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Phase Space for Retarded Equations with Infinite Delay

By

Jack K. HALE* and Junji KATO

(Brown University, U.S.A. and Tohoku University, Japan)

Introduction.

Suppose $0 \leq r \leq +\infty$ is given. If $x: [\sigma-r, \sigma+A) \rightarrow R^n$, $A>0$, is a given function, let $x_t: [-r, 0] \rightarrow R^n$, for each $t \in [\sigma, \sigma+A)$, be defined by $x_t(\theta)=x(t+\theta)$, $-r \leqq \theta \leqq 0$. In the theory of retarded functional differential equations

$$(1) \qquad \dot{x}(t)=f(t, x_t),$$

the appropriate choice for the space of initial data for a solution x at $t=\sigma$ is never very clear. If $r<+\infty$, the development of a general qualitative theory is not too sensitive to this choice since a solution is generally defined to be a continuous function for $t \geqq \sigma$. Therefore, after one delay interval r, the state x_t, $t \geqq \sigma+r$, belongs to the space of continuous functions. However, even for $r<+\infty$, there are situations where one must consider spaces other than continuous functions. If $r=+\infty$, the state x_t always contains part of the initial functions. As a result of this, each different phase space requires a new and separate development for the theory.

It is the purpose of this paper to examine initial data from a general Banach space. We develop a theory of existence, uniqueness, continuous dependence, and continuation by requiring that the space B only satisfies some general qualitative properties. Also, we impose conditions of B which will at least indicate the feasibility of a qualitative theory as general as the one presently available for retarded equations with finite delay in the space of continuous functions. In particular, this will imply that bounded orbits should be precompact and that the essential spectrum of the solution operator for a linear autonomous equation should be inside the unit circle for $t>0$. Also, we impose conditions which imply the definitions of asymptotic stability in R^n and B are equivalent and that the ω-limit set of a precompact orbit for an autonomous equation should be compact, connected and invariant.

The beginnings of such a theory appeared for the first time in [1], but there were only a few axioms, no proofs, and, therefore, several confusions and omissions. However, the ideas stimulated some interest in the subject and several papers have

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appeared [2~6]. The present paper resulted from a careful study of these papers. In the paper we take $r=+\infty$ although everything is equally valid for $r<\infty$.

A few remarks at the beginning will assist the reader in understanding why our axioms prevent the norm in the space from imposing any differentiability properties on the initial functions. In the applications, it is certainly convenient at times to require the initial functions to belong to a Banach space of functions which have some derivative satisfying specified properties. However, if one considers all differential equations whose right hand sides are continuous or continuously differentiable in such a space, then the equations will be of neutral type; that is, the derivatives of the independent variable will also contain delays. The theory for such systems certainly should be developed, but it will require much more sophistication than the one described in the text.

Our axioms are imposed to ensure that only retarded equations will be considered. This does not mean that one cannot consider these retarded equations with the initial data restricted to certain Banach spaces which impose conditions on the derivatives. In fact, one can consider such spaces which can be continuously imbedded (completely continuously embedded) in a space satisfying the axiomes in the text to obtain existence and uniqueness. The other properties could be investigated using the differential equation itself and the fact that everything is known in the larger space. These remarks are not meant to imply that the task is trivial but are merely suggestions as to how one could possibly proceed.

§ 1. Notations, definitions, fundamental axioms and examples.

At first, we need to identify certain equivalence classes in some normed linear spaces in order to state the fundamental axioms which always will be imposed. This requires some cumbersome notations which can be simplified later.

Let $\hat{B}$ be a linear real vector space of functions mapping $(-\infty, 0]$ into R^n with elements designated by $\hat{\phi}, \hat{\psi}, \cdots$ and $\hat{\phi}=\hat{\psi}$ means $\hat{\phi}(t)=\hat{\psi}(t)$ for all $t\leqq 0$. Assume that a semi-norm $|\cdot|_{\hat{B}}$ is given in $\hat{B}$, and assume that

$$B=\hat{B}/|\cdot|_{\hat{B}}$$

is a Banach space with the norm $|\cdot|_B$ which is naturally induced by $|\cdot|_{\hat{B}}$. Elements of B are denoted by $\phi, \psi, \cdots$ and correspond to equivalence classes of $\hat{B}$. For any $\phi\in B$, any element of the corresponding equivalence class will be denoted by $\hat{\phi}$, and hence $\phi=\psi$ in B means $|\hat{\phi}-\hat{\psi}|_{\hat{B}}=0$.

For a $\beta\geqq 0$ and a $\hat{\phi}\in\hat{B}$, let $\hat{\phi}^\beta$ denote the restriction of $\hat{\phi}$ to $(-\infty, -\beta]$, and let $|\cdot|_\beta$ be a semi-norm in B defined by

$$(1.1) \qquad |\phi|_\beta=\inf_{\hat{\eta}\in\hat{B}}\left\{\inf_{\hat{\psi}\in\hat{B}}\{|\hat{\psi}|_{\hat{B}}: \hat{\psi}^\beta=\hat{\eta}^\beta\}: \eta=\phi\right\}.$$

Since $|\phi|_\beta \leqq |\phi|_B$, $\{\phi \in B: |\phi|_\beta = 0\}$ is a subspace of B and hence

$$B^\beta = B/|\cdot|_\beta$$

becomes a Banach space with the norm $|\cdot|_\beta$ (naturally induced by the semi-norm $|\cdot|_\beta$ using the same notation). If

$$\{\phi\}_\beta = \{\psi \in B; |\phi - \psi|_\beta = 0\}$$

is a representative element of B^β, $\psi \in \{\phi\}_\beta$ if $\hat{\psi}^\beta = \hat{\phi}^\beta$.

For an R^n-valued function $\hat{x}$ defined on an interval $(-\infty, \sigma)$ and for a $t \in (-\infty, \sigma)$, let $\hat{x}_t$ be a function defined on $(-\infty, 0]$ such that

$$\hat{x}_t(\theta) = \hat{x}(t+\theta), \qquad \theta \leqq 0.$$

Given an $A > 0$ and a $\hat{\phi} \in \hat{B}$, let $F_A(\hat{\phi})$ be the set of all functions $\hat{x}$ defined on $(-\infty, A]$ such that $\hat{x}_0 = \hat{\phi}$ and $\hat{x}(t)$ is continuous on $[0, A]$, and denote

$$F_A = \bigcup \{F_A(\hat{\phi}) : \hat{\phi} \in \hat{B}\}.$$

Our first axiom is the following.

(α_1) $\hat{x}_t \in \hat{B}$ for all $\hat{x} \in F_A$ and all $t \in [0, A]$.

Now, we shall denote by x_t the element of B corresponding to $\hat{x}_t$. Under this axiom, for any $\beta \geqq 0$ and $\hat{\phi} \in \hat{B}$ it is possible to find a $\hat{\psi} \in \hat{B}$ such that

$$\hat{\psi}(\theta) = \hat{\phi}(\theta + \beta) \qquad \text{for } \theta \in (-\infty, -\beta].$$

By $\hat{\tau}^\beta$, $\beta \geqq 0$, we shall denote a linear operator on $\hat{B}$ into

$$\hat{B}^\beta = \{\{\hat{\psi} \in \hat{B} : \hat{\psi}^\beta = \hat{\phi}^\beta\} : \hat{\phi} \in \hat{B}\} \subset 2^{\hat{B}}$$

such that $\hat{\psi} \in \hat{\tau}^\beta \hat{\phi}$ if and only if

$$\hat{\psi}(\theta) = \hat{\phi}(\theta + \beta) \qquad \text{for } \theta \in (-\infty, -\beta].$$

Our second axiom is:

(α_2) If $\phi = \psi$ in B, then $|\eta - \xi|_\beta = 0$ for any $\beta \geqq 0$, where $\hat{\eta} \in \hat{\tau}^\beta \hat{\phi}$ and $\hat{\xi} \in \hat{\tau}^\beta \hat{\psi}$.

Under this axiom, it is possible to consider a linear operator $\tau^\beta : B \to B^\beta$ defined by

$$\tau^\beta \phi = \{\psi\}_\beta \qquad \text{for a } \psi \in B \text{ such that } \hat{\psi} \in \hat{\tau}^\beta \hat{\phi}.$$

In a similar way as $|\cdot|_\beta$, we can introduce a semi-norm $|\phi|_{(\beta)}$, $\beta \geqq 0$, in B by

$$|\phi|_{(\beta)} = \inf_{\hat{\eta} \in \hat{B}} \left\{ \inf_{\hat{\psi} \in \hat{B}} \{|\hat{\psi}|_{\hat{B}} : \hat{\psi}(\theta) = \hat{\eta}(\theta) \text{ on } [-\beta, 0]\} : \eta = \phi \right\}$$

and it will be quite natural to assume

(α_3) $|\phi|_B \leqq |\phi|_{(\beta)} + |\phi|_\beta$ for any $\beta \geqq 0$.

By this axiom, we can see that if $x_0 = y_0$ and $\hat{x}(t) = \hat{y}(t)$ on $[0, A]$, then $x_t = y_t$, which makes it possible to consider $x \in F_A(\phi)$ rather than $\hat{x} \in F_A(\hat{\phi})$. However, to do this, we must have $\hat{\phi}(0) = \hat{\psi}(0)$ when $\phi = \psi$. Therefore, we make the following assumption

(α_4') $|\hat{\phi}(0)| \leqq K |\hat{\phi}|_{\hat{B}}$ for any $\hat{\phi} \in \hat{B}$ and some constant K.

(α_4') shows that $\hat{\psi}(0)$ should be the same for every $\hat{\psi}$ with $\psi = \phi$ and therefore can be denoted by $\phi(0)$. We now can rewrite the axiom (α_4') as follows:

(α_4) $|\phi(0)| \leqq K |\phi|_B$ for any $\phi \in B$ and some K.

These are fundamental axioms on B, and we suppose that Axioms ($\alpha_1 \sim \alpha_4$) are always satisfied hereafter without any special notice. Under these fundamental axioms, there will be no confusions in using the same letter ϕ to denote an element of B and also of $\hat{B}$. Thus, we shall omit the symbol 'hat' for the element of $\hat{B}$ and even for the space $\hat{B}$ unless special considerations are required.

A *retarded functional differential equation on Ω*, referred to as an RFDE(f), or RFDE(f, Ω), is a relation

$$\dot{x}(t) = f(t, x_t), \tag{1.2}$$

where Ω is an open set in $R \times B$ and $f: \Omega \to R^n$ is a given continuous function. By a *solution of* RFDE(f) on an interval $I \subset R$, we mean a function $x: \bigcup \{(-\infty, t]: t \in I\} \to R^n$ such that $(t, x_t) \in \Omega$ for $t \in I$, $x(t)$ is continuously differentiable and satisfies (1.2) on I. For a given $(\sigma, \phi) \in \Omega$, we say $x(\sigma, \phi)$ is a *solution of* RFDE(f) *through* (σ, ϕ) if there is an $A > \sigma$ such that $x(\sigma, \phi)$ is a solution of RFDE(f) on $[\sigma, A]$ and $x_\sigma(\sigma, \phi) = \phi$. Here, we should note that $x(\sigma, \sigma, \phi) = \phi(0)$ is a definite value in R^n by Axiom (α_4), where $x(t, \sigma, \phi)$ denotes the value in R^n of $x(\sigma, \phi)$ at $t \geqq \sigma$.

To discuss the local and global theory of retarded functional differential equations, additional hypotheses will be needed. However, it is instructive to consider some specific examples of spaces that satisfy the fundamental axioms above.

Example 1.1. *Spaces of integrable functions.* Suppose $g: (-\infty, 0] \to R$ is a nonnegative locally integrable function satisfying ess. sup $\{g(s): t \leqq s \leqq 0\} < \infty$ for any $t < 0$ and

$$g(t+s) \leqq G(t) g(s) \qquad \text{for all } t \leqq 0 \text{ and } s \in (-\infty, 0] - N_t \tag{1.3}$$

for a set $N_t \subset (-\infty, 0]$ with measure zero and for a nonnegative function $G: (-\infty, 0] \to R$. For example, (1.3) is satisfied if g is nondecreasing. Under the conditions above, for any $\gamma < \sup\{(1/s)\log G(s): s<0\}$ there exists a constant $C(\gamma)$ such that

$$g(t) \leqq C(\gamma)e^{\gamma t} \qquad \text{a.e. } t \leqq 0. \tag{1.4}$$

In fact, for $\gamma < \sup\{(1/s)\log G(s): s<0\}$ choose an $s = s_\gamma < 0$ so that $(1/s_\gamma)\log G(s_\gamma) \geqq \gamma$; that is, $G(s_\gamma) \leqq e^{s_\gamma \gamma}$. Let N^γ be the set of $t \leqq 0$ such that $t - ks_\gamma \in N_{s_\gamma}$ for some integer k. Then, clearly N^γ is of measure zero, and we have

$$g(t) \leqq G(s_\gamma)g(t-s_\gamma) \leqq G(s_\gamma)^m g(t-ms_\gamma) \qquad \text{for } t \in (-\infty, 0] - N^\gamma$$

if $t - ms_\gamma \leqq 0$. Therefore, we have (1.4) with

$$C(\gamma) = \text{ess}\cdot\sup\{g(s); 0 \geqq s \geqq s_\gamma\}\cdot\max\{1, e^\gamma\}.$$

Remark 1.1. If we take

$$G(s) = \operatorname*{ess\cdot sup}_{t\leqq 0} \frac{g(s+t)}{g(t)},$$

then G itself satisfies Relation (1.3); that is,

$$G(t+s) \leqq G(s)G(t) \qquad \text{for all } t, s \leqq 0.$$

Therefore, G itself satisfies (1.4); that is,

$$\operatorname*{ess\cdot sup}_{t\leqq 0} \frac{g(t+s)}{g(t)} \leqq C(\gamma)e^{\gamma s} \qquad \text{if } \gamma < \sup_{s\leqq 0} \frac{1}{s} \log \operatorname*{ess\cdot sup}_{t\leqq 0} \frac{g(t+s)}{g(t)}.$$

For the boundedness of $\text{ess}\cdot\sup\{G(s); 0 \geqq s \geqq s_\gamma\}$, refer [7].

Let $\hat{B} = \{\hat{\phi}: (-\infty, 0] \to R^n$, measurable and $|\hat{\phi}|_{\hat{B}} < \infty\}$, where

$$|\hat{\phi}|_{\hat{B}} = |\hat{\phi}(0)| + \int_{-\infty}^0 g(\theta)|\hat{\phi}(\theta)|\,d\theta.$$

The corresponding space B is a Banach space which satisfies all of Axioms $(\alpha_1 \sim \alpha_4)$.

To verify (α_1), suppose $\hat{x} \in F_A$, $A>0$. Then, $\hat{x}_0 \in \hat{B}$, and for $t \in [0, A]$, the function $\hat{x}_t(\theta)$ is measurable in θ and

$$\begin{aligned} |\hat{x}_t|_{\hat{B}} - |\hat{x}(t)| &= \int_{-\infty}^0 g(\theta - t)|\hat{x}_0(\theta)|\,d\theta + \int_{-t}^0 g(\theta)|\hat{x}(t+\theta)|\,d\theta \\ &\leqq G(-t)|\hat{x}_0|_{\hat{B}} + \sup_{0\leqq s\leqq t}|x(s)| \int_{-t}^0 g(\theta)d\theta < \infty. \end{aligned}$$

Thus, $\hat{x}_t \in \hat{B}$.

Axiom (α_4) is satisfied with $K=1$, and Axiom (α_2) is obviously satisfied since $\phi = \psi$ in B if and only if $\hat{\phi}(\theta) = \hat{\psi}(\theta)$ a.e. on $\{\theta: g(\theta)>0\}$. Here, we should note that by (1.3) the set $\text{Cl}\{\theta: g(\theta)>0\}$ is an interval of the form $[-r, 0]$ if $g \not\equiv 0$. It is not difficult to see that $|\phi|_\beta = |\hat{\psi}|_{\hat{B}}$ and $|\phi|_{(\beta)} = |\hat{\phi} - \hat{\psi}|_{\hat{B}}$, where

$$\hat{\psi}(\theta)=\begin{cases}\hat{\phi}(\theta), & \theta\leqq-\beta\\ 0, & -\beta\leqq\theta\leqq 0\end{cases}$$

which belongs to $\hat{B}$, and hence Axiom (α_3) is satisfied.

Example 1.2. It will be left for the reader to verify that Axioms $(\alpha_1\sim\alpha_4)$ are satisfied for $1\leqq p<\infty$ and $\hat{B}=\{\hat{\phi}:(-\infty,0]\to R^n$, measurable on $(-\infty,-r]$, continuous on $[-r,0]$, $|\hat{\phi}|_{\hat{B}}<\infty\}$, where $r\geqq 0$ and

$$|\hat{\phi}|_{\hat{B}}=\left\{\sup_{-r\leqq\theta\leqq 0}|\hat{\phi}(\theta)|^p+\int_{-\infty}^{0}g(\theta)\,|\hat{\phi}(\theta)|^p\,d\theta\right\}^{1/p}$$

with $g:(-\infty,0]\to R$ as given in Example 1.1.

Example 1.3. *Spaces of continuous functions.* For any $\gamma\in R$, let

$$\hat{B}=\{\hat{\phi}\in C((-\infty,0],R^n):e^{\gamma\theta}\hat{\phi}(\theta)\to\text{a limit as }\theta\to-\infty\},$$

and let

$$|\hat{\phi}|_{\hat{B}}=\sup_{-\infty<\theta\leqq 0}e^{\gamma\theta}\,|\hat{\phi}(\theta)|. \tag{1.5}$$

To verify (α_1), simply observe that if $e^{\gamma\theta}\hat{\phi}(\theta)\to a_{\hat{\phi}}$ as $\theta\to-\infty$, then $e^{\gamma\theta}\hat{\phi}(t+\theta)\to a_{\hat{\phi}}e^{-\gamma t}$ as $\theta\to-\infty$.

Axiom (α_4) is satisfied with $K=1$. Since

$$|\hat{\phi}|_{\hat{B}}\leqq\max\left\{\sup_{-\beta\leqq\theta\leqq 0}e^{\gamma\theta}\,|\hat{\phi}(\theta)|,\ \sup_{\theta\leqq-\beta}e^{\gamma\theta}\,|\hat{\phi}(\theta)|\right\}\leqq\max\{|\phi|_{(\beta)},|\phi|_{\beta}\},$$

Axiom (α_3) is satisfied. Axiom (α_2) holds obviously because $\phi=\psi$ means $\hat{\phi}(\theta)\equiv\hat{\psi}(\theta)$.

§ 2. Axioms for the local theory.

In addition to the fundamental axioms $(\alpha_1\sim\alpha_4)$, the following hypotheses will be needed on the space B.

(β_1) There is a continuous function $K_1(\beta)$ of $\beta\geqq 0$ such that

$$|\phi|_{(\beta)}\leqq K_1(\beta)\,|\phi|_{[-\beta,0]},\qquad \beta\geqq 0,$$

where

$$|\phi|_{[-\beta,0]}=\inf_{\hat{\psi}\in\hat{B}}\left\{\sup_{-\beta\leqq\theta\leqq 0}|\hat{\psi}(\theta)|:\psi=\phi\right\}.$$

(β_2) τ^β is a bounded linear operator for any $\beta\geqq 0$ for which the norm of τ^β,

$$M_1(\beta)=\sup_{|\phi|_B=1}|\tau^\beta\phi|_\beta,$$

is *locally bounded*; that is, for any $\beta \in [0, \infty)$ there exists a neighborhood U of β such that

$$\sup_{t \in U \cap [0,\infty)} M_1(t) < \infty.$$

(β_3) If $x \in F_A$, $A > 0$, then x_t is continuous in t on $[0, A]$.

To estimate the solutions or, more generally, a function $x: (-\infty, A) \to R^n$ such that $x_\sigma \in B$ for a $\sigma \in (-\infty, A)$ and $x(t)$ is continuous on $[\sigma, A)$, the following inequality plays an important role under Hypotheses ($\beta_1 \sim \beta_2$):

$$(2.1) \qquad |x_t|_B \leqq K_1(t-\sigma) \sup_{\sigma \leqq s \leqq t} |x(s)| + M_1(t-\sigma)|x_\sigma|_B$$

for all $t \in [\sigma, A)$, which can be obtained by observing

$$|x_t|_B \leqq |x_t|_{(\beta)} + |x_t|_\beta \leqq K_1(\beta)|x_t|_{[-\beta,0]} + |\tau^\beta x_{t-\beta}|_\beta$$

and setting $\beta = t - \sigma$.

Now, we have the following lemma.

Lemma 2.1. *Suppose Hypotheses* ($\beta_1 \sim \beta_3$) *are satisfied, and let*

$$F_A^L(\Gamma) = \bigcup_{\phi \in \Gamma} \{x \in F_A(\phi): x(t) \text{ is } L\text{-Lipschitzian on } [0, A]\},$$

where $x(t)$ is said to be L-Lipschitzian on I, if

$$|x(t) - x(t')| \leqq L|t - t'| \qquad \text{on } I.$$

If $\Gamma \subset B$ is compact and $A < \infty$, then the set $\Gamma_0 = \{x_t: t \in [0, A],\ x \in F_A^L(\Gamma)\}$ is compact and x_t, $x \in F_A^L(\Gamma)$, is equi-continuous in t.

Proof. Choose any sequence $\{x_{t_k}^k: t_k \in [0, A],\ x^k \in F_A^L(\Gamma)\}$. By extracting a subsequence if necessary, we may assume that $t_k \to \sigma \in [0, A]$, $x_0^k \to \phi \in \Gamma$, $x^k(t) \to x^0(t)$ uniformly on $[0, A]$, because $x_0^k \in \Gamma$, Γ is compact and $x^k(t)$ are L-Lipschitzian on $[0, A]$, where we should note that by Axiom (α_4) $x^k(0) \to \phi(0) = x^0(0)$ and $x^k(t)$ are uniformly bounded on $[0, A]$.

Define

$$x(t) = \begin{cases} x^0(t), & t > 0 \\ \phi(t), & t \leqq 0. \end{cases}$$

Then, clearly $x \in F_A^L(\Gamma)$. Therefore, by applying the fundamental inequality (2.1) to $x^k - x$ we have

$$|x_t^k - x_t|_B \leqq K_1(t) \sup_{0 \leqq s \leqq t} |x^k(s) - x^0(s)| + M_1(t)|x_0^k - \phi|_B.$$

Hence, for a given $\varepsilon>0$ we can find an N_1 such that

$$|x_t^k - x_t|_B < \varepsilon/2 \qquad \text{for any } t \in [0, A] \qquad \text{if } k \geqq N_1.$$

On the other hand, since x_t is continuous on $[0, A]$ by (β_3), there exists a $\delta>0$ such that

$$|x_t - x_s|_B < \varepsilon/2 \qquad \text{if } |t-s|<\delta \text{ and } t, s \in [0, A].$$

Thus, choosing N_2 so that $|t_k-\sigma|<\delta$ for $k \geqq N_2$, we have

$$|x_{t_k}^k - x_\sigma|_B \leqq |x_{t_k}^k - x_{t_k}|_B + |x_{t_k} - x_\sigma|_B < \varepsilon \qquad \text{for } k \geqq \max(N_1, N_2).$$

This completes the proof of the first part of the conclusion.

To prove the second part, consider the function $(\phi) \in F_\infty(\phi)$ defined by

$$(\phi)(t) = \phi(0), \qquad t \geqq 0. \tag{2.2}$$

By applying Inequality (2.1), we have

$$\begin{aligned} |(\phi)_t - (\psi)_t|_B &\leqq K_1(t)|\phi(0)-\psi(0)| + M_1(t)|\phi-\psi|_B \\ &\leqq \{K_1(t)K + M_1(t)\}|\phi-\psi|_B; \end{aligned}$$

that is, $(\phi)_t$ satisfies the Lipschitz condition in ϕ. Since $(\phi)_t$ is continuous in t by Hypothesis (β_3), $(\phi)_t$ is continuous in (t, ϕ). Therefore, there exists a $\delta(\varepsilon)>0$ for a given $\varepsilon>0$ such that $|(\phi)_t-(\phi)_s|_B<\varepsilon$ for any $\phi \in \Gamma_0$, $A \geqq t, s \geqq 0$ with $|t-s|<\delta(\varepsilon)$, since Γ_0 is compact. Hence, if $s \leqq t < s+\delta(\varepsilon)$

$$\begin{aligned} |x_t - x_s|_B &\leqq |(x_s)_{t-s} - x_s|_B + |x_t - (x_s)_{t-s}|_B \\ &\leqq \varepsilon + K_1(t-s) \sup_{s \leqq r \leqq t} |x(r)-x(s)| \leqq \varepsilon + K_1(t-s)L|t-s|, \end{aligned}$$

which proves the equi-continuity in t.

Remark 2.1. Obviously from the proof, if Γ consists of a single element, then we can omit Hypothesis (β_2) in the first part of the lemma.

The following lemma is obvious.

Lemma 2.2. *Hypothesis* (β_3) *implies that any solution of* RFDE(f) *through* (σ, ϕ) *must satisfy the integral relation*

$$\begin{aligned} &x_\sigma = \phi \\ &x(t) = \phi(0) + \int_\sigma^t f(s, x_s)ds, \qquad t \geqq \sigma \end{aligned}$$

and conversely.

From this lemma the following result is immediately obtained.

Lemma 2.3. *Suppose that Hypothesis* (β_3) *is satisfied. Let* $x^k(t)$ *be a solution of* RFDE(f_k, Ω) *on* $[0, A]$, *and assume that there exists an* $x \in F_A$ *such that* $x^k(t) \to x(t)$, $x_t^k \to x_t$ *on* $[0, A]$, $f_k(t, \phi) \to f(t, \phi)$ *uniformly on a set* $\Omega_0 \subset \Omega$, *which contains* $\{(t, x_t^k); t \in [0, A], k \geqq 1\}$.

Then, $x(t)$ *is a solution of* RFDE(f, Ω) *on* $[0, A]$.

Theorem 2.1 (Existence). *For any* $(\sigma, \phi) \in \Omega$, *Hypotheses* ($\beta_1$) *and* ($\beta_3$) *imply the existence of a solution of* RFDE(f, Ω) *through* (σ, ϕ).

Proof. We only give the main steps since the procedure is classical. Let $(\phi) \in F_\infty(\phi)$ be the function defined by (2.2). Then, Hypothesis (β_3) implies $(\phi)_t$ is continuous in t for $t \geqq 0$. Clearly, x is a solution of RFDE(f) through (σ, ϕ) on $[\sigma, \sigma + A]$ if and only if y defined by $y(t) = x(t+\sigma) - (\phi)(t)$ satisfies

$$y_0 = 0$$
$$y(t) = \int_0^t f(s+\sigma; (\phi)_s + y_s)ds, \qquad t \in [0, A].$$

For any $\delta > 0$ and $\eta > 0$, let

$$A(\delta, \eta) = \{\zeta : (-\infty, \delta] \to R^n, \text{ continuous}, \zeta(t) = 0 \text{ for } t \leqq 0, |\zeta(t)| \leqq \eta \text{ for } t \in [0, \delta]\}.$$

Then, $A(\delta, \eta)$ is a closed bounded convex subset of the Banach space $C((-\infty, \delta], R^n)$ of all bounded continuous functions from $(-\infty, \delta]$ into R^n.

For any $\zeta \in A(\delta, \eta)$, Hypothesis ($\beta_3$) implies ζ_t is continuous in t for $t \in [0, \delta]$. Suppose U is an open neighborhood in B of ϕ and $\delta_0 > 0$ is fixed. For any $\varepsilon > 0$, Hypothesis (β_1) implies there is an $\eta_0 > 0$ such that $|\zeta_t|_B < \varepsilon$ for $t \in [0, \delta_0]$, $\zeta \in A(\delta_0, \eta_0)$. Therefore, we may assume δ_0, η_0 are so small that $\phi + \zeta_t \in U$ for $t \in [0, \delta_0]$, $\zeta \in A(\delta_0, \eta_0)$.

Since f is assumed to be continuous and $(\phi)_t$ is continuous in t, the above implies there are positive δ_0, η_0 such that the function $T\zeta$, $\zeta \in A(\delta, \eta)$, defined by

$$[T\zeta](t) = \begin{cases} 0, & t < 0 \\ \int_0^t f(\sigma + s, (\phi)_s + \zeta_s)ds, & t \geqq 0 \end{cases}$$

belongs to $C((-\infty, \delta], R^n)$ for $0 < \delta < \delta_0$, $0 < \eta < \eta_0$. One now proceeds in the usual way to show there exist positive δ, η such that $TA(\delta, \eta) \subset A(\delta, \eta)$ and T is a compact mapping. Using Hypothesis (β_1) again, one shows that T is continuous. The Schauder fixed point theorem completes the proof.

Theorem 2.2 (Uniqueness). *Suppose Hypotheses* ($\beta_1 \sim \beta_3$) *are satisfied, and assume that there exists a constant L such that*

(2.3) $$|f(t,\phi)-f(t,\psi)|\leqq L|\phi-\psi|_B \qquad on\ \Omega.$$

Then, there exists a continuous function $L(t)$ *for which we have*

$$|x_t(\sigma,\phi)-x_t(\sigma,\psi)|_B\leqq L(t-\sigma)|\phi-\psi|_B, \qquad t\geqq\sigma.$$

In particular, the solution of RFDE(f) *through* (σ,ϕ) *is unique under the condition* (2.3).

Proof. This theorem is due to K. Sawano.

If $u(t)=|x_t(\sigma,\phi)-x_t(\sigma,\psi)|_B$, then Relation (2.3), the fundamental inequality (2.1) and Lemma 2.2 imply

$$u(t)\leqq K_1(t-\sigma)\left\{|\phi(0)-\psi(0)|+\int_\sigma^t Lu(s)ds\right\}+M_1(t-\sigma)|\phi-\psi|_B.$$

Hence, Axiom (α_4) implies

$$u(t)\leqq\{K_1(t-\sigma)K+M_1(t-\sigma)\}|\phi-\psi|_B+K_1(t-\sigma)L\int_\sigma^t u(s)ds.$$

Thus, the well-known Gronwall's lemma yields the conclusion in the theorem.

Theorem 2.3 (Continuation). *If Hypotheses* $(\beta_1\sim\beta_3)$ *are satisfied and* x *is a non-continuable solution of* RFDE(f,Ω) *on* $[\sigma_0,\delta)$, *then for any compact set* W *in* Ω *there is a* t_W *such that* $(t,x_t)\notin W$ *for* $t_W\leqq t<\delta$.

This theorem is due to K. Sawano.

Proof. Suppose that the conclusion is false for a compact set $W\subset\Omega$. Then, there is a sequence $\{t_k\}$, $t_k\to\delta^-$, such that $(t_k,x_{t_k})\in W$. Since W is compact, we may assume that (t_k,x_{t_k}) converges to a (δ,ϕ). The compactness of W also assures that there are constants $\varepsilon_0>0$ and L for which U, the ε_0-neighborhood of W, is contained in Ω and $|f(s,\psi)|\leqq L$ for $(s,\psi)\in U$.

If we can prove that x_t converges to ϕ as $t\to\delta^-$, then clearly the solution $x(t)$ should be continuable beyond δ, which contradicts the assumption. Suppose that there is a sequence $\{t'_k\}$ such that $t'_k\to\delta^-$ and $|x_{t'_k}-x_{t_k}|_B=\varepsilon$ for an $\varepsilon>0$. Clearly, we may assume $\varepsilon<\varepsilon_0$, $t_k<t'_k$, $|x_t-x_{t_k}|_B<\varepsilon$ for $t_k\leqq t<t'_k$. Hence, the functions x^k, defined by

$$x^k(t)=\begin{cases}x(t+t_k), & t\leqq t'_k-t_k\\ x(t'_k), & t\geqq t'_k-t_k,\end{cases}$$

belongs to $F_\infty^L(\Gamma)$, where $\Gamma=\mathrm{Cl}(\bigcup_k\{x_{t_k}\})$ is compact. Therefore, by Lemma 2.1, noting $x_t^k = x_{t+t_k}$ for $t \leqq t'_k - t_k$

$$\varepsilon = |x_{t'_k} - x_{t_k}|_B \to 0 \qquad \text{as } t'_k - t_k \to 0,$$

which arises a contradiction.

Theorem 2.4 (Continuation). *Under the same assumption as in Theorem 2.3, if f takes closed bounded sets of Ω into bounded sets, then for any closed bounded set W in Ω there exists a sequence $t_k \to \delta^-$ such that $(t_k, x_{t_k}) \notin W$.*

Moreover, if $\Omega = R \times B$ or if B is bestowed the hypothesis;

(*) *there exist $r > 0$ and K^* such that*

$$|\phi|_{[-r,0]} \leqq K^* |\phi|_B,$$

then there is a t_W such that $(t, x_t) \notin W$ for $t_W \leqq t < \delta$.

Proof. Let W be a closed bounded set in Ω.

Suppose that the first part of this theorem is false for W. Then, there exists a $\sigma < \delta$ such that $(t, x_t) \in W$ for all $t \in [\sigma, \delta)$, which shows that $\dot{x}(t)$ together with $x(t)$ is bounded on $[\sigma, \delta)$ by the assumption on f. Hence, $x(t)$ has a continuous extension beyond δ. Therefore, (t, x_t) converges to a limit (δ, ϕ) in W as $t \to \delta^-$ by Lemma 2.1. Theorem 2.1 ensures the existence of a solution of RFDE(f) through (δ, ϕ), which is a contradiction to the fact that $x(t)$ is non-continuable beyond δ.

Now, suppose that the second part is false for W. Then, there exists a sequence $\{t_k\}$, $t_k \to \delta^-$, such that $(t_k, x_{t_k}) \in W$. First of all, we shall show that this fact implies that

$$\Gamma = \mathrm{Cl}\,\{(t, x_t) : t \in [\sigma, \delta)\}$$

is a bounded set in Ω. If this is not the case and if $\Gamma \subset \Omega$ (this is always true when $\Omega = R \times B$), then Γ is unbounded. Hence, we can choose a sequence $\{s_k\}$ so that $s_k \to \delta^-$ and $|x_{s_k}|_B = C$, where, for $C_1 = \sup\{|\phi|_B : (t, \phi) \in W \text{ for a } t \in [\sigma, \delta)\}$,

$$C = \{1 + K_1(0)K + \overline{\lim_{\beta \to 0+}}\, M_1(\beta)\}C_1 + 1.$$

Clearly, we may assume that $s_{k-1} < t_k < s_k$, $|x_t|_B < C$ for $t \in [t_k, s_k)$. Set

$$\Gamma_0 = \mathrm{Cl}\,\{(t, x_t) : t \in J\}, \qquad J = \bigcup_k\, [t_k, s_k].$$

Since $\Gamma_0 \subset \Gamma \subset \Omega$ and Γ_0 is closed bounded, $f(t, x_t)$ is bounded by an L on J, and hence by Inequality (2.1) we have

$$\begin{aligned} |x_{s_k}|_B &\leqq K_1(\beta_k)\{|x(t_k)| + L\beta_k\} + M_1(\beta_k)\,|x_{t_k}|_B \\ &\leqq \{K_1(\beta_k)K + M_1(\beta_k)\}C_1 + K_1(\beta_k)L\beta_k \end{aligned}$$

for $\beta_k = s_k - t_k$. From this inequality, it follows that $|x_{s_k}|_B < C$ for large k since $\beta_k \to 0$ as $k \to \infty$. This is a contradiction. Therefore, now let $\Gamma \cap \partial\Omega \neq \emptyset$, and there exists a sequence $(\sigma_k, x_{\sigma_k}) \to (\delta, \phi) \in \Gamma \cap \partial\Omega$. By Hypothesis (*), $|x_{\sigma_k} - \phi|_{[-r,0]} \to 0$ for an $r > 0$, $r < \delta - \sigma$. Therefore, $x(t)$ has a continuous extension beyond δ and Γ is compact by Lemma 2.1. Since $W_0 = W \cap \Gamma$ is a compact subset of Ω, Theorem 2.3 insists the existence of t_{W_0} such that $(t, x_t) \notin W_0$ for $t \in [t_{W_0}, \delta)$, which implies that $(t, x_t) \notin W$ for $t \in [t_{W_0}, \delta)$, since $(t, x_t) \in \Gamma$ for all $t \in [\sigma, \delta)$. This contradicts the existence of the sequence $\{t_k\}$, which implies that Γ is a closed bounded subset of Ω.

Again $\dot{x}(t)$ is bounded together with $x(t)$ on $[\sigma, \delta)$, and Γ is compact. Repeating the same argument yields a contradiction. This completes the proof of the theorem.

Hypothesis (*) in Theorem 2.4 follows from

(β_0) there exists a $\theta_0 < 0$ such that

$$|\phi(\theta_0)| \leqq K_0 |\phi|_B \qquad \text{for any } \phi \in B \text{ and some constant } K_0.$$

This will be shown by the following lemma.

Lemma 2.4. *Under Hypotheses ($\beta_0 \sim \beta_2$), there is a constant K^* such that*

$$|\phi(\theta)| \leqq K^* |\phi|_B \tag{2.4}$$

for any $\theta \in [\theta_0, 0]$ and any $\phi \in B$.

Proof. For $\phi \in B$, let $(\phi) \in F_\infty(\phi)$ be defined by (2.2). Then, $(\phi)_t \in B$ for $t \geqq 0$ by Axiom (α_1), and Hypothesis (β_0) implies $|(\phi)_t(\theta_0)| \leqq K_0 |(\phi)_t|_B$. On the other hand, Inequality (2.1) together with Axiom (α_4) implies

$$|(\phi)_t|_B \leqq \{K_1(t)K + M_1(t)\} |\phi|_B,$$

and $(\phi)_t(\theta_0) = \phi(t + \theta_0)$ if $t + \theta_0 \leqq 0$. Therefore, we have (2.4) by setting $t = \theta - \theta_0$ for $\theta \in [\theta_0, 0]$ and

$$K^* = K_0 \sup_{0 \leqq t \leqq -\theta_0} \{K_1(t)K + M_1(t)\}.$$

Theorem 2.5 (Continuous dependence). *Suppose the solution $x(\sigma, \phi)$ of* RFDE (f, Ω) *through (σ, ϕ), defined on $[\sigma, \sigma + A]$, $A > 0$, is unique and Hypotheses ($\beta_1 \sim \beta_3$) are satisfied. Then, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $(s, \psi) \in \Omega$, $|s - \sigma| < \delta(\varepsilon)$, $|\psi - \phi|_B < \delta(\varepsilon)$, then*

$$|x_t(s, \psi) - x_t(\sigma, \phi)|_B < \varepsilon \qquad \text{for all } t \in [\max\{s, \sigma\}, \sigma + A],$$

where $x(s, \psi)$ is any solution of RFDE (f) *through (s, ψ).*

Proof. This theorem is proved by Hino [3] under stronger conditions.

Since $\{(t, x_t(\sigma, \phi)): t \in [\sigma, \sigma+A]\}$ is compact by (β_3), there exist $\varepsilon>0$, $L>0$ such that $(s, \psi) \in W$ implies $|f(s, \psi)| \leqq L$, where $W=\{(s, \psi): |t-s| \leqq \varepsilon, |\psi - x_t(\sigma, \phi)|_B \leqq \varepsilon$ for a $t \in [\sigma, \sigma+A]\}$. Since W is a closed set in $R \times B$, by Tietze's extension theorem there exists a continuous function $g(s, \psi)$ on $R \times B$ such that $|g(s, \psi)| \leqq L$ on $R \times B$ and $g=f$ on W.

Suppose that the conclusion of the theorem is false. Then, there exist sequences σ_k, ϕ^k, t_k such that $(\sigma_k, \phi^k) \in W$, $|\sigma_k - \sigma| < 1/k$, $|\phi^k - \phi|_B < 1/k$, $t_k \in [\max\{\sigma, \sigma_k\}, \sigma+A]$ and $|x_{t_k}(\sigma_k, \phi^k) - x_{t_k}(\sigma, \phi)|_B = \varepsilon_0$, where we may assume that $t_k \to t_0$, $\varepsilon_0 = \varepsilon$ and $|x_t(\sigma_k, \phi^k) - x_t(\sigma, \phi)|_B < \varepsilon_0$ for $t \in [\max\{\sigma, \sigma_k\}, t_k)$. For convenience, we shall denote $\sigma = \sigma_0$, $\phi = \phi^0$.

By Theorem 2.4, $x(\sigma_k, \phi^k)$ has a continuous extension beyond t_k up to $\sigma+A+1$ as a solution of RFDE$(g, R \times B)$, which we shall denote by x^k. Clearly, y^k, defined by $y^k(t) = x^k(t+\sigma_k)$, is a solution of RFDE$(g_k, R \times B)$ through $(0, \phi^k)$ on $[0, A]$, where $g_k(t, \psi) = g(t+\sigma_k, \psi)$. Since $\Gamma = \{\phi^k: k \geqq 0\}$ is compact and $|g_k| \leqq L$, $y^k \in F_A^L(\Gamma)$, which is given in Lemma 2.1. Therefore, $\{(t, y_t^k): t \in [0, A], k \geqq 0\} = W_0$ is compact and we can extract a sequence $\{y^{k_j}\}$ so that $y_t^{k_j}$ together with $y^{k_j}(t)$ converges on $[0, A]$.

Thus, since g_k converges to g_0 uniformly on W_0, by Lemma 2.3, y^{k_j} converges to a solution y of RFDE(g_0) through $(0, \phi)$. Clearly, x, defined by $x(t) = y(t-\sigma)$ is a solution of RFDE(g) through (σ, ϕ) which satisfies $|x_{t_0} - x_{t_0}(\sigma, \phi)|_B = \varepsilon_0$ and $(t, x_t) \in W$ for $[\sigma, t_0]$. From this, x is a solution of RFDE(f) on $[\sigma, t_0]$, and there arises a contradiction. This completes the proof of the theorem.

Using the same proof as above, one obtains,

Theorem 2.6 (Continuous dependence). *If the conditions of Theorem* 2.5 *are satisfied and* $f = f_\lambda$ *in* RFDE(f) *continuously depends on a parameter* λ *in a Banach space, then the solution* x^λ *of* RFDE(f_λ) *through* (σ, ϕ) *is continuous in* (λ, σ, ϕ).

It is not difficult to see that Hypotheses $(\beta_1 \sim \beta_3)$ are satisfied in the spaces given in Examples 1.1 ~ 1.3. For the space in Example 1.2 (including the one in Example 1.1 as a special case with $p=1$, $r=0$). We can put

$$(2.5)\qquad \begin{aligned} &K_1(\beta) = \left[1 + \int_{-\beta}^{0} g(\theta) d\theta\right]^{1/p}, \\ &M_1(\beta) = \max\left\{\chi_r(\beta), \operatorname*{ess\cdot sup}_{s \leqq 0} \left[\frac{g(s-\beta)}{g(s)}\right]^{1/p}\right\}, \qquad \chi_r(\beta) = \begin{cases} 0, & \beta > r \\ 1, & \beta \leqq r, \end{cases} \end{aligned}$$

and in Example 1.3,

$$(2.6)\qquad K_1(\beta) = \sup_{-\beta \leqq \theta \leqq 0} e^{r\theta}, \qquad M_1(\beta) = e^{-r\beta}.$$

On the other hand, Hypothesis (β_0) is not valid for the space in Example 1.2 if $r=0$, while it is valid with any $\theta_0<0$ for the space in Example 1.3 and with $\theta_0 \in [-r, 0]$, when $r>0$, for the one in Example 1.2.

The space given in Example 1.2 with $r=0$ is discussed by Coleman-Mizel [8] and Coleman-Owen [9] (also refer the references in [9]). However, in [9] the elements in $\hat{B}$ take $(-\infty, 0]$ into a Banach space with possibly infinite dimension. Therefore, Lemma 2.1 in our text may not be true for their case, while they need the Lipschitz condition and

$$\text{(2.7)} \qquad \operatorname{ess\cdot sup}_{s\leqq 0} \frac{g(s)}{g(s-\beta)} < \infty \qquad \text{for each } \beta \geqq 0$$

for the existence theorem, the continuous dependence and so on. The condition (2.7) requires, in particular, $g(s)>0$ for almost all $s \leqq 0$.

§ 3. Axioms for the global theory.

In order to estimate solutions as $t\to\infty$ by applying the fundamental inequality (2.1) and to expect the effect of the initial data in the norm to fade as $t\to\infty$, the following hypotheses seem to be acceptable.

(γ_1) In (β_1), $K_1=K_1(\beta)$ is constant for all $\beta \geqq 0$.

(γ_2) In (β_2), $M_1(\beta_0)<1$ for a $\beta_0>0$.

(γ_3) In (β_2), $M_1(\beta) \leqq M_1$ for all $\beta \geqq 0$ and some M_1.

Moreover, in much of examples of the space B including Examples 1.1～1.3, the following hypothesis is always satisfied.

(γ_4) If $\{\hat{\phi}^k\}$, $\hat{\phi}^k \in \hat{B}$, converges to $\hat{\phi}$ uniformly on any compact set in $(-\infty, 0]$ and if $\{\phi^k\}$ is a Cauchy sequence in B, then $\hat{\phi} \in \hat{B}$ and $\phi^k \to \phi$.

For the spaces given in Examples 1.1 and 1.2, Hypothesis (γ_1) requires

$$\text{(3.1)} \qquad \int_{-\infty}^{0} g(\theta)d\theta < \infty$$

by (2.5). Hypothesis (γ_2) is satisfied if $G(-\beta_0)<1$ for a $\beta_0>r$, while (γ_3) requires $\operatorname{ess\cdot sup}_{s\leqq 0} G(s)<\infty$. However, by Remark 1.1 if $M_1(\beta_0)<1$ for a $\beta_0>0$, then

$$M_1(\beta) \leqq C(\gamma)e^{-\gamma\beta} \qquad \text{for all } \beta \geqq 0$$

with $\gamma>0$, since $-(1/\beta_0)\log M_1(\beta_0)>0$.

In Example 1.3, Hypotheses (γ_1) and (γ_3) are satisfied if $\gamma \geqq 0$, while (γ_2) requires $\gamma>0$.

These facts suggest that Hypothesis (γ_2) is equivalent to the condition

$$(\gamma_2^*) \qquad M_1(\beta)\to 0 \qquad \text{as } \beta\to\infty.$$

This will be established in the following lemma.

Lemma 3.1. *Hypotheses* (γ_2) *and* (γ_2^*) *are equivalent, and* (γ_2) *implies* (γ_3).

Proof. The second part is clear from the first part and the fact that $M_1(\beta)$ is locally bounded.

To prove the first part, it is sufficient as in Remark 1.1 to show that

$$(3.2) \qquad M_1(t+s)\leqq M_1(t)M_1(s), \qquad t,\, s\geqq 0.$$

First of all, we recall that $\tau^t\phi=\{\psi\}_t$ means that for any given $\varepsilon>0$ we can choose $\hat{\phi}$, $\hat{\psi}, \hat{\eta}\in\hat{B}$ so that $\hat{\eta}\in\hat{\tau}^t\hat{\phi}$ and $|\hat{\psi}-\hat{\eta}|_{\hat{B}}<\varepsilon$, where $\hat{\phi}$, $\hat{\psi}$ are suitable representations of ϕ, ψ. $\hat{\eta}\in\hat{\tau}^t\hat{\phi}$ means $\hat{\eta}(\theta)=\hat{\phi}(\theta+t)$ for $\theta\leqq -t$. Now, choose $\hat{\xi}\in\hat{B}$ so that $\hat{\xi}(\theta)=\hat{\eta}(s+\theta)$ for $\theta\leqq -s$, which shows that $\{\xi\}_s=\tau^s\eta$ and $\hat{\xi}(\theta)=\hat{\phi}(\theta+s+t)$ for $\theta\leqq -s-t$, and hence $\{\xi\}_{t+s}=\tau^{t+s}\phi$.

On the other hand, for $\{\zeta\}_s=\tau^s\psi$ and $\{\xi\}_s=\tau^s\eta$, Hypothesis (β_2) shows that

$$|\zeta-\xi|_s\leqq M_1(s)\,|\psi-\eta|_B=M_1(s)\,|\hat{\psi}-\hat{\eta}|_{\hat{B}}\leqq M_1(s)\varepsilon,$$

which implies that $|\zeta-\xi|_{s+t}\leqq M_1(s)\varepsilon$ for arbitrary $\varepsilon>0$, and hence $\{\zeta\}_{s+t}=\tau^{t+s}\phi$. Thus, we have

$$\{\psi\}_t=\tau^t\phi,\ \{\zeta\}_s=\tau^s\psi \quad \text{imply} \quad \{\zeta\}_{s+t}=\tau^{t+s}\phi.$$

From this, it follows that

$$|\tau^{t+s}\phi|_{t+s}=|\zeta|_{t+s}\leq|\zeta|_s=|\tau^s\psi|_s\leqq M_1(s)\,|\psi|_B,$$

which implies (3.2) since we can choose ψ with $\{\psi\}_t=\tau^t\phi$ so that $|\psi|_B<|\psi|_t+\varepsilon$ for any given $\varepsilon>0$. Here, we should note $|\psi|_t=|\tau^t\phi|_t\leqq M_1(t)\,|\phi|_B$. This completes the proof of the lemma.

Lemma 3.2. *Suppose Hypotheses* (β_2) *and* (γ_1) *are satisfied. If* $\hat{\phi}(\theta)$ *is bounded and continuous on* $(-\infty, 0]$, *then there exists a sequence* $\{\hat{\phi}^k\}$ *in* $\hat{B}$ *such that* $\hat{\phi}^k$ *converges to* $\hat{\phi}$ *uniformly on any compact set in* $(-\infty, 0]$ *and satisfies*

$$(3.3) \qquad |\phi^k|_\beta\leqq K_1M_1(\beta)\sup_{\theta\leqq 0}|\hat{\phi}(\theta)|$$

for any β *and large* k.

Moreover, if Hypothesis (γ_2) *is satisfied, then* $\{\phi^k\}$ *is a Cauchy sequence in* B, *and the additional hypothesis* (γ_4) *implies that* $\hat{\phi}\in\hat{B}$ *and* $|\phi|_B\leqq K_1\sup_{\theta\leqq 0}|\hat{\phi}(\theta)|$.

Proof. Let $\hat{\phi}(\theta)$ be bounded, continuous on $(-\infty, 0]$, and define continuous functions $\hat{\phi}^k$ by

$$\hat{\phi}^k(\theta)=\begin{cases}\hat{\phi}(\theta), & -k\leqq\theta\leqq 0\\ \text{linear}, & -k-1\leqq\theta\leqq -k\\ 0, & 0\leqq -k-1,\end{cases}$$

which clearly converge to $\hat{\phi}$ uniformly on any compact interval and $\hat{\phi}^k_t \in \hat{B}$ for any $t\leqq 0$ by Axiom (α_1).

Since $\{\psi\}_\beta=\tau^\beta\psi_{-\beta}$ if $\psi_{-\beta}\in B$, we have

$$|\phi^k|_\beta=|\tau^\beta\phi^k_{-\beta}|_\beta\leqq M_1(\beta)\,|\phi^k_{-\beta}|_B\leqq M_1(\beta)K_1 \sup_{\theta\leqq 0}|\hat{\phi}(\theta)|$$

by Hypotheses (β_2) and (γ_1), which shows (3.3).

Since $|\phi^k-\phi^l|_{(\beta)}=0$ for $k, l\geqq\beta$,

$$|\phi^k-\phi^l|_B\leqq|\phi^k-\phi^l|_{(\beta)}+|\phi^k|_\beta+|\phi^l|_\beta\leqq 2K_1M_1(\beta)\sup_{\theta\leqq 0}|\hat{\phi}(\theta)|$$

for $k, l\geqq\beta$. Therefore, the remaining parts of the proof are obvious from Lemma 3.1.

Corollary 3.1. *Under Hypothesis (γ_4), Hypotheses (γ_1), (γ_2) imply that all constant functions belong to $\hat{B}$.*

In the following, by $F^{C,L}_\infty(\Gamma)$ we shall denote the set of all functions $x\in F_\infty$ such that $x_0\in\Gamma$, $|x(t)|\leqq C$ for $t\geqq 0$ and $|x(t)-x(s)|\leqq L|t-s|$ on $[0, \infty)$, where Γ is a subset of B and C, L are constants.

Theorem 3.1. *If Hypotheses (γ_1) and (γ_2) are satisfied, then for any bounded set $\Gamma\subset B$ and constants C, L the set*

$$W=\bigcap_{t\geqq 0}\mathrm{Cl}\,\{x_s: x\in F^{C,L}_\infty(\Gamma),\ s\geqq t\}$$

is compact.

Proof. To prove this, we shall show that any given sequence $\{x^k_{t_k}\}$, with $x^k\in F^{C,L}_\infty(\Gamma)$, $t_k\to\infty$, contains a convergent subsequence.

Since $\{x^k(t_k+\theta)\}$ are uniformly bounded and equi-continuous on any compact set in $(-\infty, 0]$ for large k, it contains a subsequence, denoted by $\{x^k(t_k+\theta)\}$ again, which converges to a $\phi(\theta)$ uniformly on any compact set in $(-\infty, 0]$. Clearly, $\phi(\theta)$ is bounded by C and continuous on $(-\infty, 0]$, and hence there exists a sequence $\{\phi^k\}$ as mentioned in Lemma 3.2. Thus, by Inequality (2.1) and Relation (3.3),

$$\begin{aligned}|x^k_{t_k}-\phi^k|_B&\leqq K_1|x^k_{t_k}-\phi^k|_{[-\beta,0]}+M_1(\beta)\,|x^k_{t_k-\beta}|_B+|\phi^k|_\beta\\ &\leqq K_1\{|x^k_{t_k}-\phi|_{[-\beta,0]}+|\phi-\phi^k|_{[-\beta,0]}\}\\ &\quad+M_1(\beta)\{K_1C+C_\Gamma\ M_1\}+K_1M_1(\beta)C\end{aligned}$$

since by applying Inequality (2.1) again,

$$|x^k_{t_k-\beta}|_B \leqq K_1 \sup_{0\leqq\theta\leqq t_k-\beta} |x^k(\theta)| + M_1(t_k-\beta)\,|x^k_0|_B \leqq K_1C + C_\Gamma M_1,$$

where C_Γ is a bound for ϕ in Γ and M_1 is a bound for $M_1(\beta)$. Therefore, by (γ_2) if ϕ^k converges to ψ, then so does $x^k_{t_k}$. This completes the proof of the theorem.

Corollary 3.2. *Under Hypotheses* $(\beta_3), (\gamma_1), (\gamma_2)$, *the set* $\{x_t : \hat{x} \in F^{C,L}_\infty(\Gamma),\ t \geqq 0\}$ *is precompact in B for any compact set* $\Gamma \subset B$ *and constants C, L.*

Proof. The result follows immediately from Theorem 3.1 and Lemma 2.1.

Remark 3.1. When Γ consists of a single element, Hino [4] has proved Corollary 3.2 and related results under a weaker condition than (γ_2); that is, he has assumed that

(3.4) $$|\tau^\beta\phi|_\beta \to 0 \text{ as } \beta\to\infty \quad \text{for any } \phi \in B.$$

However, he also assumed that (∗∗) if $\{\hat{\phi}^k\} \subset \hat{B}$ is uniformly bounded and converges to $\hat{\phi}$ uniformly on any compact set in $(-\infty, 0]$, then $\hat{\phi} \in \hat{B}$ and $|\phi^k - \phi|_B \to 0$ as $k\to\infty$. While these conditions are satisfied for the space in Example 1.1 when $g(s) = 1/(1+s^2)$, the condition (∗∗) is a pretty strong condition. In fact, the condition (∗∗) immediately implies that any bounded, continuous function belongs to $\hat{B}$ and that Relation (3.4) holds if $\hat{\phi}$ is bounded.

Therefore, our space B is slightly different from the one in [2], [4].

Consider the RFDE $(f, R\times B)$, and assume that f takes $R\times$(bounded sets) into bounded sets. Then, Corollary 3.2 shows that any bounded orbit of RFDE (f) is precompact. Therefore, for autonomous equations we can expect same properties for the ω-limit set as in the case of ordinary differential equations.

Definition 3.1. For a solution x of an autonomous equation, the *ω-limit set* L^+ of x is defined by

$$L^+ = \bigcap_{t\geqq 0} \mathrm{Cl}\,\{x_s : s \geqq t\}$$

and consists of the limits of $\{x_{t_k}\}$ for $t_k \to \infty$.

Definition 3.2. A set W in Ω is said to be *invariant* for RFDE (f, Ω) if for any $(\sigma, \phi) \in W$ there exists a solution $x(\sigma, \phi)$ of RFDE (f) through (σ, ϕ) defined on an interval I, which contains σ as an interior point, such that $(t, x_t(\sigma, \phi)) \in W$ for all $t \in I$. If we can take R as I, then W is said to be *R-invariant*.

Theorem 3.2. *Under Hypotheses* $(\gamma_1), (\gamma_2)$ *the ω-limit set L^+ of a bounded solution* *x of an autonomous* RFDE(f), *with f completely continuous, is nonempty, compact, connected and invariant.*

Furthermore, if (γ_4) *is satisfied, then* L^+ *is R-invariant.*

Remark 3.2. Under Axiom (α_4), Hypotheses (γ_1), (γ_3) there is no difference between boundedness in R^n, $\sup_{t\geqq 0}|x(t)|<\infty$, and boundedness in B, $\sup_{t\geqq 0}|x_t|_B<\infty$, because by Inequality (2.1)

$$(1/K)|x(t)|\leqq|x_t|_B\leqq K_1\sup_{0\leqq s\leqq t}|x(s)|+M_1|x_0|_B.$$

Proof. Let $\sup_{t\geqq 0}|x(t)|\leqq C$. Then, there is an $L>0$ for which $|\dot{x}(t)|\leqq L$ on $[0,\infty)$ by the complete continuity of f. Therefore, $x\in F_\infty^{C,L}(x_0)$ and Corollary 3.2 shows that $\{x_t: t\geqq 0\}$ is precompact. From this, all conclusions except the invariance follow immediately.

For a given $\phi\in L^+$, choose t_k, $t_k\to\infty$, so that $x_{t_k}\to\phi$. Clearly, we can assume that $\{x(t_k+t)\}$ converges to a $y(t)$ uniformly on any compact set in $(-\infty,\infty)$, while $\{x_{t_k-s}\}$ contains a subsequence which converges to a ψ for any given $s>0$. Put

$$z(t)=\begin{cases} y(t-s), & t>0\\ \psi(t), & t\leqq 0.\end{cases}$$

Then, $\psi(0)=y(-s)$ by Axiom (α_4), and hence $z\in F_\infty(\psi)$. Applying Inequality (2.1), we have

$$\begin{aligned}|z_t-x_{t-s+t_k}|_B&\leqq K_1|z_t-x_{t-s+t_k}|_{[-t,0]}+M_1|z_0-x_{t_k-s}|_B\\&\leqq K_1\sup_{0\leqq\theta\leqq t}|y(\theta-s)-x(t_k+\theta-s)|+M_1|\psi-x_{t_k-s}|_B,\end{aligned}$$

which converges to 0 as $k\to\infty$. Therefore, $z_t\in L^+$ for all $t\geqq 0$ and $z_s=\phi$. Lemma 2.3 implies that x, defined by $x(t)=z(t+s)$, is a solution of RFDE(f) through $(0,\phi)$ on $[-s,\infty)$.

For the second part, it is sufficient to note that $y_t\in B$ for all $t\in(-\infty,\infty)$ and we can take y_{-s} as ψ so that y is a solution of RFDE(f) through $(0,\phi)$ on $(-\infty,\infty)$.

§ 4. Operator $S(t)$.

By Corollary 3.1 we know that all constant functions belong to $\hat{B}$ under Hypotheses (γ_1), (γ_2), (γ_4). Here, we shall take this as one of our hypotheses.

(δ_0) All constant functions belong to $\hat{B}$.

Under this hypothesis, we can find a constant K_2 such that

$$|\bar{a}|_B\leqq K_2|a|\qquad\text{for any } a\in R^n,\tag{4.1}$$

where $\bar{a}$ denotes the constant function $\bar{a}(\theta)\equiv a$.

If we allow Hypothesis (δ_0), we can consider the decomposition

$$x_t=(\phi)_t-\overline{\phi(0)}+U(t)x_t \tag{4.2}$$

for $x\in F_\infty(\phi)$, where (ϕ) is the function defined by (2.2) associated with ϕ and $U(t)x_t$ denotes the remainder. Clearly, $(\phi)_t-\overline{\phi(0)}$ belongs to B_0 for all $t\geqq 0$, where $B_0=\{\phi\in B\colon \phi(0)=0\}$.

Therefore, it will be very useful to introduce a linear operator $S(t)$ on B_0 by

$$[S(t)\phi](\theta)=\begin{cases}0, & -t<\theta\leqq 0\\ \phi(t+\theta), & \theta\leqq -t\end{cases} \tag{4.3}$$

for $t\geqq 0$, which will be used fully in the next section.

Now, we have the following lemma.

Lemma 4.1. *$S(t)$, defined by* (4.3) *for $t\geqq 0$, is a semigroup of linear operators: $B_0\to B_0$ and satisfies*

$$|S(t)\phi|_{B_0}=|\tau^t\phi|_t \qquad \textit{for any } \phi\in B_0, \tag{4.4}$$

where $|\phi|_{B_0}$ denotes $\phi\in B_0$ and $|\phi|_{B_0}=|\phi|_B$.

Proof. The first part is obvious. Since $\{S(t)\phi\}_t=\tau^t\phi$ for any $\phi\in B_0$, the second part is proved by

$$|S(t)\phi|_{B_0}\leqq|\tau^t\phi|_t=|S(t)\phi|_t\leqq|S(t)\phi|_{B_0},$$

where the first inequality follows from Axiom (α_3).

We shall impose the following hypotheses on $S(t)$. However, Relation (4.4) suggests that the operator $S(t)$ plays the same role as τ^t, and it will be shown that the hypotheses on $S(t)$ are equivalent to those on τ^t under Hypothesis (δ_0).

(δ_1) $S(t)$ is strongly continuous in $t\geqq 0$.

(δ_2^*) $S(t)$ is a bounded operator for each $t\geqq 0$.

Hypothesis (δ_2^*) means that

$$M(t)=\sup_{|\phi|_{B_0}=1}|S(t)\phi|_{B_0}<\infty \qquad \text{for each } t\geqq 0.$$

Clearly, (δ_1) and (δ_2^*) imply the following property (δ_2).

(δ_2) In (δ_2^*), $M(t)$ is locally bounded.

(δ_3) In (δ_2), $M(t_0)<1$ for a $t_0>0$.

(δ_3^*) In (δ_2), $M(t)\to 0$ as $t\to\infty$.

Lemma 4.2. *Hypothesis* (β_3) *implies* (δ_1), *and the converse is also true if Hypothesis* (β_1) *and* (δ_0) *hold.*

Proof. The first part is obvious.

To prove the second part, suppose $x\in F_A$, $A>0$. Let $\phi=x_0-\overline{x(0)}$. Then, $\phi\in B_0$ by (δ_0) and (4.2) yields

$$x_t=S(t)\phi+U(t)x_t,$$

where

$$[U(t)\phi](\theta)=\begin{cases}\phi(-t), & \theta<-t\\ \phi(\theta), & -t\leqq\theta\leqq 0.\end{cases}$$

By (δ_1), we need only to prove that $U(t)x_t$ is continuous in t. By noting that $[U(t)x_t-U(s)x_s](\theta)=0$ for $\theta\leqq-\beta$, $\beta=\max\{t,s\}$, we can apply Inequality (2.1) without assuming (β_2). Hence, we have

$$|U(t)x_t-U(s)x_s|_B\leqq K_1(\beta)\sup_{0\leqq t',t''\leqq\beta}\{|x(t')-x(t'')|: |t'-t''|\leqq|t-s|\}$$

and this can be smaller than any given number, if $|t-s|$ is sufficiently small.

Lemma 4.3. *Hypotheses* (δ_0), (δ_2) *imply* (β_2), *while Hypothesis* (β_2) *implies* (δ_2).

Proof. For any $\psi\in B$, $\phi=\psi-\overline{\psi(0)}\in B_0$ by (δ_0). Therefore, by Relation (4.4),

$$|\tau^\beta\psi|_\beta\leqq|\tau^\beta\phi|_\beta+|\tau^\beta\overline{\psi(0)}|_\beta\leqq|S(\beta)\phi|_{B_0}+|\overline{\psi(0)}|_B, \tag{4.5}$$

which implies that

$$|\tau^\beta\psi|_\beta\leqq\{M(\beta)(1+K_2K)+K_2K\}|\psi|_B,$$

since $|\overline{\psi(0)}|_B\leqq K_2|\psi(0)|\leqq K_2K|\psi|_B$ by (4.1), Axiom (α_4). This completes the proof of the first part.

The converse is clear by Relation (4.4).

The following lemma is obvious from the semi-group property of $S(t)$.

Lemma 4.4. *Hypotheses* (δ_3) *and* (δ_3^*) *are equivalent to each other.*

Lemma 4.5. *Hypothesis* (γ_2) *implies* (δ_3), *and the converse is true if* (δ_0) *holds.*

Proof. The first part is obvious from Relation (4.4). To prove the second part, by Relation (4.5) it is sufficient to show that $|\tau^t\bar{a}|_t\to 0$ as $t\to\infty$ for any $a\in R^n$.

For $a\in R^n$, define a continuous function x by

$$x(\theta)=\begin{cases} a, & \theta\leqq -1 \\ \text{linear}, & -1\leqq\theta\leqq 0 \\ 0, & \theta\geqq 0. \end{cases}$$

Clearly, $x_t \in B_0$ for $t\geqq 0$, $\tau^t\bar{a}=\{x_{t-1}\}_t$ and $\{x_{t-1}\}_{t-1}=\tau^{t-1}x_0$ for $t\geqq 1$. Therefore, for $t\geqq 1$

$$\begin{aligned}|\tau^t\bar{a}|_t&=|x_{t-1}|_t\leqq|x_{t-1}|_{t-1}\\ &=|\tau^{t-1}x_0|_{t-1}\leqq|S(t-1)x_0|_{t-1}\leqq|S(t-1)x_0|_B\leqq M(t-1)\,|x_0|_B.\end{aligned}$$

This completes the proof of the lemma.

§ 5. The solution operator.

Suppose B satisfies $(\delta_0\sim\delta_3)$, (γ_1), $\Omega=R\times B$ and there is a unique solution $x(\sigma,\phi)$ of RFDE(f) through $(\sigma,\phi)\in R\times B$, defined for $t\geqq\sigma$, and let the *solution operator* $T(t,\sigma)\colon B\to B$, $t\geqq\sigma$, of RFDE(f) be defined by

$$T(t,\sigma)\phi=x_t(\sigma,\phi), \qquad t\geqq\sigma.$$

In this section, we discuss some properties of the solution operator.

Definition 5.1. If W is a bounded set in a Banach space X, the *Kuratowskii measure of noncompactness* of W, $\alpha(W)$, is defined as

$$\alpha(W)=\inf\{d>0\colon W \text{ has a finite cover of diameter less than } d\}.$$

Some properties of $\alpha(W)$ that are needed in following are:

(i) $\alpha(W)=0$ if and only if W is precompact.
(ii) $\alpha(W_1+W_2)\leqq\alpha(W_1)+\alpha(W_2)$.

Definition 5.2. If X is a Banach space and $T\colon X\to X$ is a continuous mapping, then we say T is a *conditional α-contraction* if there is a $k\in[0,1)$ such that $\alpha(TW)\leqq k\alpha(W)$ for all bounded set $W\subset X$ for which TW is bounded. If T is a conditional α-contraction and takes bounded sets into bounded sets, then T is said to be an *α-contraction.*

If T is an α-contraction linear operator on X, we define

$$\alpha(T)=\inf\{k\in R\colon \alpha(TW)\leqq k\alpha(W) \text{ for all bounded sets } W\subset X\}.$$

Definition 5.3. If X is a Banach space and $U(t,\sigma)\colon X\to X$ is defined for $t\geqq\sigma$, then $U(t,\sigma)$ is said to be *conditionally completely continuous* if $U(t,\sigma)\phi$ is continuous in (t,σ,ϕ) and, for any bounded set $W\subset X$, there is a compact set $W^*\subset X$ such that $U(t,\sigma)\phi\in W$ for $\tau\in[\sigma,t]$ implies $U(t,\sigma)\phi\in W^*$. If $U(t,\sigma)$ is conditionally completely continuous and for any bounded set $W \subset X$ and any compact set $I \subset [\sigma, \infty)$, there is a bounded set $W_0 \subset X$ such that $U(t, \sigma)W \subset W_0$ for $t \in I$, then $U(t, \sigma)$ is completely continuous in the usual sense.

We are now able to prove a fundamental property of the solution operator.

Theorem 5.1. *If f is completely continuous, if Hypotheses (β_1), (δ_0), (δ_1), (δ_2) are satisfied and if $U(t, \sigma)$ is defined by*

$$T(t, \sigma)\phi = S(t-\sigma)(\phi - \overline{\phi(0)}) + U(t, \sigma)\phi, \qquad t \geqq \sigma,$$

then $U(t, \sigma)$ is conditionally completely continuous for $t \geqq \sigma$. Furthermore, for any bounded set $W \subset B$ for which $T(s, \sigma)W$ is uniformly bounded for $\sigma \leqq s \leqq t$, we have

$$\alpha(T(t, \sigma)W) \leqq M(t-\sigma)\alpha(W),$$

where $M(t)$ is given in (δ_2) as the norm of $S(t)$.

If, in addition, (δ_3) is satisfied, then $T(t_0+\sigma, \sigma)$ is a conditionally α-contraction.

Proof. Since $\|S(t)\| = M(t)$, we know that $\alpha(S(t)) \leqq M(t)$. Therefore, we need only show that $U(t, \sigma)$ is conditionally completely continuous in order to prove the first part of the theorem. But this is clear from Ascoli's theorem, Hypothesis (β_1) and the fact that

$$[U(t, \sigma)\phi](\theta) = \begin{cases} \phi(0), & t+\theta < \sigma \\ \phi(0) + \int_\sigma^{t+\theta} f(s, T(s, \sigma)\phi)ds, & t+\theta \geqq \sigma \end{cases}$$

and $S(t)$ is a linear operator with norm $M(t)$ locally bounded. The second part is obvious and the theorem is proved.

Theorem 5.1 has important applications for the development of a general qualitative theory for RFDE (f). To see how such properties have been used in the theory of retarded equations and neutral equations with finite delay, see Hale [10]. Many of the results known for these cases should carry over to the abstract space B satisfying $(\delta_0 \sim \delta_3)$, (γ_1). We do not exploit this situation here but content ourselves with pointing out the implications for linear equations.

Consider the linear equation

$$\dot{x}(t) = Lx_t, \tag{5.1}$$

where $L: B \to R^n$ is a bounded linear operator and B satisfies Hypotheses (γ_1), $(\delta_0 \sim \delta_2)$. One can show that the solution operator $T(t, \sigma)$, $t \geqq \sigma \geqq 0$, is well defined and satisfies $T(t; \sigma) = T(t-\sigma, 0)$. If we define $T(t) = T(t, 0)$; that is,

$$T(t)\phi = x_t(0, \phi), \qquad t \geqq 0, \ \phi \in B, \tag{5.2}$$

then $T(t): B \to B$, $t \geqq 0$, is a strongly continuous semi-group of bounded linear operator by Theorem 2.2 (also, refer the lemmas in § 4). As a corollary to Theorem 5.1, we have:

Corollary 5.1. *If* $T(t): B \to B$, $t \geqq 0$, *is the strongly continuous semi-group of bounded linear operators defined by the linear equation* (5.1) *and Relation* (5.2), *then*

$$\alpha(T(t)) \leqq M(t),$$

where $M(t)$ *is the norm of* $S(t)$ *defined in* (4.3). *If, in addition,* (δ_3) *is satisfied, then* $T(t_0)$ *is an* α*-contraction.*

For the space B in Example 1.2, Theorem 5.1 and Corollary 5.1 are due to Hale [11].

For any bounded linear operator T on a Banach space X, it is known (see, Nussbaum [12]) that the radius $r_e(T)$ of the smallest closed disk in the complex plane with center zero which contains the essential spectrum of T is given by $\alpha(T)$. Therefore, Corollary 5.1 implies

$$r_e(T(t)) = M(t). \tag{5.3}$$

For any fixed t_0 and any $r_0 > M(t_0)$, it follows from Equation (5.3) that there are only a finite number of elements μ in the spectrum $\sigma(T(t_0))$ of $T(t_0)$ with $|\mu| > r_0$, each such μ must be in the point spectrum $P_\sigma(T(t))$ with a generalized eigenspace of a finite dimension and there exists an integer k such that

$$B = \mathfrak{N}(T(t_0) - \mu I)^k \oplus \mathfrak{R}(T(t_0) - \mu I)^k, \tag{5.4}$$

where $\mathfrak{N}$, $\mathfrak{R}$ denote the null space and range, respectively.

The importance of Equation (5.4) is well known for studying the local behavior of the solutions of differential equations near an equilibrium point or for studying perturbed linear systems. However, for Equation (5.4) to be of practical value from the point of view of studying specific equations we must be able to compute elements of $P_\sigma(T(t_0))$ as well as the Decomposition (5.4). From the theory of semi-groups of linear transformations, it is known that this information is obtained from the infinitesimal generator A. More specifically, $\mu \in P_\sigma(T(t_0))$, $\mu \neq 0$, implies there is a $\lambda \in P_\sigma(A)$ such that $\mu = \exp \lambda t_0$. Furthermore, $\mathfrak{N}(T(t_0) - \mu_0 I)^k$ is the space of the generalized eigenspace of $A - \lambda I$ for all $\lambda \in P_\sigma(A)$ such that $\mu = e^{\lambda t_0}$.

At the present time, the infinitesimal generator A of $T(t)$ is not known although it is certainly reasonable to conjecture that $A\phi$ is the operator of differentiation. For the space considered in Example 1.2, Naito [5] has shown that this is actually the case and has characterized the domain of A. Naito [6] as well as Burns and

Herdman [13] have also developed the adjoint theory for Equation (5.1) in the space of Example 1.2.

Fortunately, it is still possible to obtain information about Decomposition (5.4) without knowing explicitly the infinitesimal generator. The following interesting result was communicated to the authors by Naito.

Let $T(t)$ be the solution operator associated with (5.1) and defined by Equation (5.2).

Theorem 5.2. *If A is the infinitesimal generator of $T(t)$, then $P_\sigma(A)=\{\lambda: \det\Delta(\lambda)=0\}$ and $e^{\lambda\cdot}b\in B$ for all $b\in R^n$ (which stands for n-complex space here) such that $\Delta(\lambda)b=0$ and $\mathfrak{N}(A-\lambda I)=\{e^{\lambda\cdot}b: \Delta(\lambda)b=0\}$ for any $\lambda\in P_\sigma(A)$, where*

$$\Delta(\lambda)=\lambda I-Le^{\lambda\cdot}I.$$

Proof. If $\lambda\in P_\sigma(A)$, then there exists a $\phi\neq 0$, $\phi\in D(A)$, such that $A\phi=\lambda\phi$. Therefore,

$$\frac{d}{dt}T(t)\phi=AT(t)\phi=T(t)A\phi=\lambda T(t)\phi.$$

Since $T(t)$ is a strongly continuous semi-group, this implies $T(t)\phi=e^{\lambda t}\phi$, $t\geqq 0$ (in particular, $e^{\lambda t}\in P_\sigma(T(t))$). Since $T(t)\phi=x_t(\phi)$, it follows that

$$[T(t)\phi](\theta)=[x_t(\phi)](\theta)=x(t+\theta,\phi)=[x_{t+\theta}(\phi)](0)=[T(t+\theta)\phi](0)$$

for all $t\geqq-\theta$, $\theta\in(-\infty,0]$. Since $T(t)\phi=e^{\lambda t}\phi$, this implies

$$e^{\lambda t}\phi(\theta)=e^{\lambda(t+\theta)}\phi(0) \quad\text{and}\quad \phi(\theta)=e^{\lambda\theta}\phi(0) \qquad \text{for all } \theta.$$

Since $\phi\neq 0$, this implies $\phi(0)\neq 0$. Also, the function $e^{\lambda(t+\theta)}\phi(0)=x_t(\phi)(\theta)$ is a solution of Equation (5.1). Therefore, $\Delta(\lambda)\phi(0)=0$. This implies $\det\Delta(\lambda)=0$ since $\phi(0)\neq 0$.

Conversely, for any vector $b\neq 0$, $b\in R^n$, and λ such that $\Delta(\lambda)b=0$ and $e^{\lambda\cdot}b\in B$, one easily shows that the function $e^{\lambda t}b$ is a solution of Equation (5.1) for all $t\in R$. Therefore, if $\phi(\theta)=e^{\lambda\theta}b$, then $\phi\neq 0$ by Axiom (α_4) and $[T(t)\phi](\theta)=[x_t(\phi)](\theta)=e^{\lambda t}e^{\lambda\theta}b=e^{\lambda t}\phi(\theta)$, $e^{\lambda t}\in P_\sigma(T(t))$, $dT(t)\phi/dt=\lambda T(t)\phi$ exists for all $t\geqq 0$. Thus, $\phi\in D(A)$ and $A\phi=\lambda\phi$. Thus, $\lambda\in P_\sigma(A)$. The proof is completed.

A repeated application of the type of argument used in the proof of Theorem 5.2 will characterize $\mathfrak{N}(A-\lambda I)^k$ for any integer k if $\lambda\in P_\sigma(A)$. To state this characterization, let

$$P_{j+1}=P_{j+1}(\lambda)=\frac{1}{j!}\Delta^{(j)}(\lambda)=\frac{1}{j!}\frac{d^j\Delta(\lambda)}{d\lambda^j},$$

$$A_k = \begin{bmatrix} P_1 & P_2 \cdots P_k \\ 0 & P_1 \quad P_{k-1} \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \quad P_1 \end{bmatrix}.$$

Theorem 5.3. *If $\lambda \in P_\sigma(A)$, then $\mathfrak{N}(A-\lambda I)^k$ coincides with the space of functions $\phi \in B$ of the form*

$$\phi(\theta) = \sum_{j=0}^{k-1} \gamma_{j+1} \frac{\theta^j}{j!} e^{\lambda\theta}, \qquad -\infty < \theta \leqq 0,$$

where $\gamma_{j+1} \in R^n$ for each j and $\gamma = \mathrm{col}(\gamma_1, \cdots, \gamma_k)$ satisfies $A_k\gamma = 0$.

Proof. For $k=1$, this has been proved in Theorem 5.2. Suppose $k=2$ and $\phi \in \mathfrak{N}(A-\lambda I)^2$. Then the same proof as in the proof of Theorem 5.2 shows that $(A-\lambda I)\phi = e^{\lambda\cdot}b$, where $\Delta(\lambda)b=0$. Therefore,

$$\frac{d}{dt}T(t)\phi = T(t)A\phi = \lambda T(t)\phi + T(t)e^{\lambda\cdot}\phi = \lambda T(t)\phi + e^{\lambda t}e^{\lambda\cdot}b$$

and

$$T(t)\phi = e^{\lambda t}\phi + te^{\lambda t}e^{\lambda\cdot}b.$$

Using the fact that $[T(t)\phi](\theta) = x(t+\theta, \phi)$, we obtain $\phi(\theta) = e^{\lambda\theta}c$ for some vector $c \in R^n$. Since $T(t)\phi$ must satisfy Equation (5.1), a few computations show that

$$P_1c + P_2b = 0, \qquad P_1b = 0,$$

which is the assertion of the theorem for $k=2$. The same type of calculations and an induction argument will complete the proof of the theorem for any integer k.

The computational procedure for obtaining the complementary subspace in Decomposition (5.4) is not complete. Effective procedures similar to the ones for functional differential equations with finite delays need to be desired. We do not pursue this direction any further except to say that the use of Laplace transform seems very natural. Also, the procedure in Hale [11, Section 7.3] could possibly be generalized.

If (δ_3) is satisfied, then we choose r_0 so that $M(t_0) < r_0 < 1$ for $t_0 > 0$ mentioned in (δ_3). The nature of the spectrum of $T(t_0)$ outside the circle with center zero and radius r_0 and Theorems 5.2 and 5.3 make these problems look very similar to the ones that have been discussed for retarded and certain types of neutral equations with finite delay. It should be possible to extend all of that theory.

Although the above comments have been sketchy, it is hoped that they have been sufficiently specific to point out some among the numerous possibilities for further research.

We conclude this section with a short discussion of the spaces in Examples 1.1 and 1.3. For Hypothesis (δ_3), since $M(t)=\|S(t)\|$ should satisfy

$$M(t)=\sup_{\theta\leqq 0}\frac{g(\theta-t)}{g(\theta)} \qquad \text{for Example 1.1} \tag{5.5}$$

and

$$M(t)=e^{-rt} \qquad \text{for Example 1.3,}$$

the same conditions as for τ^t are obtained, as was expected, and Hypothesis (δ_3) will not be satisfied for $g(\theta)=\theta^{-2}$, where $M(t)=1$ for all $t\geqq 0$ (refer Remark 3.1, too).

When $M(t)=1$, this implies, in particular, for the linear equation (5.1) that $r_e(T(t))=1$. The difficulties encountered in this case can be easily be seen from the numerous investigations in Volterra integral equations (see Miller [14]).

For the spaces of continuous functions in Example 1.3, Hypothesis (δ_3) will be satisfied if and only if $\gamma>0$. Thus, this hypothesis is not satisfied for the space C_0 of continuous functions on $(-\infty, 0]$ which approach limits at $-\infty$. The space C_0 has been used in a number of papers, but from our point of view it is very undesirable. In fact, our goal is to lay the foundation for a general qualitative theory of functional differential equations in a Banach space B. This naturally implies that orbits should be considered in B and trajectories in $R\times B$ and not R^n and $R\times R^n$, respectively. The limitations of the latter approach even with finite delays is amply demonstrated in Hale [11]. For infinite delay, the space C_0 has other additional disadvantages. For example, it happens that $x_t\not\to 0$ in C_0 even if $x\in F_\infty$ and $x(t)\to 0$ as $t\to\infty$. This implies it is never possible to define asymptotic stability in C_0 (for the definition, see § 6). If $x\in F_\infty$ and $x(t)$ is bounded and uniformly continuous for $t\geqq 0$, then the set $\{x_t, t\geqq 0\}\subset C_0$ is not necessarily compact. In fact take $x_0=\bar{1}$, the constant function one, and suppose $x(t)\to 0$ as $t\to\infty$. Then for any sequence $t_k\to\infty$ as $k\to\infty$, we have $|x_{t_k}-x_{t_l}|_{C_0}\to 1$ as $k, l\to\infty$. This is very undesirable since the only way of checking precompactness of orbits in B is to investigate the behavior of the solution $x(t)$ for $t\geqq 0$. In the next section, we investigate some of the implications of Hypothesis (δ_3), or equivalently (γ_2), for the development of a qualitative theory of stability. We shall return to the same topics again (see Remark 6.1 below).

§ 6. Relationship between the stability in R^n and in B.

Razumikhin type Liapunov theorems can be applied to the stability problem of retarded functional differential equations with infinite delay, see [15], [16], [17], [18]. Such theorems and practical phenomena involved in the equations intimate an advantage of the adoption of the concept of the stability in R^n rather than in B under Hypotheses (β_3), ($\gamma_1 \sim \gamma_2$).

Definition 6.1. A solution $u(t)$ of RFDE(f) defined on $[0, \infty)$ is said to be *stable in B*, if for $\sigma \geqq 0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\phi - u_\sigma|_B < \delta$ implies

$$(6.1) \qquad |x_t(\sigma, \phi) - u_t|_B < \varepsilon$$

for all $t \geqq \sigma$, where $x(\sigma, \phi)$ denotes a solution of RFDE(f) through (σ, ϕ).

If, in addition to the stability, for $\sigma \geqq 0$ there is a constant $\delta_0 > 0$ such that $|\phi - u_\sigma|_B < \delta_0$ implies

$$(6.2) \qquad |x_t(\sigma, \phi) - u_t|_B \to 0 \qquad \text{as } t \to \infty,$$

then $u(t)$ will be said to be *asymptotically stable in B.*

Definition 6.2. In Definition 6.1 if the B-norm $|x_t(\sigma, \phi) - u_t|_B$ in Relations (6.1), (6.2) is replaced by the R^n-norm $|x(t, \sigma, \phi) - u(t)|$, then we say that $u(t)$ is *stable in* R^n and *asymptotically stable* in R^n, respectively.

Definition 6.3. In Definition 6.1 (Definition 6.2) if the convergence in (6.2) is uniformly for solutions, that is, for any $\sigma \geqq 0$ and $\varepsilon > 0$ there exists a T such that (6.1) holds when $t \geqq \sigma + T$, then $u(t)$ is said to be *equi-asymptotically stable in B* (in R^n, respectively).

Moreover, if the numbers δ, δ_0 appearing in Definition 6.1 (Definition 6.2) and T in the above can be chosen independently of $\sigma \geqq 0$, then we say the stabilities are *uniform.*

Definition 6.4. A function $f(t, \phi)$, defined on $R \times B$, is said to have the *compact hull,* if for any compact set $W \subset R \times B$, any sequence $\{t_k\}$, $t_k \geqq 0$, contains a subsequence such that $\{f(t + t_k, \phi)\}$ converges uniformly for $(t, \phi) \in W$.

The hull $H(f)$ denotes the set of pairs (g, Ω), $\Omega \subset R \times B$, such that there exists a sequence $\{t_k\}$ $t_k \geqq 0$, for which $\{f(t + t_k, \phi)\}$ converges to $g(t, \phi)$ for $(t, \phi) \in \Omega$. If $\Omega = R \times B$, then we shall write $g \in H(f)$ simply, instead of $(g, R \times B) \in H(f)$.

The following lemma can be proved by the same way as in [19].

Lemma 6.1. *f has the compact hull if and only if for any compact set $\Gamma \subset B$, $f(t, \phi)$ is bounded and uniformly continuous on $R \times \Gamma$.*

If f has the compact hull and if $\Gamma_k \subset B$ are compact sets for $k = 1, 2, \cdots$, then for any $(g^0, \Omega_0) \in H(f)$, there exists a g such that $(g, \Omega) \in H(f)$ for a $\Omega \supset \Omega_0 \cup \{R \times \bigcup_k \Gamma_k\}$ $g|_{\Omega_0} = g^0$. Therefore, if B is separable, we can choose $R \times B$ as Ω in the above.

Definition 6.5. $u(t)$ is said to be *uniformly attracting in the hull,* if there exists a constant $\eta_0>0$ such that if $(v, g, \Omega) \in H(u,f)$ and $|\phi-v_\sigma|_B<\eta_0$ for a $\sigma\geqq 0$, then $|y_t(\sigma, \phi)-v_t|_B \to 0$ as $t\to\infty$ for any solution $y(\sigma, \phi)$ of

$$\dot{x}(t)=g(t, x_t) \tag{6.3}$$

through (σ, ϕ) as long as $(t, y_t(\sigma, \phi)) \in \Omega$.

Now, we can state the following theorems.

Theorem 6.1. *If a solution $u(t)$ of* RFDE(f) *is (uniformly) (asymptotically) stable in B, then so is it in R^n.*

If we suppose Hypotheses (γ_1), (γ_3) (also (γ_2) for the case of the asymptotic stability), then the converse is true.

Proof. The first part is obvious by Axiom (α_4).

Since there is no special assumption for f, we may consider $u(t)\equiv 0$. By Inequality (2.1) and Hypotheses (γ_1), (γ_3) we have

$$|x_t|_B \leqq K_1 \sup_{\sigma\leqq s\leqq t} |x(s)|+M_1|x_\sigma|_B, \qquad t\geqq\sigma, \tag{6.4}$$

for solutions $x(t)$ defined for $t\geqq\sigma$, and hence if the zero solution is (uniformly) stable in R^n, then so is it in B.

Let the zero solution be asymptotically stable in R^n. Again using (6.4) we can assume

$$|x_\sigma|_B \leqq \delta_0(\sigma) \text{ implies } |x_t|_B \leqq C(\sigma),\ t\geqq\sigma,$$

where $C(\sigma)=K_1+M_1\delta_0(\sigma)$ which depends on σ only through δ_0. For any $\beta>0$ Hypothesis (γ_1) implies

$$|x_t|_B \leqq K_1 \sup_{t-\beta\leqq s\leqq t} |x(s)|+|\tau^\beta x_{t-\beta}|_\beta \qquad t\geqq\beta+\sigma. \tag{6.5}$$

From this, it follows

$$|x_t|_B \leqq K_1 \sup_{t-\beta\leqq s\leqq t} |x(s)|+M_1(\beta)C(\sigma), \qquad t\geqq\beta+\sigma.$$

Hence, for any $\varepsilon>0$ if we choose β_0 so that $M_1(\beta)C(\sigma)<\varepsilon/2$ for all $\beta\geqq\beta_0$ by Hypothesis (γ_2) and Lemma 3.1, then the fact $|x(t)|<\varepsilon/2K_1$ for $t\geqq\sigma+T$ implies $|x_t|_B<\varepsilon$ for $t\geqq\sigma+\beta_0+T$.

The procedure above shows that equi-ness and uniformity are preserved.

Remark 6.1. From Relation (6.5) it is clear that for the simple asymptotic stability we can replace Hypothesis (γ_2) by

(γ_5) $\quad |\tau^\beta\phi|_\beta \to 0$ as $\beta\to\infty$ for each $\phi\in B$.

In fact, for any $\varepsilon>0$ choose $s_0\geqq\sigma$ so that $|x(s)|<\varepsilon/2K_1$ for $s\geqq s_0$, and then (6.5) implies

$$|x_t|_B\leqq K_1\sup_{s_0\leqq s\leqq t}|x(s)|+|\tau^{t-s_0}x_{s_0}|_{t-s_0}<\varepsilon/2+|\tau^{t-s_0}x_{s_0}|_{t-s_0}$$

which shows that $|x_t|_B\to 0$ as $t\to\infty$.

In Example 1.1, Hypothesis (γ_5) is satisfied when $g(\theta)=\theta^{-2}$, since

$$\int_{-\infty}^{-\beta}g(\theta)|\phi(\theta+\beta)|\,d\theta\leqq\int_{-\infty}^{-\eta}g(\theta)|\phi(\theta)|\,d\theta+\frac{\eta^2}{(\eta+\beta)^2}\int_{-\infty}^{0}g(\theta)|\phi(\theta)|\,d\theta$$

and the first integrand tends to 0 as $\eta\to\infty$ while the second tends to 0 as $\beta\to\infty$ for each fixed $\eta>0$.

Theorem 6.2. *Suppose Hypotheses (β_3), (γ_1), (γ_2) are satisfied, let* RFDE $(f, R\times B)$ *have a bounded solution $u(t)$ defined on* $[0,\infty)$, *and let f in* RFDE (f) *be continuous on $R\times B$, have the compact hull and be bounded by L on*

$$\Omega_0=\{(t,\phi): t\geqq 0, |\phi-u_t|_B\leqq\eta\}$$

for an $\eta>0$.

If $u(t)$ is uniformly stable in B and uniformly attracting in the hull, then it is uniformly asymptotically stable in B.

Proof. Let δ, η_0 be the numbers appearing in the definitions of the uniform stability and the uniform attractor in the hull, and let

$$\delta_0=\delta(\min\{\eta,\eta_0\}). \tag{6.6}$$

Since we have the uniform stability, in order to prove the uniformly asymptotic stability it is sufficient to show the existence of T for given ε such that

$$|x_\sigma-u_\sigma|_B\leqq\delta_0 \quad\text{implies}\quad \inf_{\sigma\leqq t\leqq\sigma+T}|x_t-u_t|_B<\delta(\varepsilon)$$

for any $\sigma\geqq 0$ and any solution $x(t)$ of RFDE (f).

Suppose it is not the case. Then, there exist sequences $\{\sigma_k\}$, $\sigma_k\geqq 0$, $\{x^k(t)\}$, solutions of RFDE (f), such that $|x^k_{\sigma_k}-u_{\sigma_k}|_B\leqq\delta_0$ but $|x^k_t-u_t|_B\geqq\delta(\varepsilon)$ for $t\in[\sigma_k,\sigma_k+2k]$. Since $|x^k_t-u_t|_B\leqq\eta$ for all $t\geqq\sigma_k$ by (6.6), $|\dot{x}^k(t)|\leqq L$ for all $k, t\geqq\sigma_k$ by the assumption on f and $|x^k_t|_B\leqq\eta+\sup_{t\geqq 0}|u_t|_B<\infty$, which implies the uniform boundedness of $x^k(t+\sigma_k)$ by Axiom (α_4). Therefore, for $t_k=\sigma_k+k$, we can assume that $x^k_{t_k}$ and u_{t_k} converge and, hence

$$\Gamma=\mathrm{Cl}\{x^k_t, u_t: t\geqq t_k, k=1,2,\cdots\}$$

is a compact subset of B by Corollary 3.2. Hence, we can assume that $f(t+t_k,\phi)$

converges to a $g(t, \phi)$ uniformly on $[0, A]\times\Gamma$ for any $A>0$. By the same argument as in the proof of Theorem 3.2., it is allowed to assume that $u(t+t_k)$ and $x^k(t+t_k)$ converge to $v(t)$ and $y(t)$, respectively, solutions of RFDE (g), and then clearly

$$|y_0-v_0|_B\leqq\delta_0 \quad \text{but} \quad |y_t-v_t|_B\geqq\delta(\varepsilon) \qquad \text{for all } t\geqq 0,$$

which contradicts the uniform attractor in the hull. This completes the proof of the theorem.

Corollary 6.1. *Suppose Hypotheses* $(\beta_3), (\gamma_1), (\gamma_2)$.

If f is continuous and periodic in t and if the zero solution of RFDE (f) *is asymtotically stable in B, then it is uniformly asymptotically stable in B.*

Proof. It is possible to find an $\eta>0$ so that $f(t, \phi)$ is bounded on $R\times\{\phi: |\phi|_B \leqq\eta\}$, because $[0, p]\times\{0\}$ is compact, where $p>0$ is a period of f.

Since the zero solution must be unique for the initial value problem, we can find a $\rho(\varepsilon)>0$ for any given $\varepsilon>0$ such that

$$|x_\sigma|_B\leqq\rho(\varepsilon),\ \sigma\in[0, p), \quad \text{implies} \quad |x_p|_B\leqq\varepsilon$$

for any solution $x(t)$ of RFDE (f) by Theorem 2.5. From this, it follows that the zero solution is uniformly stable in B and uniformly attracting in the hull, where the associated numbers $\delta(\varepsilon), \eta_0$ can be chosen by

$$\delta(\varepsilon)=\rho(\delta(p, \varepsilon)), \qquad \eta_0=\rho(\delta_0(p))$$

for $\delta(p, \varepsilon)$, $\delta_0(p)$ appearing in the definition of the asymptotic stability in B at $\sigma=p$. Thus, the rest of proof follows immediately from Theorem 6.2.

Here, we should note that $H(f)=\{(f(t+s, \phi), R\times B);\ s\in[0, p]\}$, which is compact.

Corollary 6.2. *Under Hypotheses* $(\beta_3), (\gamma_1), (\gamma_2)$, *if the zero solution of a periodic system is asymptotically stable in* R^n, *then it is uniformly asymptotically stable in* R^n.

Proof. Under the assumption, the zero solution is asymptotically stable in B by Theorem 6.1, and hence it is uniformly asymptotically stable in B by Corollary 6.1. By applying Theorem 6.1 again, the proof is finished.

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nuna adreso:
Jack K. Hale
Division of Applied Mathematics
Brown University
Providence, RI 02912
U.S.A.
Junji Kato
Mathematical Institute
Tohoku University
Sendai
Japan

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