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On Linear Games with Subspace Target

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We shall consider a linear two controller system specified by

$$\dot{x}=Ax+Bu+Cv\,; \qquad t\geqslant 0. \tag{1}$$

Here $A\in \boldsymbol{R}^{n\times n}$, $B\in \boldsymbol{R}^{n\times p}$ and $C\in \boldsymbol{R}^{n\times e}$ (where $\boldsymbol{R}^{i\times j}$ denotes the space of real $i\times j$ matrices). The space U of *pursuer controls u* consists of measurable $\boldsymbol{R}^{p}$-valued functions defined on $[0, \infty)$ which are bounded on bounded intervals. Analogously V is the space of ($\boldsymbol{R}^{e}$-valued) *evader controls*. The model is completed by specifying as *target* a linear subspace $S\subset \boldsymbol{R}^{n}$.

By a class E of *evader strategies* we refer to pairs (T, e) where $T\in \boldsymbol{R}$, $T>0$, and e is a mapping $e: U\rightarrow V$ which is causal in the sense that $u_1(t)=u_2(t)$ a.e. on $[0, t]$ implies $e(u, (t))=e(u_2(t))$ a.e. on $[0, t]$. The inclusion of T in the general definition of evader strategy signifies the fact that we view the evader as the announcer of the game's duration. (As will be seen below, this is not crucial but only a formal convenience.) We consider three types of evader strageties:

(E_1) The map e is constant, i.e. there exists a fixed control function $v\in V$ such that $e(u)=v$ for all $u\in U$ (the "blind" or open loop case).

(E_2) The map e is specified through a linear (time invariant) state feedback law $v(t)=Kx(t)$ where $K\in \boldsymbol{R}^{e\times n}$.

(E_3) The map e is given by a combination of E_1 and E_2 strategies, that is, $v(t)=Kx(t)+\tilde{v}(t)$ where $K\in \boldsymbol{R}^{e\times n}$ and where $\tilde{v}\in V$ is a fixed control.

The class of games under consideration is the following: The pursuer's goal is to steer x to S and the evader's goal is to prevent this. Accordingly, the evader selects a strategy (T, e) and announces it to the pursuer. The pursuer then specifies a control $u\in U$. Hence, pursuer strategies are mappings $p: E\rightarrow U$. We shall consider games G_1, G_2 and G_3 where G_i is the game in which the evader employs stragety E_i $(i=1, 2, 3)$.

Consider a single input process

$$\dot{x}=\mathrm{D}x+\mathrm{E}w\,; \qquad t\geqslant 0 \tag{2}$$

where $\mathrm{D}\in \boldsymbol{R}^{n\times n}$, $\mathrm{E}\in \boldsymbol{R}^{n\times q}$ and where $w\in W$, a space of control functions. We shall adopt the terminology of [7] and say that a subspace $N \subset \boldsymbol{R}^n$ is (D, E)-*invariant* if $\mathrm{D}N \subset N + R(\mathrm{E})$ where $R(\mathrm{E})$ denotes the column span (or range) of E. For a subset $M \in \boldsymbol{R}^n$ we denote by core (M) the set of initial states $x_0 \in M$ such that there exists $w \in W$ for which the associated solution of (1.2) satisfying $x(0) = x_0$ also satisfies $x(t) \in M$ for all $t \geqslant 0$. Clearly, if M is a subspace of $\boldsymbol{R}^n$ so is core (M). Relevant facts concerning (D, E)-invariant subspaces of M and core (M) are summarized in the following.

Proposition 1. (i) *A subspace* $N \subset \boldsymbol{R}^n$ *is* (D, E)-*invariant if and only if there exists* $Q \in \boldsymbol{R}^{q \times n}$ *such that* $(D + EQ)N \subset N$.

(ii) *Let* $M \subset \boldsymbol{R}^n$ *be a subspace. Then* core (M) *is the supremal* (D, E)-*invariant subspace of* M.

The elementary proof of (i) can be found in [7]. The proof of (ii) can be found in [1] (see also [2]). Further properties of cores may be found in [3], [4] and [8].

We denote by $(\mathrm{D} \mid \mathrm{E})$ the $n \times nq$ matrix $[\mathrm{E}, \mathrm{DE}, \cdots, \mathrm{D}^{n-1}\mathrm{E}]$. Also, we let $\{\mathrm{D} \mid \mathrm{E}\} := \{\mathrm{D} \mid R(\mathrm{E})\} := R([\mathrm{D} \mid \mathrm{E}]) = R(\mathrm{E}) + \mathrm{D}R(\mathrm{E}) + \cdots + D^{n-1}R(\mathrm{E})$. The following well known facts are used repeatedly.

$$\{\mathrm{D} \mid \mathrm{E}\} = \bigcup_{w \in W} e^{AT} \int_0^T e^{-At} \mathrm{E} w(t) dt \quad \forall T > 0. \tag{3}$$

$$\{\mathrm{D} \mid \mathrm{E}\} = \{\mathrm{D} + \mathrm{EQ} \mid \mathrm{E}\} \quad \forall \mathrm{Q} \in \boldsymbol{R}^{q \times n}. \tag{4}$$

For a subspace $M \subset \boldsymbol{R}^n$ and $\mathrm{D} \in \boldsymbol{R}^{n \times n}$ we denote by M_{D} the supremal D-invariant subspace of M. We denote the empty set by ϕ.

An initial state $x_0 \in \boldsymbol{R}^n$ is called *capturable in game* G_i provided that for every announced evader strategy $(T, e) \in E_i$ $(i = 1, 2, 3)$ there exists a control $u \in U$ for which the associated solution of (1) emanating from $x(0) = x_0$ satisfies $x(T) \in S$. The set of capturable initial states for game G_i is denoted X_i $(i = 1, 2, 3)$.

Theorem 2.

$$X_1 = \begin{cases} (S + \{A \mid B\})_A & \textit{if and only if } \{A \mid C\} \subset S + \{A \mid B\} \\ \phi & \textit{otherwise.} \end{cases}$$

Proof. Assume $x_0 \in X_1$ and let the evader choose $v = 0$ and $T > 0$. Then there exists $u \in U$ such that

$$e^{AT} x_0 + e^{AT} \int_0^T e^{-At} B u(t) dt \in S. \tag{5}$$

Hence, by (3)

$$e^{AT} x_0 \in S + \{A \mid B\} \quad \forall T > 0, \tag{6}$$

which is equivalent to $x_0 \in (S+\{A \mid B\})_A$. Thus $X_{11} \neq \phi$ implies $X_{11} \subset (S+\{A \mid B\})_A$. Consider now

$$x(T)=e^{AT}x_0+e^{AT}\int_0^T e^{-At}Bu(t)dt+e^{AT}\int_0^T e^{-At}Cv(t)dt \tag{7}$$

with $x_0 \in (S+\{A \mid B\})_A$ chosen arbitrarily. By (3) and (6) it then follows that

$$x(t)-e^{AT}\int_0^T e^{-At}Cv(t)dt \in S+\{A \mid B\}$$

so that $x(T) \in S+\{A \mid B\}$ if and only if $e^{AT}\int_0^T e^{-At}Cv(t)dt \in S+\{A \mid B\}$. Since $v \in V$ and $T>0$ were arbitrarily chosen, the proof is complete $\square$

Remark 3. Note that $\{A \mid C\} \subset S+\{A \mid B\}$ if and only if $\{A \mid C\} \subset (S+\{A \mid B\})_A$. Thus X_1, if nonempty, contains $\{A \mid C\}$.

Remark 4. Theorem 2 generalizes a result in [5] which deals with "max-min controllability" to target $S=0$.

Corollary 5.

$$X_2 = \bigcap_{K \in R^{e \times n}} (S+\{A+CK \mid B\})_{A+CK}$$

Proof. Consider the system

$$\dot{x}=(A+CK)x+Bu. \tag{8}$$

For each announced K, the set of states controllable to S by the pursuer is $(S+\{A+CK \mid B\})_{A+CK}$, due to the specialization of Theorem 2 to the single input case. $\square$

Remark 6. In general, $X_2 \neq \bigcap_{K \in R^{e \times n}} (S+\{A+CK \mid B\})$. Consider for example $S \neq S_A$, $B=C=0$.

Remark 7. Consider the target $S=0$. In view of Proposition 1 (i) it follows that X_2 is equal to the intersection of all (A, C)-invariant subspaces of R^n which contain $R(B)$. Hence, $R(B) \subset X_2$ always. Should $R(B)=X_2$ (a "best" situation from the evader is viewpoint) it does not follow that there exists a "best" evader strategy, i.e., a $\tilde{K} \in R^{e \times n}$ such that $X_2=\{A+C\tilde{K} \mid B\}$. To see this, consider the following example:

$$A=\begin{bmatrix}0&1&0\\0&0&0\\1&0&0\end{bmatrix},\quad B=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad C=\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

It is easily checked that $R(B)$ is not (A, C)-invariant, while the spaces

$$S_1=R\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\quad\text{and}\quad S_2=R\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right)$$

are (A, C)-invariant. (The first space is A-invariant while the second is $(A+CK)$-invariant with $K=(1\ \ 0\ \ 0)$.) Now note that the intersection of these spaces in $R(B)$.

Remark 8. If $X_1=\phi$, then the evader prefers to play in game G_1 rather than in G_2, since always $X_2\neq\phi$. If, however, $X_1\neq\phi$, then the preference is reversed, as $X_2\subset X_1$. In fact, the inclusion can be proper as in the following example with $S=0$: Let

$$A=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad B=\begin{bmatrix}1\\1\end{bmatrix},\quad C=\begin{bmatrix}0\\1\end{bmatrix}.$$

Then $X_1=\boldsymbol{R}^2$ while $X_2=R(B)$.

Theorem 9.

$$X_3=\begin{cases}X_2 & \text{if and only if } \{A\,|\,C\}\subset X_2\\ \phi & \text{otherwise.}\end{cases}$$

Proof. Consider the system

$$\dot{x}=(A+CK)x+Bu+Cv. \tag{9}$$

Fix $K\in\boldsymbol{R}^{e\times n}$ and consider the resulting G_1-type game. Then a state $x_0\in\boldsymbol{R}^n$ is capturable if and only if $x_0\in(S+\{A+CK\,|\,B\})_{A+CK}$ and $\{A+CK\,|\,C\}=\{A\,|\,C\}\subset S+\{A+CK\,|\,B\}$, or equivalently, $\{A\,|\,C\}\subset(S+\{A+CK\,|\,B\})_{A+CK}$. Upon varying K over $\boldsymbol{R}^{e\times n}$ the result follows. ☐

It is interesting to compare the evader's capabilities in the three types of games on a somewhat more intuitive level. In game G_3, it may be that the evader has at his disposal a $\hat{K}\in\boldsymbol{R}^{e\times n}$ which renders the set of capturable states of the resulting G_1-type game empty. Such $\hat{K}$ as above are completely characterized by the property

$$\{A\,|\,C\}\not\subset S+\{A+C\hat{K}\,|\,B\}. \tag{10}$$

Indeed, consider the G_3 game with $S=0$ and system matrices as in Remark 8 above with $\hat{K}=(0, -1)$. The reader can easily verify that in this case $X_1=\boldsymbol{R}^2$, $X_3=R(B)$

and $X_3 = \phi$.

The next theorem gives a necessary and sufficient condition for the existence of $\hat{K}$ satisfying (10) in the case where the target space is A-invariant. This generalizes a result of [5] where the case $S=0$ was dealt with.

Theorem 10. *Consider game* G_1 *with* $S=S_A$. *Then a necessary and sufficient condition for the existence of* $\hat{K}$ *such that* (10) *holds is*

$$R(C) \not\subset S_A + R(B). \tag{11}$$

Proof. Assume first that (11) fails to hold and that $R(C) \subset S_A + R(B)$. Then

$$\{A \mid C\} \subset S_A + \{A \mid B\} = \{A \mid S_A + R(B)\} = \{A + \hat{A} \mid S_A + R(B)\}$$

for every $\hat{A} \in \boldsymbol{R}^{n \times n}$ satisfying $R(\hat{A}) \subset S_A + R(B)$. This condition is satisfied (by hypothesis) by every $\hat{A}$ with $R(\hat{A}) \subset R(C)$ and thus there is no $\hat{K}$ for which (10) holds.

Conversely, assume that (11) holds. Then there exists a nonzero subspace $Z \subset R(C)$ such that $Z \cap [S_A + R(B)] = 0$. Write $\boldsymbol{R}^n \in Z \oplus Y$ for a subspace Y such that $S_A + R(B) \subset Y$, and denote by P the projection of $\boldsymbol{R}^n$ onto Z along Y. Choose $\hat{K} \in \boldsymbol{R}^{e \times n}$ so that $C\hat{K} = -PA$ and note that $A + C\hat{K} = (I-P)A$. We will show that

$$S_A + \{(I-P)A \mid B\} \subset Y. \tag{12}$$

This is enough, for then $Z \subset Y$ and $Z \subset R(C)$ imply $R(C) \subset S_A + \{A + CK \mid B\}$. This, in turn, implies (10). To verify (12), observe that $S_A + R(B) \subset Y$ implies $\{(I-P)A \mid S_A + R(B)\} \subset Y$. Now $\{(I-P)A \mid S_A + R(B)] = S_A + R(B) + (I-P)A[S_A + R(B)] + \cdots + (I-P)A^{n-1}[S_A + R(B)]$ (using the fact that $I-P$ is idempotent) which in turn equals $S_A + \{(I-P)A \mid B\}$ (since $PS_A = 0$). □

We conclude the paper with some remarks on a slightly different class of games. Suppose that in each case (i.e. evader strategies (E_1, E_2, or E_3)) the pursuer is required to respond with a constant feedback (of the form $u(t) = Fx(t)$) rather than being allowed to use open loop controls. Denoting the corresponding game $\bar{G}_i$, we say that $x_0 \in S$ is capturable in $\bar{G}_1$ if there exists $F \in \boldsymbol{R}^{p \times n}$ such that against every evader strategy $(T, e) \in E_i$ the pursuer feedback law $u = Fx$ insures $x(T) \in S$ (where $x(0) = x_0$). It is easily noted that in this case the set of capturable states is independent of the evader strategy and is given by

$$\bar{X}_i = \begin{cases} \widehat{\text{core}}\,(S) & \text{if and only if } R(C) \subset \widehat{\text{core}}\,(S) \\ \phi & \text{otherwise} \end{cases}$$

for all $i = 1, 2, 3$, where $\widehat{\text{core}}\,(S)$ denotes the core of S in the "pursuer" system $\dot{x} = Ax + Bu$.

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