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On Inverse Scattering for the Klein-Gordon Equation with Small Potentials[*]

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§ 1. Introduction

There are many interesting problems in mathematical physics in which the questions reduce to determine the coefficients of a partial differential equation from the knowledge of functionals of its solution. Let us simply mention a few problems in this direction:

1) In Quantum Mechanics, people usually ask if it is possible that, given the spectrum of a second order differential operator, is it possible to find the operator. It turns out that this problem must be posed somewhat differently since a single spectrum is insufficient to determine the differential operator. See Gel'fand-Levitan [1], Berezanskii, Ju [2] and Faddeev, L.D. [3].

2) The seismological inverse Kinematic problem which can be roughly described as follows: In a region Ω bounded by a surface S, waves are generated by sources at points of the surface S and their travel-times are "recorded" at various points of the boundary of Ω. By using this information, find the propagation velocities of the disturbances inside Ω. This problem is very important because even partial answers could make it possible to make inferences about the Earth's internal structure from seismological data. A nice approach to this with references can be found in Romanov's book [4].

In this note we would like to present some results concerning the so-called Inverse Scattering problem for the Klein-Gordon equation with an external potential $q(x)$:

$$\Box u + m^2 u + q(x) u = 0 \tag{1.1}$$

$(m>0)$ in $\Omega = R^3$, $-\infty < t < +\infty$. Here $\Box = \partial^2/\partial t^2 - \Delta$, $\Delta = \sum_{j=1}^{3} \partial^2/\partial x_j^2$. Suitably conditions on the potential $q(x)$ will be given later. It is important to observe that $q(x)$ will not necessarily be spherically symmetric. In a previous work (see [5]) we have treated equation (1.1) in the particular case in which $m=0$ and we proved that the scattering operator determines uniquely the potential $q(x)$. In what follows we generalize our previous results to the case $m>0$. The relativistic wave equation

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(1.1) was treated by J. Chadam in [6] who proved the existence and completeness the wave operators and the unitarity of the scattering operator associated with (1.1). Related results concerning equation (1.1) were given by Ju Berezanskii [2], T. Schonbek [7] and more recently by G. Avila [8].

§ 2. Preliminaries

We denote by $\boldsymbol{R}^3$ the three-dimensional Euclidean space. The space of C^∞-functions with compact support in $\boldsymbol{R}^3$ is denoted by $C_0^\infty(\boldsymbol{R}^3)$. By $L^p(\boldsymbol{R}^3)$ the space of (classes of) measurable functions $f(x)$ in $\boldsymbol{R}^3$ whose p'th powers are integrable, with the norm

$$\|f\|_p=\left(\int |f(x)|^p\,dx\right)^{1/p} \qquad (p\geq 1)$$

and by $L^\infty(\boldsymbol{R}^3)$ the space of measurable, essentially bounded functions in $\boldsymbol{R}^3$, with the norm

$$\|f\|_\infty=\operatorname{ess\,sup}|f(x)|.$$

Let f be a function on R^3 with partial derivatives. We will denote the gradient of f by grad $f(x)$ where

$$\operatorname{grad} f(x)=\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

and $|\operatorname{grad} f(x)|^2=\sum_{j=1}^3 |\partial f/\partial x_j|^2$. An integral sign to which *no domain is attached* will be understood to be taken over all space $\Omega=\boldsymbol{R}^3$.

The real valued function $q(x)$ in (1.1) is assumed to be non-negative and belongs to $L^1\cap L^\infty(\boldsymbol{R}^3)$. The initial data at $t=0$ for the equation (1.1) are in $C_0^\infty(\boldsymbol{R}^3)$.

In particular, the above conditions on $q(x)$ imply that the associated (Schrödinger) wave operators

$$W_\pm=s-\lim_{t\to\pm\infty}\exp(i(m^2+q(x)-\Delta)t)\exp(-i(m^2-\Delta)t)$$

are complete and invariance principle holds. Also, the scattering operator $S=W_+^{-1}W_-$ associated with (1.1) exists and is unitarily implementable in Fock space. (See J. Chadam [6]). Let us present some technical lemmas concerned with energy inequalities

Lemma 2.1. *Let* $u=u(x,t)$ *be a classical solution of* (1.1) *in all space-time, which has* $C_0^\infty(\boldsymbol{R}^3)$ *initial data at* $t=0$. *Then*

a) *The total energy of u is constant, i.e.*

$$\|u\|_E^2=\frac{1}{2}\int\left[|\operatorname{grad} u|^2+\left(\frac{\partial u}{\partial t}\right)^2+m^2u^2+q(x)u^2\right]dx$$

is independent of t.

b) $\iint_{Ch}(m^2u^2+q(x)u^2)dS\leq 2\sqrt{2}\,\|u\|_E^2$

where Ch denotes any characteristic cone and dS the surface measure on Ch.

Proof. By multiplying the equation (1.1) by $\partial u/\partial t$ we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2+m^2u^2+q(x)u^2\right)-\frac{\partial u}{\partial t}\Delta u=0. \tag{2.1}$$

Also, direct computation shows the identity

$$\frac{\partial u}{\partial t}\Delta u=-\frac{1}{2}\frac{\partial}{\partial t}|\operatorname{grad} u|^2+\operatorname{grad}\cdot\left(\frac{\partial u}{\partial t}\operatorname{grad} u\right). \tag{2.2}$$

Substitution of (2.2) in (2.1) gives us

$$\frac{1}{2}\frac{\partial}{\partial t}\left(|\operatorname{grad} u|^2+\left(\frac{\partial u}{\partial t}\right)^2+m^2u^2+q(x)u^2\right)=\operatorname{grad}\cdot\left(\frac{\partial u}{\partial t}\operatorname{grad} u\right). \tag{2.3}$$

Integration (in space) of (2.3), followed by the divergence theorem, gives us *a*).

Now, let *Ch* be a characteristic cone with vertex at a point (z_0, t_0) and let T be such that $t_0\leq T$. We consider the integrals only on the upward cone (for $t_0\leq t\leq T$) then integration in space-time of (2.3) gives us:

$$\begin{aligned}\iint_{Ch_T}&\left[\frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2+|\operatorname{grad} u|^2+m^2u^2+q(x)u^2\right)-\frac{\partial u}{\partial|x|}\frac{\partial u}{\partial t}\right]dS\\&=\frac{\sqrt{2}}{2}\int_{S_T}\left[\left(\frac{\partial u}{\partial t}\right)^2+|\operatorname{grad} u|^2+m^2u^2+q(x)u^2\right]dx\end{aligned} \tag{2.4}$$

where $\partial u/\partial|x|$ denotes the "radial" derivative, i.e. $\operatorname{grad} u\cdot x/|x|$, Ch_T denotes the surface of the upward characteristic cone (for $t_0\leq t\leq T$) and $S_T=\{(x,t), |x-z_0|\leq T-t_0\}$. Since $(\partial u/\partial|x|)^2\leq|\operatorname{grad} u|^2$, it follows that

$$\frac{\partial u}{\partial|x|}\frac{\partial u}{\partial t}\leq\frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2+|\operatorname{grad} u|^2\right]. \tag{2.5}$$

By combining (2.5) with (2.4) and using part a) we get

$$\iint_{Ch_T}(m^2u^2+q(x)u^2)dS\leq\sqrt{2}\,\|u\|_E^2 \tag{2.6}$$

letting $T\to+\infty$ and considering the other half of the characteristic cone we get part b).

We will need also a calculus type lemma:

Lemma 2.2. *Let* $g(x)\in L^1(\boldsymbol{R}^3)\cap L^r(\boldsymbol{R}^3)$ *for some* $r\geq 2$, *then* $F(x,y)=|y|^{-1}g(x+y)$ *is as a function of "y" in* $L^1(\boldsymbol{R}^3)$ *and*

$$\|F(x,\cdot)\|_1\leq C\|g\|_r^{1/(4-s)}\|g\|_1^{(3-s)/(4-s)}$$

where C is a positive constant, depending only on s (here $1/s+1/r=1$, $s>0$).

Proof. If $g\equiv 0$, it is trivial. If $g\not\equiv 0$, let $\varepsilon>0$.

Then, the integral $\int|y|^{-1}|g(x+y)|\,dy$ may be broken in two parts. The first part is the integral of $|y|^{-1}|g(x+y)|$ over the region $|y|>\varepsilon$, which is less than or equal to $(1/\varepsilon)\|g\|_1$. The second part can be estimated as follows:

$$\int_{|y|\leq\varepsilon}|y|^{-1}|g(x+y)|\,dy\leq\frac{4\pi}{3-s}\varepsilon^{3-s}\|g\|_r$$

because of Hölder's inequality and the fact that $\int_{|y|\leq\varepsilon}|y|^{-s}\,dy=\frac{4\pi}{3-s}\varepsilon^{3-s}$. Therefore, we have the inequality

$$\int|y|^{-1}|g(x+y)|\,dy\leq\frac{1}{\varepsilon}\|g\|_1+\frac{4\pi}{3-s}\varepsilon^{3-s}\|g\|_r$$

for any $\varepsilon>0$. By choosing

$$\varepsilon=\|g\|_1^{1/(4-s)}\left(\frac{4\pi}{3-s}\|g\|_r\right)^{1/(s-4)}$$

we get the desired inequality with

$$C=\left(\frac{4\pi}{3-s}\right)^{1/(4-s)}+\left(\frac{4\pi}{3-s}\right)^{(3-s)/(4-s)}.$$

§ 3. The inverse problem

As we mentioned in § 2, our hypotheses on $q(x)$ imply the existence of the scattering operator S associated with (1.1). The "action" of this operator S is essentially the following: For each solution u of (1.1) with $C_0^\infty(\boldsymbol{R}^3)$ initial data at $t=0$, there exists a unique pair $u_\pm$ solutions of (1.1) with $q\equiv 0$ such that $\lim_{t\to\pm\infty}\|u-u_\pm\|_E=0$. In this case $Su_-=u_+$. In what follows we shall show that this operator $S=S(q)$ determines *uniquely* the potential, at least when q satisfies our previous hypothesis, plus continuity and is "small" in a sense we will specify later. Thus, we will give a *local* kind of result via the time-dependent approach. The question of uniqueness or reconstruction from S for general q's seems to be very hard and remains, as far as we know, an open problem. Although, a few years ago, L.D. Faddeev, announced and presented a sketch of similar problems for the three-dimensional Schrödinger equation, (see [3]), there are some crucial steps in his proof which remain obscure to many physicists and mathematicians (see [9]).

Let u be a solution of (1.1) with $C_0^\infty(\mathbf{R}^3)$ initial data at $t=0$, then a useful way to rewrite the operator S is (see [5], Lemma 3.1)

$$Su_-(x,t)-u_-(x,t)=\int_{-\infty}^{\infty} R(x,t-s)*q(x)u(x,s)ds \tag{3.1}$$

where $*$ denotes spatial convolution and R is the Riemann function associated with $\Box u+m^2u=0$ $(m>0)$, $x \in \mathbf{R}^3$, which, in this case, is given by

$$\begin{aligned} R(x,t)&=\frac{\delta(|x|-t)}{4\pi t}+m\theta(x,t)\frac{J_1(m\sqrt{t^2-|x|^2})}{\sqrt{t^2-|x|^2}} \\ &=R_{\Box}(x,t)+R_m(x,t) \end{aligned} \tag{3.2}$$

for $t>0$. Here $\delta(x)$ denotes the three-dimensional Dirac-delta function,

$$\theta(x,t)=\begin{cases} 1, & |x|<t \\ 0, & |x|>t \end{cases}$$

and $J_1=-J_0^{(1)}$ is the Bessel function of 1-th order. On the other hand we can also write u as

$$u(y,s)=u_-(y,s)+Pu(y,s) \tag{3.3}$$

where

$$Pu(y,s)=-\int_{-\infty}^{s}\int R(y-z,s-r)q(z)u(z,r)dzdr.$$

Now we need an estimate on $Pu(y,s)$:

Lemma 3.1. *Let u and Pu be as above, then*

$$|Pu(y,s)|\leq \mathit{constant}\,(\|q\|_\infty^{1/3}\|q\|_1^{1/6}+\|q\|_\infty^{1/2}\|q\|_1^{1/2})\,\|u\|_E.$$

Proof. As in (3.2) we write $R=R_{\Box}+R_m$. We already estimated the first integral in Lemma 4.2 of [5], then

$$\left|\int_{-\infty}^{s}\int R_{\Box}(y-z,s-r)q(z)u(z,r)dzdr\right|\leq \mathit{const.}\,\|q\|_\infty^{1/3}\|q\|_1^{1/6}\|u\|_E. \tag{3.4}$$

For the second integral we have

$$\begin{aligned}&\left|\int_{-\infty}^{s}\int R_m(y-z, s-r)q(z)u(z, r)dzdr\right|\\&\qquad\leq const.\int_{-\infty}^{s}\int_{|z-y|<s-r}(1+\sqrt{(s-r)^2-|z-y|^2})^{-3/2}q(z)\,|u(z, r)|\,dzdr\end{aligned}\tag{3.5}$$

because of the well known estimative

$$\left|\frac{J_1(\rho)}{\rho}\right|\leq\frac{constant}{(1+\rho)^{3/2}}.$$

The right hand side of (3.5) can be estimated by

$$\begin{aligned}&\leq const.\int_{-\infty}^{s}\int_{|z-y|<s-r}(1+(s-r)-|z-y|)^{-3/2}q(z)\,|u(z, r)|\,dzdr\\&=const.\int_{0}^{\infty}\int_{Ch_\xi}(1+\xi)^{-3/2}q(y+\rho\omega)\left|u\left(y+\rho\omega, s-\frac{\eta+\xi}{2}\right)\right|dSd\xi\end{aligned}\tag{3.6}$$

where $\rho=|z-y|$, $\xi=s-r-\rho$, $\eta=s-r+\rho$, Ch_ξ denotes the surface of the characteristic cone with $\xi=$constant, $|\omega|=1$ and dS the surface measure on Ch_ξ.
It is easy to see that

$$\int_{Ch_\xi} qdS\leq\sqrt{2}\,\|q\|_1.\tag{3.7}$$

The inequality (3.7) together with Lemma 1.1 gives us

$$\int_{Ch_\xi} q\,|u|\,dS\leq constant\,\|q\|_\infty^{1/2}\,\|q\|_1^{1/2}\,\|u\|_E.\tag{3.8}$$

Finally, combining (3.6)–(3.7) and (3.8) we get

$$\left|\int_{-\infty}^{s}\int R_m(y-z, s-r)q(z)u(z, r)dzdr\right|\leq constant\,\|q\|_\infty^{1/2}\,\|q\|_1^{1/2}\,\|u\|_E$$

which together with (3.4) implies the desired inequality.

Theorem 3.2. *If the scattering operator S associated with the equation* (1.1) *is the identity operator, then*

$$\langle q\rangle\leq C\,\|u\|_E\,(\|q\|_\infty^{1/3}\,\|q\|_1^{1/6}+\|q\|_1^{1/2}\,\|q\|_\infty^{1/2})(\|q\|_\infty^{1/3}\,\|q\|_1^{2/3}+\|q\|_1)$$

where C is a positive constant, u as in Lemma 1.1,

$$\langle q\rangle=\sup_{u_-}\sup_{x,t}\left|\int_{-\infty}^{\infty}\int R(x-y, t-s)q(y)u_-(y, s)dyds\right|$$

u_- denotes any incoming free solution of (1.1) (*with* $q\equiv 0$) *and R is given by* (3.2).

Proof. It is clear that (3.1) and the above hypotheses give us

$$\langle q\rangle \leq const.\, \|u\|_E(\|q\|_\infty^{1/3}\|q\|_1^{1/6}+\|q\|_1^{1/2}\|q\|_\infty^{1/2})\int_{-\infty}^{\infty}\int |R|\,qdyds \tag{3.9}$$

because of Lemma 3.1. Now, the integral

$$\int_{-\infty}^{\infty}\int |R_m|\,qdyds$$

may be broken in two parts. The first part is

$$\begin{aligned}\int_{-\infty}^{t}\int |R_m|\,qdyds &\leq const.\int_{-\infty}^{t}\int_{|x-y|<t-s}(1+(s-t)^2-|x-y|^2)^{-3/2}qdyds\\ &\leq const.\int_0^\infty\Big(\int_{Ch_{\eta_1}}qdS\Big)\frac{d\eta_1}{(1+\eta_1)^{3/2}}\end{aligned}$$

where $\eta_1=t-s-|x-y|$, $\xi_1=t-s+|x-y|$ and dS denotes the surface measure on Ch_{η_1} (recall that Ch_{η_1} is the surface of the characteristic cone with $\eta_1=$constant). By using (3.7) we obtain

$$\int_{-\infty}^{t}\int |R_m|\,qdyds\leq const.\,\|q\|_1\int_0^\infty(1+\eta_1)^{-3/2}d\eta_1\leq const.\,\|q\|_1.$$

Similarly

$$\int_t^\infty\int |R_m|\,qdyds\leq const.\,\|q\|_1$$

thus

$$\int_{-\infty}^{\infty}\int |R_m|\,qdyds\leq const.\,\|q\|_1. \tag{3.10}$$

We also observe that an easy application of Lemma 4.2 in [5] gives us

$$\int_{-\infty}^{\infty}\int |R_\Box|\,qdyds\leq const.\,\|q\|_\infty^{1/3}\|q\|_1^{2/3}. \tag{3.11}$$

From (3.9)–(3.10) and (3.11) the proof of the theorem is complete.

Corollary 3.3. *Let q_1 and q_2 be spatial potentials which are non-negative, and belong to $L^1\cap L^\infty(R^3)$. Let $S(q_1)$ and $S(q_2)$ denote the scattering operators associated with $\Box u_1+q_1u_1=0$ and $\Box u_2+q_2u_2=0$ respectively. If $S(q_1)=S(q_2)$ then*

$$\begin{aligned}\langle q_1-q_2\rangle\leq const.\,&\|u_1\|_E\,(\|q_1\|_\infty^{1/3}\|q_1\|^{1/6}+\|q_1\|_1^{1/2}\|q_1\|_\infty^{1/2})(\|q_1\|_\infty^{1/3}\|q_1\|_1^{2/3}+\|q_1\|_1)\\ +const.\,&\|u_2\|_E(\|q_2\|_\infty^{1/3}\|q_2\|_1^{1/6}+\|q_2\|_1^{1/2}\|q_2\|_\infty^{1/2})(\|q_2\|_\infty^{1/3}\|q_2\|_1^{2/3}+\|q_2\|)\end{aligned} \tag{3.12}$$

where $\langle\ \rangle$ *is given as in Theorem* 3.2.

Proof. In fact, because of (3.1) we have

$$(S(q_j)-I)u_-(x,t)=\int_{-\infty}^{\infty}\int R(x-y,t-s)q_j(y)u_j(y,s)dyds$$

$j=1,2$, so it follows that

$$\int_{-\infty}^{\infty}\int R(x-y,t-s)[q_1(y)u_1(y,s)-q_2(y)(y,s)]dyds=0.$$

Now, by using (3.3), Lemma 3.1 and the estimatives given in Theorem 3.2, the desired inequality follows.

The above theorem gives us a good idea about uniqueness in the inverse problem for small potentials. In fact, suppose we have a fixed function $q_0(x)$ that satisfies all of our previous hypotheses, let us consider the function $q_\varepsilon(x)=\varepsilon q_0(x)$ $(\varepsilon>0)$ by assuming all hypotheses in Theorem 3.2 are satisfied and plot q_ε in the conclusion of the mentioned theorem to get

$$\langle q_0\rangle\leq const.\ \|u_\varepsilon\|_E\,\varepsilon^{1/2}(\|q_0\|_\infty^{1/3}\|q_0\|_1^{1/6}+\|q_0\|_1^{1/2}\|q_0\|_\infty^{1/2})(\|q_0\|_\infty^{1/3}\|q_0\|_1^{2/3}+\|q_0\|_1)$$

by taking $\|u_\varepsilon\|_E<1$ and letting $\varepsilon\to 0$ we get that $\langle q_0\rangle=0$. If we assume that q is continuous and then we choose a sequence $\{u_-^\rho\}_{\rho>0}$ of free solutions with initial data $u_-^\rho(x,0)=0$, $(\partial u_-^\rho/\partial t)(x,0)=\varphi_\rho(x)$ with $\varphi_\rho(x)\to\delta(x)$, where φ_ρ is the standard δ-sequence: $\varphi_\rho\in C_0^\infty$, $\varphi_\rho\geq 0$, $\int\varphi_\rho=1$ and with support equal to $\{\xi\in \boldsymbol{R}^3, |\xi|\leq\rho\}$ then it follows by an argument quite similar to Proposition 4.5 of [5] that $q_0\equiv 0$. Thus, in this sense, if S is the identity operator, then q must be identically zero. Observe also that because of Corollary 3.3 and the above remarks we get

Theorem 3.4. *Let* q_1 *and* q_2 *be as in Corollary* 3.3 *then, if* $q_1(x)\neq q_2(x)$ *for some* $x\in\boldsymbol{R}^3$ *and* $\langle q_1-q_2\rangle>a(q_1,q_2)$, *where* $a(q_1,q_2)$ *is the right hand side of* (3.12) *then* $S(q_1)\neq S(q_2)$.

Final remarks

1) It should not be very hard by using recent results, for example the work of Sh. Agmon (Ann. Scuo. Nor. Pisa, 152–218, Vol. II. 2, 1975) to improve the above results to the case in which q is of order $0(|x|^{-1-\varepsilon})$, $\varepsilon>0$ and q being a "little" negative so that

$$\|\operatorname{grad} u\|_2^2+\|mu\|_2^2+\int q(x)u^2dx+\|u_t\|_2^2$$

is still a norm.

2) Observe that by using (3.1) we can write

$$S(q)u_- - u_- - \int_{-\infty}^{\infty} R(x, t-s)*q(x)u_-(x, s)ds = \int_{-\infty}^{\infty} R(x, t-s)*q(x)Pu(x, s)ds.$$

Therefore, by Lemma 3.1 and Theorem 3.1 it follows that

$$\left|S(q)u_- - u_- - \int_{-\infty}^{\infty} R*qu_-\right|$$
$$\leq const.\ \|u\|_E [\|q\|_\infty^{1/3}\|q\|_1^{1/6} + \|q\|_1^{1/2}\|q\|_\infty^{1/2}][\|q\|_\infty^{1/3}\|q\|_1^{2/3} + \|q\|_1].$$

By choosing $\|u\|_E < 1$ and plotting ε_q ($\varepsilon > 0$) instead of q in the above inequality we get

$$\left|\varepsilon^{-1}(S(\varepsilon q)u_- - u_-) - \int_{-\infty}^{\infty} R*qu_-\right| \leq \varepsilon^{1/2} const. \to 0$$

as $\varepsilon \to 0$. Thus

$$S'(0)u_- - \int_{-\infty}^{\infty} R*qu_- = 0 \tag{3.13}$$

for any incoming free solution u_-. Therefore we may suspect that (3.13) should allow us to determine q pointwise in terms of $S'(0)$.

In a forthcoming work we will treat both of these remarks.

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