} \leq e^{-(t-s)} |\phi|_{C([-1, s])}.$

It will be seen (see [5] and also Theorem 6 below) that a corresponding Liapunov function $V(t, \phi)$ can be defined by
\[ V(t, \phi) = \sup_{u \geq 0} |x_{t+u}(t, \phi)|_{C([-\tau, 0])} e^{c u} \]

for a \( c \in (0, 1) \). Hence, we have

\[ V(t, \phi) = \max \left\{ \sup_{u \geq 1} e |\phi(0)| e^{(e-1) u}, \sup_{0 \leq u \leq 1} |\phi(s)| e^{c u} \right\}, \]

which is not so simple as was expected. However, if \( x(u) \) is a solution at least on the interval \([t-1, t]\), then

\[ V(t, x_t) = |x_t|_{C([-\tau, 0])} = e |x(t)|. \]

Thus, the following theorem is expected to be more effective. Such a theorem has been given by Barnea [6] for the uniform stability of an autonomous system with finite delay.

**Theorem 2.** Suppose that the condition (L) is satisfied for the solutions of (E).

Then, in Theorem 1 it is sufficient for \( V(t, \phi) \) to satisfy the condition (C) under the case

\[(P) \quad \phi = x_t \text{ for a function } x(u), x(t) \neq 0, \text{ which is a solution of (E) at least on} \]

the interval \([p(t, V(t, x_t)), t]\),

where \( p(t, r) \) is a continuous function of \( t \geq 0, r > 0, \) increasing in \( t \), non-decreasing in \( r \) and \( p(t, r) \leq t \). Here, for the uniform stability we assume that the condition (UL) is satisfied and that \( p(t, r) \) in the condition (P) is of the form

\[(6) \quad p(t, r) = t - q(r), \]

which will be referred to as the condition (UP).

This theorem is a special case of the following theorem, where \( X(\tau) = \{(t, \phi) : \phi \in B(t - \tau), t \geq \tau\} \) and \( B(\tau), \tau \geq 0 \), denotes a subspace of \( B \) such that \( \phi \in B(\tau) \) if and only if \( \phi_{-\tau} \in B \) and \( \phi(s) \) is continuous on \([-\tau, 0]\), and we shall introduce a new semi-norm \(|.|_{B(\tau)}\) in \( B(\tau) \) by

\[ |\phi|_{B(\tau)} = \max_{-\tau \leq s \leq 0} |\phi_s|_B, \]

which is well-defined by the assumptions on \( B \).

**Theorem 3.** Suppose that the condition (L) is satisfied for the solution of (E) and that for any \( \tau \geq 0 \) there exists a continuous function \( V(t, \phi; \tau) \) defined on \( X_{H(\tau)} = \{(t, \phi) \in X(\tau) : |\phi|_{B(t - \tau)} < H_\tau\} \) which satisfies the conditions (A), (B) and (C) under (P), that is, \( p(t, V(t, \phi; \tau)) \geq \tau \), where \( a(r) \) in (A), \( c(t, r) \) in (C), \( p(t, r) \) in (P) are independent of \( \tau \), while \( b \) in (B) may depend on \( \tau \), that is,
Then, the zero solution of (E) is asymptotically stable in $R^n$ under the condition (D).

Moreover, if (UL), (UB) with $b$ in (B') independent of $t$, $\tau$, (UD) and (UP) are assumed, then the zero solution of (E) is uniformly asymptotically stable in $R^n$.

Proof. Let $x(t)$ be a solution of (E) starting at $t = \tau$, and let $\varepsilon > 0$, $\varepsilon < H_0$ be given.

Suppose that $V(\tau, x_i; \tau) \leq \frac{1}{2} a(\epsilon)$ and $V(t, x_i; \tau) \geq a(\epsilon)$ for a $t > \tau$, where we may assume that $|x_i|_B < H_0$ for $s \in [\tau, t]$, that is, $|x_i|_{B(t-\varepsilon)} < H_0$. Then, there exist $t_1$ and $t_2$, $\tau \leq t_2 < t_1$, such that

$$t_1 = \inf \{ t > \tau; V(t, x_i; \tau) \geq a(\epsilon) \},$$

$$t_2 = \sup \{ t < t_1; V(t, x_i; \tau) \leq \frac{a(\epsilon)}{2} \}.$$

The condition (L) can be read as

$$|x_i|_{B(t-\varepsilon)} \leq L(t, \tau, |x_i|_B) \quad \text{for} \quad t \geq \tau,$$

while the condition (B') assures that there exists a continuous function $\gamma(\epsilon, t, \tau) > 0$ such that

$$|\phi|_{B(t-\varepsilon)} < \gamma(\epsilon, t, \tau) \quad \text{implies} \quad V(t, \phi; \tau) < \epsilon,$$

since $b(t, \tau, 0) = 0$. Since $L(t, \tau, r) \to 0$ as $r \to 0$, we can choose $\delta = \delta(\epsilon, \tau) > 0$ so that

$$L(t, \tau, \delta) < \gamma\left(\frac{a(\epsilon)}{2}, t, \tau\right) \quad \text{for any} \quad t \in \left[\tau, p_i^{-1}(\tau, \frac{a(\epsilon)}{2})\right],$$

where $p_i^{-1}(\tau, r)$ denotes the inverse function of $p(t, r)$ with respect to $t$ for a fixed $r > 0$, which is clearly increasing in $\tau$, non-increasing in $r$ and satisfies $p_i^{-1}(\tau, r) \geq \tau$. Therefore, if $|x_i|_B < \delta$, then $V(t, x_i; \tau) \geq a(\epsilon)/2$ only when

$$t > p_i^{-1}\left(\tau, \frac{a(\epsilon)}{2}\right) \geq p_i^{-1}(\tau, V(t, x_i; \tau)),$$

and hence, if $|x_i|_B < \delta$,

$$p(t, V(t, x_i; \tau)) > \tau \quad \text{for} \quad t \in [t_2, t_1]$$

and, especially, $t_2 > \tau$. Thus, by the assumptions $V(t, x_i; \tau)$ is non-increasing on $[t_2, t_1]$, which is a contradiction. This shows that the zero solution of (E) is stable in $R^n$. If (UL), (UB), (UP) are satisfied, then obviously $\delta$ can be chosen independent of $\tau$, since $p_i^{-1}(\tau, r) = \tau + q(r)$.
Next, we shall prove the asymptotic stability.

Let \( \delta(\tau) = \delta(\frac{H_0}{2}, \tau) \) for the \( \delta \) in the stability, and let \( T_1 = T_1(\varepsilon, \tau) \geq 0 \) be so that

\[
\int_{\sigma}^{\sigma + T_1} c(s, \varepsilon) ds > C(\sigma, \tau) - \varepsilon,
\]

where \( \sigma = \frac{1}{r} \tau(\tau, \varepsilon) \) and

\[
C(\sigma, \tau) = \max \left\{ b(\sigma, \tau, \varepsilon) : \sup_{s \leq \tau \leq \sigma + \tau} \left[ K(s) \frac{H_0}{2} + M(s) \delta(\tau) \right] \right\}.
\]

Let \( T(\varepsilon, \tau) = T_1(\varepsilon, \tau) + \sigma(\varepsilon, \tau) - \tau \) and suppose that for a \( t_1 > T(\varepsilon, \tau) + \tau \) we have \( V(t_1, x_{t_1}; \tau) \leq \varepsilon \). Clearly

\[
p(t_1, V(t_1, x_{t_1}; \tau)) \leq p(t, \varepsilon) \geq p(\sigma, \varepsilon) = \tau.
\]

Let

\[
t_2 = \sup \{ t \in [\tau, t_1] : p(t, V(t, x_t; \tau)) \leq \tau,\}
\]

which exists since \( p(\tau, \tau) \leq \tau \) for any \( \tau > 0 \). Then, by the condition \((C)\) under \((P)\), \( V(t, x_t; \tau) \) is non-increasing on \([t_2, t_1]\). Hence, we have

\[
p(t_2, V(t_2, x_{t_2}; \tau)) \leq p(t_2, V(t_1, x_{t_1}; \tau)) \leq p(t_2, \varepsilon),
\]

which implies \( \tau \geq p(t_2, \varepsilon) \), that is, \( \sigma = \frac{1}{r} \tau(\tau, \varepsilon) \geq t_2 \). Therefore, for \( t \in [\sigma, t_1] \)

\[
\dot{V}(t, x_t; \tau) \leq - c(t, V(t, x_t; \tau)) \quad \text{and} \quad V(t, x_t; \tau) \leq \varepsilon,
\]

and hence

\[
\varepsilon \leq V(t_1, x_{t_1}; \tau) \leq V(\sigma, x_\sigma; \tau) - \int_{\sigma}^{t_1} c(s, V(s, x_s; \tau)) ds
\]

\[
\leq V(\sigma, x_\sigma; \tau) - \int_{\sigma}^{t_1} c(s, \varepsilon) ds.
\]

From this, we have

\[
\int_{\sigma}^{t_1} c(s, \varepsilon) ds \leq b(\sigma, \tau, |x_\sigma|_{B(\sigma - \tau)}) \geq C(\sigma, \tau) - \varepsilon
\]

if \( |x_t| < \delta(\tau) \), because

\[
|x_t|_B \leq K(s - \tau) \sup_{t \leq s \leq x} |x(t)| + M(s - \tau) |x_t|_B
\]

\[
\leq K(s - \tau) \frac{H_0}{2} + M(s - \tau) \delta(\tau)
\]
for any $s \in [\tau, \sigma]$, that is,

$$|x_s|_{B(\sigma-t)} \leq \sup_{0 \leq \varepsilon \leq \sigma - \tau} \left\{ K(s) \frac{H_0}{2} + M(s)\delta_\varepsilon(t) \right\},$$

which contradicts $t_{i \geq \sigma + T_{i}(s, \tau)}$.

It is not difficult to see that we can choose $T$ together with $\delta_\varepsilon$ independent of $\tau$ under the hypotheses $(UL), (UB), (UP), (UD)$. Q.E.D.

Now we shall give a Razumikhin type theorem for the equation $(E)$. Such theorems have been given in [4], [7], [8]. Here we shall state the following theorem by extending the ideas in [9] and [10].

**Theorem 4.** Suppose that there is a continuous function $V(t, \phi)$ defined on $[0, \infty) \times B_{H_0}$ satisfying the conditions $(A)$, $(B)$ and the condition $(C)$ under the condition

$$(F) \quad \phi = x_s \text{ satisfies } V(s, x_s) \leq F(V(t, \phi)) \text{ for any } s \in [p(t, V(t, \phi)), t] \text{ in addition to the condition } (P),$$

where $p(t, r)$ satisfies the same conditions as in $(P)$ and, moreover, the function $q(t, r) = t - p(t, r)$ is positive, non-decreasing in $t$ and

$$(7) \quad \int_t^{t+T} \frac{ds}{q(p^{-1}(s, r), r)} \to \infty \quad \text{as} \quad T \to \infty$$

while $F(r)$ is a continuous function of $r > 0$ such that $F(r) > r$ and $F(r)/r$ is non-decreasing.

Suppose that $c(t, r)$ in $(C)$ satisfies

$$\sup_{t \geq 0} c(t, r) < \infty$$

for any $r \geq 0$.

Then, under the condition $(L)$ and $(D)$ the zero solution of $(E)$ is asymptotically stable in $\mathbb{R}^n$ if either $c(t, r)$ in $(C)$ or $q(t, r)$ in $(F)$ is independent of $t$. The latter case corresponds to the case $(6)$ and will be referred to as $(UF)$.

Furthermore, under the conditions $(UL), (UB), (UD)$ and $(UF)$, the zero solution of $(E)$ is uniformly asymptotically stable in $\mathbb{R}^n$.

**Proof.** The proof of this theorem will be given by constructing a Liapunov function of the type mentioned in Theorem 3.

Let $a, b, c, q, p, F$ be the functions relating to the conditions $(A), (B), (C), (F)$ for $V(t, \phi)$, and define
\[ \alpha(t, r) = \frac{1}{q \left( p^{-1}(t, F^{-1}\left( \frac{r}{2} \right)), F^{-1}\left( \frac{r}{2} \right) \right)} \log \frac{r}{F^{-1}(r)}. \]

Clearly, \( \alpha(t, r) \) is a continuous function of \( t \geq 0, r > 0, \) \( \alpha(t, r) > 0 \) for \( r > 0, \)
non-decreasing in \( r \) and non-increasing in \( t, \) and put \( \alpha(t, 0) = \lim_{r \to 0^+} \alpha(t, r). \)

For any \( \tau \geq 0 \) and any \( (t, \phi) \in X_{K_0}(\tau) \) define
\[ V(t, \phi; \tau) = \sup_{t \leq s \leq 0} V(t + s, \phi_0) e^{\alpha(t + s, V(t + s, \phi_0)) t}. \]
Since \( \alpha(t, r) \geq 0, \) obviously this function satisfies the conditions \((A)\) and \((B^\prime)\) with the same \( a(r) \) and
\[ b(t, \tau, r) = \sup_{t \leq s \leq t} b(s, r). \]

Now, we shall prove that \( V(t, \phi; \tau) \) satisfies
\[ \dot{V}((E); t, \phi; \tau) \leq -d(t, V(t, \phi, \tau)) \]
under the condition \((P), \) where \( p(t, r) \) in \((P)\) is the one in \((F)\) and
\[ d(t, r) = \min\{c(t, r), r \alpha(t, r)\}. \]

Suppose that for a given \((t, \phi)\) there exists a solution \( x(u) \) of \((E)\) starting at \( \tau \)
such that \( x_t = \phi, \) \( p(t, V(t, \phi; \tau)) \geq \tau. \)

For the brevity set
\[ V(u) = V(u, x_u), \quad W(u) = V(u, x_u; \tau), \]
\[ P(s, u) = V(s) e^{\alpha(s, V(s))(s-u)}, \]
and then we can find an \( s(u) \in [\tau, u] \) so that
\[ W(u) = P(s(u), u) \geq P(s, u) \quad \text{for} \quad s \in [\tau, u]. \]
Clearly we may assume that \( s(u) \) is continuous at \( u = t. \)

Let \( \{h_k\}, h_k \to 0^+, \) be any sequence. By extracting a subsequence we may assume
that \( s(t + h_k) \leq t \) for all \( k \) or \( s(t + h_k) \geq t \) for all \( k. \)

Case 1. \( s(t + h) \leq t \) for any \( h \in \{h_k\}. \) In this case, since \( W(t) \geq P(s(t + h), t), \)
\[ \frac{W(t + h) - W(t)}{h} \leq P(s(t + h), t + h) - P(s(t + h), t) \]
\[ \leq W(t + h) \frac{1}{h} \left\{ 1 - e^{\alpha(s(t + h), V(s(t + h))) h} \right\} \]
\[ \leq -W(t) \alpha(s(t), V(s(t))) + o(1) \]
\[ \leq -W(t) \alpha(t, W(t)) + o(1), \]
where we note the monotonicity of \( \alpha(t, r) \) in \( t, r \) and the fact that \( V(s(t)) \geq W(t) \).

**Case 2.** \( t \leq s(t + h) \leq t + h \) for all \( h \in [h_0] \).

Then, clearly \( s(t) = t \), and hence \( V(t) = W(t) \geq p(s, t) \) for any \( s \in [\tau, t] \), which assures

\[
(9) \quad V(t) \geq V(s)e^{-\alpha(s, V(s))q(t, V(t))} \quad \text{for} \quad s \in [p(t, V(t)), t],
\]

where we note \( p(t, V(t)) = p(t, W(t)) \geq \tau \). If we can prove that for an \( s \in [p(t, V(t)), t] \)

\[
(10) \quad V(t) \geq F^{-1}\left( \frac{V(s)}{2} \right),
\]

then immediately we have

\[
t \leq p^{-1}\left( s, F^{-1}\left( \frac{V(s)}{2} \right) \right),
\]

and hence by the definition of \( \alpha(t, r) \)

\[
\alpha(s, V(s))q(t, V(t)) \leq \log \frac{V(s)}{F^{-1}(V(s))},
\]

which implies \( V(t) \geq F^{-1}(V(s)) \) by the relation (9). Since \( V(t) \) is continuous and the relation (10) holds at \( s = t \), this fact also assures (10) for all \( s \in [p(t, V(t)), t] \), and hence

\[
F(V(t)) \geq V(s) \quad \text{for all} \quad s \in [p(t, V(t)), t],
\]

that is, \((t, \phi), \phi = x_t\), satisfies the condition \((F)\). By assumptions, we have \( \dot{V}(t) \leq -c(t, V(t)) = -c(t, W(t)) \). Therefore, we have

\[
\frac{W(t + h) - W(t)}{h} = V(s(t + h), \frac{1}{h}\left\{e^{\alpha(s(t + h), V(s(t + h)) (s(t + h) - t - h)} - 1\right\}
\]

\[
+ \frac{V(s(t + h)) - V(t)}{h}
\]

\[
\leq V(t)\alpha(t, V(t))\left\{ \frac{s(t + h) - t}{h} - 1\right\}
\]

\[
+ \dot{V}(t)\frac{s(t + h) - t}{h} + o(1)
\]

\[
\leq -d(t, W(t)) + o(1),
\]

where we note that \( V(t) = W(t) \) and that \( (s(t + h) - t) / h \in [0, 1] \).
Clearly, \((UB), (UP)\) for \(V(t, \phi; r)\) follow from \((UB), (UF)\), respectively, for \(V(t, \phi)\). Finally, it is also obvious that \(d(t, r)\) in (8) satisfies the condition \((D)\) under the assumptions on \(c(t, r)\) and \(q(t, r)\), and it satisfies the condition \((UD)\) if so does \(c(t, r)\) and if \(q\) is independent of \(t\).

Q.E.D.

**Example 2.** The zero solution of the equation

\[
\dot{x}(t) = -ax(t) + b(t)x(g(t)),
\]

considered on \(C\), for a \(\gamma > 0\), is asymptotically stable in \(R^n\) if \(|b(t)| \leq \beta < a\) and \(t \geq g(t) \geq \epsilon t - N\) for an \(\epsilon \in (0, 1]\) and an \(N > 0\), because we can apply Theorem 4 by putting

\[
V(t, \phi) = \phi(0)^2, \quad F(r) = \rho^2r \quad \text{for a } \rho > 1 \text{ such that } \beta \rho < a,
\]

\[
p(t, r) = \epsilon t - N.
\]

**Remark.** Driver [4] has shown that the zero solution of (11) is asymptotically stable in \(R^n\) if \(g(t) \to \infty\) as \(t \to \infty\), which contains the case where

\[
g(t) = \frac{\sqrt{1+4t} - 1}{2},
\]

but unfortunately our result does not cover this case because \(q(t) = t - g(t)\) does not satisfy the relation (7). However, in [4] the definition of the asymptotic stability allows \(T\) in (4) to depend on each solution, while our definition requires actually the equi-asymptotic stability.

**Example 3.** Consider the scalar equation

\[
\dot{x}(t) = -ax(t) + \int_{-\infty}^{0} g(t, s, x(t+s))ds,
\]

and assume that \(a > 0\) is a constant and \(g(t, s, r)\) is continuous in \((t, s, r)\) and satisfies

\[
|g(t, s, r)| \leq m(s) |r| \quad \text{for } t \geq 0, s \leq 0, r \in R.
\]

If \(M(s)\) satisfies

\[
\int_{-\infty}^{0} m(s)ds < a, \quad \int_{-\infty}^{0} m(s)e^{-\gamma t}ds \text{ exists}
\]

for a \(\gamma \geq 0\), then the zero solution of (12), considered as an equation on \([0, \infty) \times C\), is uniformly asymptotically stable in \(R^1\).

**Proof.** Choose a constant \(\rho > 1\) and a function \(q(r)\) so that \(\rho \int_{-\infty}^{0} m(s)ds < a\) and for any \(r > 0\)
\[ 2 \int_{-\infty}^{-q(r)} m(s) e^{-rt} ds \leq r \left\{ a - \rho \int_{-\infty}^{0} m(s) ds \right\} = c(r). \]

Then, \( V(t, \phi) = \phi(0)^2 \) satisfies all the conditions in Theorem 4 with \( F(r) = \rho^2 r, \quad p(t, r) = t - q(r), \quad c(t, r) = c(r), \) while \((UL)\) is satisfied by Lemma 1 because
\[
\left| -a \phi(0) + \int_{-\infty}^{t} g(t, s, \phi(s)) ds \right| \leq \left\{ a + \int_{-\infty}^{0} m(s) e^{-rt} ds \right\} |\phi|_{C_r}.
\]

**Example 4.** Consider the scalar equation

\[ \dot{x}(t) = -\int_{-\infty}^{t} g(t, s, x(t+s)) ds, \]

and assume that \( g(t, s, r) \) is a continuous function satisfying
\[ 0 \leq m'(s) \leq \frac{g(t, s, r)}{r} \leq m(s) \quad (r \neq 0). \]

If \( \int_{-\infty}^{0} m(s) e^{-rt} ds < \infty \) for a \( r > 0 \) and if \( \mu(\int_{-\infty}^{0} m(s) ds, 1) > 0 \), where
\[
\mu(\sigma, \rho) = \int_{-\sigma}^{0} m'(s)(1 + \sigma s) ds + \int_{-\sigma}^{-\sigma/2} m(s)(1 + \sigma s) ds - \rho \int_{-\infty}^{-\sigma/2} m(s) ds,
\]
then the zero solution of (13), considered on the space \([0, \infty) \times C_r\), is uniformly asymptotically stable in \( R^1 \).

Especially, if \( g(t, s, r) = g(t) e^{\alpha s} r \) for a constant \( \alpha > 0 \) and a continuous function \( g(t) \) satisfying
\[
\frac{\alpha^2 (e^{1} - e^{-2\lambda})}{\lambda(\lambda - 1 + e^{-\lambda})} < g(t) \leq \frac{\alpha^2}{\lambda}
\]
with a constant \( \lambda > 0 \) such that \( \lambda - 1 + e^{-\lambda} > 0 \), then the zero solution of (13), considered on the space \([0, \infty) \times C_r, r < \alpha\), is uniformly asymptotically stable in \( R^1 \).

**Proof.** Since \( \mu(\sigma, \rho) \) is continuous, we can find constants \( \rho > 1 \) and \( \varepsilon > 0 \) such that \( \mu(\sigma, \rho) > \varepsilon \) for \( \sigma = \rho \int_{-\infty}^{0} m(s) ds + \varepsilon \). Choose \( q(r) \) so that
\[
\int_{-\infty}^{-q(r) + \sigma/\rho} m(s) e^{-rs} ds < \varepsilon r e^{-s/\rho}.
\]
Suppose that the solution \( x(u) \) is defined on \([t - q(1/\sigma), t]\) and satisfies \( |x(u)| \leq \rho |x(t)| \) there. For any \( r \in [t - 2/\sigma, t] \) we have
\[
|\dot{x}(r)| \leq \int_{-\infty}^{r-t} m(s) |x(r+s)| ds \leq \int_{-\infty}^{r-t} m(t-r+s) |x(t+s)| ds
\]
\[
\leq \int_{-\rho/(\phi(0))}^{r-t} m(t+s-r) \rho |x(t)| ds + \int_{-\infty}^{-\rho/(\phi(0))} m(t+s-r)e^{-\rho t} ds |x_t|_{C_T}
\]
\[
\leq \rho |x(t)| \int_{-\infty}^{0} m(s) ds + \int_{-\infty}^{-\rho/(\phi(0))} m(s)e^{s/(\sigma_0-\rho)} ds |x_t|_{C_T}
\]
\[
\leq \sigma_0 |x(t)| \quad \text{if } |x_t|_{C_T} \leq 1,
\]
and hence \(x(t)x(t+s) \geq |x(t)|^2 (1+\sigma_0 s)\) on \(s \in [-2/\sigma_0, 0]\). Thus, we have
\[
\dot{x}(t)x(t) = -x(t) \int_{-\infty}^{t} g(t, s, x(t+s)) ds
\]
\[
\leq -|x(t)|^2 \int_{-1/\sigma_0}^{0} m'(s)(1+\sigma_0 s) ds
\]
\[
-|x(t)|^2 \int_{-\rho/(\phi(0))}^{-1/\sigma_0} m(s) \max \{1+\sigma_0 s, -\rho\} ds
\]
\[
+ |x(t)| \int_{-\infty}^{-\rho/(\phi(0))} m(s)|x(t+s)| ds
\]
\[
\leq -\mu(\sigma_0, \rho) |x(t)|^2 + \varepsilon |x(t)|^2
\]
if \(|x_t|_{C_T} \leq 1\), which shows that \(V(t, \phi) = |\phi(0)|^2\) satisfies the conditions in Theorem 4 with \(F(r) = \rho^2 r\). Thus, we can prove the first part.

For the second part, it is sufficient to show that if
\[
m'(s) > \frac{c^2(e^{-s}-e^{-2})}{(\lambda - 1 + e^{-s})} e^{as}, \quad m(s) \leq \frac{c^2}{\lambda} e^{as}
\]
then \(\mu(\int_{-\infty}^{0} m(s) ds, 1) > 0\).

2. Stability of perturbed systems.

In the above, we have discussed the stability in \(\mathbb{R}^n\), while in [2] we have discussed relationships between the stability in \(\mathbb{R}^n\) and the stability in \(B\). Under the hypotheses on our space \(B\) we can reproduce Theorem 6 in [2] in the following form.

**Theorem 5.** Suppose that in the relation (2) \(K(s) = K\) is a constant for \(s \geq 0\) and \(M(s) \to 0\) as \(s \to \infty\).

Then, the concept of the (uniform) (asymptotic) stability in \(\mathbb{R}^n\) and in \(B\) is equivalent. Furthermore, if \(M(t)\) in (2) satisfies
\[
M(t) \leq Me^{-\rho t}
\]

(14)
for positive constants $M, \mu$ and if the zero solution of $(E)$ is exponentially stable in $\mathbb{R}^n$, then it is exponentially stable in $B$.

**Proof.** The proofs except for the case of the exponential stability have been given in [2; Theorem 6].

Suppose that the zero solution of $(E)$ is exponentially stable in $\mathbb{R}^n$, that is, the relation (5) holds for the solution $x(u)$ of $(E)$. By the assumptions and the condition (2) we have

$$|x_t|_B \leq K \sup_{s \leq t} |x(s)| + Me^{-\rho(t-\sigma)} |x_s|_B$$

for any $\sigma \in [\tau, t]$ if $x(u)$ is a solution of $(E)$ starting at $\tau$. Putting $\sigma = \tau$ in (15) we have

$$|x_t|_B \leq (K\eta + M) |x_\tau|_B$$

if $|x_\tau|_B < \delta_0$

where $\delta_0$ is the one in (5). Next, putting $\sigma = (\mu(t + \alpha \tau)/\alpha + \mu) \in [\tau, t]$ in (15) we have

$$|x_t|_B \leq K\eta e^{-\alpha(t-\tau)} |x_\tau|_B + Me^{-\rho(t-\sigma)} |x_\tau|_B$$

$$\leq [K\eta e^{-\alpha(t-\tau)} + Me^{-\rho(t-\sigma)}(K\eta + M)] |x_\tau|_B$$

$$\leq [K\eta(1 + M) + M] e^{-\beta(t-\tau)} |x_\tau|_B$$

if $|x_\tau|_B < \delta_0$,

which proves the exponential stability in $B$, where $\beta = \alpha \mu / (\alpha + \mu)$ and we note that $\alpha(t - \sigma) = \mu(t - \alpha) = \beta(t - \tau)$ for $\sigma$ in the above. Q.E.D.

By the same way as for ordinary differential equations (for example, see [11]), it is possible to prove the following theorem (see [5] for the exponential stability).

**Theorem 6.** Suppose that $f(t, \phi)$ in $(E)$ satisfies a Lipschitz condition

$$|f(t, \phi) - f(t, \psi)| \leq L_1 |\phi - \psi|_B$$

for a constant $L_1$ and that the zero solution of $(E)$ is uniformly asymptotically stable in $B$.

Then, there exists a continuous function $V(t, \phi)$, defined on $[0, \infty) \times B_{2\delta}$ for a small $H_0 > 0$, satisfying the conditions

$$a(|\phi|_B) \leq V(t, \phi) \leq b(|\phi|_B),$$

$$\dot{V}(E_1)(t, \phi) \leq -c V(t, \phi),$$

$$|V(t, \phi) - V(t, \psi)| \leq L |\phi - \psi|_B$$

for positive-definite functions $a(r), b(r)$ and positive constants $c, L$.

Moreover, if the zero solution of $(E)$ is exponentially stable in $B$, then $a(r)$ and $b(r)$ in the above can be chosen so as to be linear in $r$. 
Applying this theorem, we can readily state theorems on the stability of the perturbed system

\[ \dot{x}(t) = f(t, x_t) + X(t, x_t) \]

of the system \((E)\), because if \(V(t, \phi)\) satisfies the relation \((17)\), we have

\[ \dot{V}_{(18)}(t, \phi) \leq \dot{V}_{(E)}(t, \phi) + LK(0)V(t, \phi). \]

**Theorem 7.** Suppose that \(K(t) = K\) is a constant and \(M(t) \to 0\) as \(t \to \infty\) in the relation \((2)\) and that \(f(t, \phi)\) satisfies the condition \((16)\).

If the zero solution of \((E)\) is uniformly asymptotically stable in \(R^n\), then there exists a continuous, positive definite function \(\varepsilon(r)\) such that if

\[ |X(t, \phi)| \leq \varepsilon(|\phi|_B), \]

then the zero solution of \((18)\) is uniformly asymptotically stable in \(B\).

Moreover, if the relation \((15)\) holds and if the zero solution of \((E)\) is exponentially stable in \(R^n\), then the zero solution of \((18)\) is exponentially stable in \(B\) under the condition

\[ |X(t, \phi)| \leq h(t)|\phi|_B \]

for a small \(\varepsilon > 0\) and a continuous function \(h(t)\) satisfying

\[ \lim_{t \to \infty} \int_t^{t+1} h(s)ds < \varepsilon. \]

**Proof.** We shall prove the first part of the theorem.

Under the assumptions, the zero solution of \((E)\) is uniformly asymptotically stable in \(B\) by Theorem 5, and hence there is a continuous function \(V(t, \phi)\) as given in Theorem 6.

Therefore, by relation \((19)\) we have

\[ \dot{V}_{(18)}(t, \phi) \leq \dot{V}_{(E)}(t, \phi) + LK\varepsilon(|\phi|_B) \]

\[ \leq -cV(t, \phi) + LK\varepsilon(|\phi|_B). \]

Put

\[ \varepsilon(r) = \frac{c}{2LK}a(r). \]

Then, clearly we have

\[ \dot{V}_{(18)}(t, \phi) \leq -\frac{c}{2}V(t, \phi), \]
which guarantees that the zero solution of (18) is uniformly asymptotically stable in $B$.

For the second part of the theorem, we note that under the assumptions, there exists a $V(t, \phi)$ with $a(r) = ar$ and $b(r) = br$ linear in $r$ by Theorems 5 and 6. Therefore, we have

$$\dot{V}(t, \phi) \leq -cV(t, \phi) + \frac{LK}{a} h(t)V(t, \phi).$$

Consider the equation

$$\dot{u} = -cu + \frac{LK}{a} h(t)u.$$

Then, we have

$$u(t) = u(\tau) \exp \left[ -c(t - \tau) + \frac{LK}{a} \int_{\tau}^{t} h(s) \, ds \right].$$

Put

$$\varepsilon = \frac{ac}{2LK}$$

in the relation (20), and then we have

$$\int_{\tau}^{t} h(s) \, ds \leq \frac{ac}{2LK} (t - \tau) + N_0$$

for an $N_0 > 0$ and all $t \geq \tau$.

Thus, by the comparison theorem we have

$$|x_t| \leq \frac{1}{a} V(t, x_t) \leq \frac{1}{a} u(t) \leq \frac{1}{a} u(\tau)e^{(LK/a)N_0} \varepsilon^{-c/2}(t-\tau)$$

$$\leq N|x_t| e^{-a(t-\tau)},$$

where $\alpha = c/2$, $N = (1/a)e^{(LK/a)N_0}b$, which shows that the zero solution of (18) is exponentially stable in $B$. Q.E.D.

Example 5. If $a > 0$, $ah < 3/2$ and if $|X(t, \phi)| \leq \varepsilon |\phi|_\alpha$, for a $\gamma > 0$ and a suitably small $\varepsilon > 0$, then the zero solution of the scalar equation

$$\dot{x}(t) = -ax(t - h) + X(t, x_t)$$

is exponentially stable in $R^1$. 

This follows immediately from Theorem 7, since as was seen in [10; Theorem B] the zero solution of the equation
\[
\dot{x}(t) = -ax(t-h)
\]
is exponentially stable in $\mathbb{R}^n$ under the conditions on $a$ and $h$.

Example 6. Consider a system
\[
\dot{x}(t) = A_1 x(t) + A_2 x(t-h) + \int_0^t A(s-t)x(s)ds
\]
and assume that $h>0$ is a constant and that $A_1, A_2, A(s)$ are square matrices satisfying
\[
\det \left( \lambda I - A_1 - A_2 e^{-\lambda h} - \int_0^\infty A(s)e^{\lambda s}ds \right) \neq 0 \quad \text{if } \Re \lambda \geq 0, \lambda \in \mathbb{C},
\]
\[
\int_{-\infty}^0 |A(s)| e^{-\gamma s}ds < \infty \quad \text{for } \gamma > 0.
\]

Then, the zero solution of (21) is exponentially stable in $\mathbb{R}^n$.

Proof. The system (21) can be written in the form
\[
\dot{x}(t) = A_1 x(t) + A_2 x(t-h) + \int_{-\infty}^0 A(s)x(t-s)ds - \int_{-\infty}^{-t} A(s)x(t+s)ds,
\]
which is a perturbed system of the autonomous linear system
\[
\dot{x}(t) = A_1 x(t) + A_2 x(t-h) + \int_{-\infty}^0 A(s)x(t+s)ds.
\]

The condition (23) assures that these systems can be considered as a functional differential equation on the space $C_r$ and that the solution operator induced by (24) has only point spectra in the half plane $\{ \lambda \in \mathbb{C}; \Re \lambda > -\gamma \}$ (see [2; Section 5]). On the other hand, the condition (22) shows that the point spectra of the solution operator have negative real parts, and hence the zero solution of (24) is exponentially stable in $\mathbb{R}^n$ (see [12; Theorem 4.4]).

Put
\[
h(t) = \int_{-\infty}^{-t} |A(s)| e^{-\gamma s}ds,
\]
and then we have $h(t) \to 0$ as $t \to \infty$ and
\[
\left| \int_{-\infty}^{-t} A(s)\phi(s)ds \right| \leq h(t) |\phi|_{C_r}.
\]
Hence, by applying Theorem 7 we can see that the zero solution of (21) is exponentially stable in $C$, and, then, in $R^n$. Q.E.D.

Remark. When $A_2=0$, Example 6 is discussed by Miller [13] under a weaker condition than (23) but he has obtained merely the uniformly asymptotic stability in $R^n$.

References


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