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On a Basic Analogue of the Generalised Laguerre Equation

By

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1. Introduction.

In a long series of papers, F. H. Jackson has studied basic analogues of differentiation and integration and various analogues of certain special functions, see, for example, [4]; a complete list of Jackson's papers is given in [1]. Instead of the usual number system, a system of what is referred to as basic number is employed. These numbers are defined by the relation

(1.1) $$[a]=(1-q^{a})/(1-q),$$

where q is any number, real of or complex, called the base.

It will be seen that, corresponding to the sequence of positive integers, 1, 2, $\cdots$, n, $\cdots$, we have the sequence

$$\begin{aligned}&[1]=1\\&[2]=1+q\\&[3]=1+q+q^{2}\\&\cdots\cdots\cdots\cdots\cdots\\&[n]=1+q+q^{2}+\cdots+q^{n-1}\\&\cdots\cdots\cdots\cdots\cdots.\end{aligned}$$

In [4], p. 255, Jackson introduces the operative symbol Δ defined by

(1.2) $$\Delta\{\varphi(x)\}=\frac{\varphi(qx)-\varphi(x)}{x(q-1)},$$

which becomes the same as ordinary differentiation in the limit as q tends to unity. Similarly, he defines basic integration as the inverse of basic differentiation, employing the symbol $\overset{b}{\underset{a}{\mathrm{S}}}$, which reduces in the limit to $\int_a^b$. These operations correspond exactly, in every way, to differentiation and integration. In order to avoid confusion with the usual difference operator Δ, we employ the operator $\hat{B}_{qx}$ to mean basic differentiation, and omit the base and independent variable provided that this does not lead to any possibility of misunderstanding.

We have the following elementary results:

$$\hat{B}x^n = [n]x^{n-1},$$
$$\hat{B}E_q(ax) = aE_q(ax),$$

where

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}, \qquad [n]! = [1][2][3]\cdots[n],$$

and corresponds to the exponential function, and,

$$\hat{B}u(x)v(x) = v(qx)\hat{B}u(x) + u(x)\hat{B}v(x). \tag{1.3}$$

The purpose of this paper is to discuss a basic analogue of the generalised Laguerre equation, and to study some properties of certain of its solutions. It is assumed in what follows that all quantities and functions are real.

2. An analogue of the Laguerre equation.

Consider the equation

$$x\hat{B}^2y + \{[\alpha+1] - q^{\alpha+1}x\}(\hat{B}y)_{qx} + [\lambda]q^{\alpha}y(qx) = 0, \tag{2.1}$$

where

$$\alpha > -1.$$

If $q \to 1$, (2.1) reduces to the generalised Laguerre equation

$$xy'' + \{\alpha+1-x\}y' + \lambda y = 0. \tag{2.2}$$

By means of (1.3), it readily follows that (2.1) may be written in the form

$$\hat{B}\{x^{\alpha+1}E(-x)\hat{B}y\} + [\lambda]q^{\alpha}x^{\alpha}E(-x)y(qx) = 0. \tag{2.3}$$

Suppose that y_m and y_n are solutions of (2.1) corresponding to λ_m and λ_n respectively, then y_m and y_n satisfy

$$\hat{B}\{x^{\alpha+1}E(-x)\hat{B}y_m\} + [\lambda_m]q^{\alpha}x^{\alpha}E(-x)y_m(qx) = 0, \tag{2.4}$$

and

$$\hat{B}\{x^{\alpha+1}E(-x)\hat{B}y_n\} + [\lambda_n]q^{\alpha}x^{\alpha}E(-x)y_n(qx) = 0, \tag{2.5}$$

so that

$$\begin{aligned}([\lambda_m] - [\lambda_n])E(-x)(qx)^{\alpha}y_m(qx)y_n(qx)\\ = y_m(qx)\hat{B}\{x^{\alpha+1}E(-x)\hat{B}y_n\} - y_n(qx)\hat{B}\{q^{\alpha+1}E(-x)\hat{B}y_m\}.\end{aligned} \tag{2.6}$$

Consider the expression

$$\hat{B}\{x^{\alpha+1}E(-x)\hat{B}(y_n)y_m - x^{\alpha+1}E(-x)\hat{B}(y_m)y_n\}, \tag{2.7}$$

which on expansion by means of (1.3) becomes

$$\begin{aligned} y_m(qx)\hat{B}\{x^{\alpha+1}E(-x)\hat{B}(y_n)\} + x^{\alpha+1}E(-x)\hat{B}(y_n)\hat{B}(y_m) \\ - y_n(qx)\hat{B}\{x^{\alpha+1}E(-x)\hat{B}(y_m)\} - x^{\alpha+1}E(-x)\hat{B}(y_m)\hat{B}(y_n). \end{aligned}$$

This last expression is identical with the right-hand member of (2.6), so that if we q-integrate between the limits of 0 and ∞, the result

$$\begin{aligned} \{[\lambda_m]-[\lambda_n]\} \mathop{\mathrm{S}}_0^\infty (qx)^\alpha E(-x) y_m(qx) y_n(qx) d(qx) \\ = \{x^{\alpha+1}E(-x)\hat{B}(y_n)y_m - x^{\alpha+1}E(-x)\hat{B}(y_m)y_n\}_0^\infty, \end{aligned} \tag{2.8}$$

follows immediately.

By hypothesis, $\alpha > -1$, so that ${}^{\alpha+1}E(-x)$ vanishes at both limits, see [5], p. 195, and if λ_m and λ_n are distinct, we have the interesting basic orthogonality relation

$$\mathop{\mathrm{S}}_0^\infty (qx)^\alpha E(-x) y_m(qx) y_n(qx) d(qx) = 0, \qquad m \neq n. \tag{2.9}$$

This result reduces to the corresponding expression for the solutions of the generalised Laguerre equation when $q \to 1$.

3. Series solution of (2.1).

Substitute

$$y = \sum_{r=0}^\infty a_r x^{r+e} \tag{3.1}$$

into (2.1), when we have the indicial equation

$$[e]\{[e-1] + [\alpha+1]q^{e-1}\} = 0. \tag{3.2}$$

Since this implies that

$$1 - q^{e-1} + q^{e-1}(1 - q^{\alpha+1}) = 0,$$

we have the two factors

$$[e] = 0 \quad \text{and} \quad [e+\alpha] = 0,$$

whence the two principal solutions of the indicial equation are

$$e = 0 \quad \text{and} \quad e = -\alpha.$$

If $e=0$, the recurrence relation for the coefficients $\{a_r\}$ is found to be

(3.3) $$[r]\{[r-1]+[\alpha+1]q^{r-1}\}a_r-\{[r-1]-[\lambda]\}q^{\alpha+r-1}a_{r-1}=0,$$

so that we have the solution

(3.4) $$y_1=\sum_{r=0}^{\infty}\frac{\prod_{j=0}^{r-1}\{[j]-[\lambda]\}q^{\alpha r+\frac{1}{2}r(r-1)}x^r}{\prod_{j=0}^{r-1}\{[j]+[\alpha+1]q^j\}[r]!}.$$

Similarly, the solution relative to the exponent $e=-\alpha$ is given by

(3.5) $$y_2=x^{-\alpha}\sum_{r=0}^{\infty}\frac{\prod_{j=0}^{r-1}\{[j-\alpha]-[\lambda]\}q^{\frac{1}{2}r(r-1)}x^r}{\prod_{j=0}^{r-1}\{[j-\alpha]+[\alpha+1]q^{j-\alpha}\}\prod_{j=0}^{r-1}\{[j+1-\alpha]\}}.$$

y_1 and y_2 may be presented more conveniently by observing that

$$[a]-[b]=\frac{1-q^a-1+q^b}{1-q}=q^b[a-b]$$

and

$$[a]+[b]q^a=\frac{1-q^a+q^a-q^{a+b}}{1-q}=[a+b].$$

Hence,

$$y_1=\sum_{r=0}^{\infty}\frac{[-\lambda]_r q^{\lambda r+\frac{1}{2}r(r-1)}}{[1+\alpha]_r}\frac{x^r}{[r]!}$$

and

$$y_2=x^{-\alpha}\sum_{r=0}^{\infty}\frac{[-\alpha-\lambda]_r q^{\lambda r+\frac{1}{2}r(r-1)}}{[1-\alpha]_r}\frac{x^r}{[r]!}$$

where

$$[a]_r=[a][a+1][a+2]\cdots[a+r-1],\qquad [a]_0=1.$$

If $\lambda=n$, a non-negative integer, then y_1 becomes a polynomial of degree n, and is written

(3.6) $$\mathscr{L}_n^{\alpha}(q\,;x)=\sum_{r=0}^{\infty}\frac{[-n]_r q^{\frac{1}{2}r(r-1)+nr}}{[1+\alpha]_r}\frac{x^r}{[r]!}.$$

These polynomials correspond with the generalised Laguerre polynomials, apart from a multiplicative constant independent of x; they are q-orthogonal with respect to the weight function $(qx)^{\alpha}E(-x)$ over the range 0 to ∞.

4. A recurrence formula for $\mathscr{L}_n^{\alpha}(q\,;x)$.

If the polynomial $p_p(x)$ possesses an orthogonality property $(p_m, p_n)=K\delta_{mn}$, where K is a constant and δ_{mn} is the Kronecker delta, then a three-term recurrence relation of the form

$$p_{n+1}(x)=(A_n x+B_n)p_n(x)-C_n p_{n-1}(x) \tag{4.1}$$

must exist, see [2], p. 158, for example.

Sister Celine's technique, [6], p. 233, is now employed to obtain such a recurrence relation for the basic Laguerre polynomial. Let

$$\mathscr{L}_n^{\alpha}(q\,;x)=f_n(x)=\sum_{k=0}^{\infty}\frac{[-n]_k q^{\frac{1}{2}k(k-1)+nk}x^k}{[1+\alpha]_k[k]!}=\sum_{k=0}^{\infty}\epsilon(k,n), \tag{4.2}$$

so that

$$f_{n-1}(x)=\sum_{k=0}^{\infty}\frac{[n-k]}{[n]}\epsilon(k,n), \tag{4.3}$$

$$f_{n-2}(x)=\sum_{k=0}^{\infty}\frac{[n-k-1][n-k]}{[n-1][n]}\epsilon(k,n) \tag{4.4}$$

and

$$xf_{n-1}(x)=-\sum_{k=0}^{\infty}\frac{[k][\alpha+k]}{[n]}q^{2-2k}\epsilon(k,n). \tag{4.5}$$

Hence,

$$\begin{aligned}[n][n-1]=&-A_{n-1}[n-1][k][\alpha+k]q^{2-2k}\\&+B_{n-1}[n-k][n-1]-C_{n-1}[n-k-1][n-k],\end{aligned} \tag{4.6}$$

and if we put k equal to n, $n-1$ and zero, in turn, we have

$$A_{n-1}=-\frac{q^{2n-2}}{[\alpha+n]},$$

$$B_{n-1}=[n]-\frac{[n-1][\alpha+n-1]q^2}{[\alpha+n]}$$

and

$$C_{n-1}=[n]-\frac{[n-1][\alpha+n-1]q^2}{[\alpha+n]}-1,$$

and so, the required recurrence relation is

$$\begin{aligned}\mathscr{L}_{n+1}^{\alpha}(q\,;x)=&\left\{[n+1]-\frac{[n][\alpha+n]q^2}{[\alpha+n+1]}-\frac{q^{2n}x}{[\alpha+n+1]}\right\}\mathscr{L}_n^{\alpha}(q\,;x)\\&+\left\{1-[n+1]+\frac{[n][\alpha+n]q^2}{[\alpha+n-1]}\right\}\mathscr{L}_{n-1}^{\alpha}(q\,;x).\end{aligned} \tag{4.7}$$

5. Expansions in series of basic Laguerre polynomials.

Suppose that the formal expansion

$$(5.1)\qquad f(x)=\sum_{r=0}^{\infty} c_r \mathscr{L}_r^{\alpha}(q\,;\,qx)$$

exists, then, if both sides of (5.1) are multiplied by $(qx)^{\alpha}E(-x)\mathscr{L}_n^{\alpha}(q\,;\,qx)$, basic integration over the range 0 to ∞ gives the expression

$$(5.2)\qquad h_r c_r=\mathop{\mathrm{S}}_0^{\infty} (qx)^{\alpha}E(-x)f(x)\mathscr{L}_r^{\alpha}(q\,;\,qx)d(qx),$$

where

$$(5.3)\qquad h_r=\mathop{\mathrm{S}}_0^{\infty} (qx)^{\alpha}E(-x)\{\mathscr{L}_r^{\alpha}(q\,;\,qx)\}^2 d(qx).$$

If $f(x)=x^m$, $m=0, 1, 2, \cdots$, then the expansion (5.1) exists, and

$$(5.4)\qquad h_r c_r^{(m)}=q^{\alpha} \mathop{\mathrm{S}}_0^{\infty} x^{\alpha+m}E(-x)\mathscr{L}_r^{\alpha}(q\,;\,qx)d(qx).$$

We now expand the basic Laguerre polynomial in the integrand, when

$$(5.5)\qquad h_r c_r^{(m)}=q^{\alpha}\sum_{k=0}^{r}\frac{[-r]_k q^{\frac{1}{2}k(k-1)+rk}}{[1+\alpha]_k[k]!}\mathop{\mathrm{S}}_0^{\infty} x^{\alpha+m+k}E(-x)d(qx),$$

and since the series terminates, the question of convergence does not arise. The inner basic integral is equal to

$$q^{-\frac{1}{2}(\alpha+m+k)(\alpha+m+k+1)}\Gamma_q(\alpha+m+k+1)\qquad \text{if } q<1$$

or to

$$\Gamma_q(\alpha+m+k+1)\qquad \text{if } q>1,$$

where $\Gamma_q(a)$ is the basic gamma function, see [3] and [5]. If $q<1$,

$$\Gamma_q(\alpha+m+k+1)=[\alpha+m+1]_k q^{(\alpha+m)k+\frac{1}{2}k(k-1)}\Gamma_q(\alpha+m+1),$$

and if $q>1$,

$$\Gamma_q(\alpha+m+k+1)=[\alpha+m+1]_k\Gamma_q(\alpha+m+1),$$

so that, in the former case, after some reduction

$$(5.6)\qquad h_r c_r^{(m)}=q^{\alpha-\frac{1}{2}(\alpha+m)^2-\frac{1}{2}(\alpha+m)}\Gamma_q(\alpha+m+1)\,{}_2\Phi_1\left(\begin{matrix}-r,\ \alpha+m+1\,;\\ 1+\alpha\,;\end{matrix}\ q^{\frac{1}{2}(k+1)+r}\right)$$

($q<1$), and in the latter case

$$(5.7) \qquad h_r c_r^{(m)} = q^{\alpha}\Gamma_q(\alpha+m+1)\,{}_2\Phi_1\left(\begin{matrix}-r, \alpha+m+1\,; \\ 1+\alpha\,;\end{matrix}\; q^{\frac{1}{2}(k-1)+r}\right)$$

($q>1$). The basic Gauss function is defined by the relation

$$(5.8) \qquad {}_2\Phi_1\left(\begin{matrix}a, b\,; \\ c\,;\end{matrix}\; x\right) = \sum_{k=0}^{\infty} \frac{[a]_k[c]_k}{[c]_k[k]!}x^k,$$

and in (5.6) and (5.7), the kth terms of the series involve $q^{\frac{1}{2}k(k+1)+kr}$ and $q^{\frac{1}{2}k(k-1)+kr}$ respectively.

The function h_r may be calculated from the recurrence relation (4.7). If k_n is the coefficient of x^n in $\mathscr{L}_n^{\alpha}(q\,;x)$, then it is shown in [2], p. 158 that

$$(5.9) \qquad C_n = \frac{k_{n+1}k_{n-1}}{k_n^2}\cdot\frac{h_n}{h_{n-1}}.$$

From (3.6), we have

$$(5.10) \qquad k_n = \frac{[-n]_n q^{\frac{1}{2}n(n-1)+n^2}}{[1+\alpha]_n[n]!}$$

which gives, after a little manipulation

$$(5.11) \qquad C_n = \frac{[n+\alpha]q^3}{[n+\alpha+1]}\,\frac{h_n}{h_{n-1}} = [n+1] - \frac{[n][\alpha+n]q^2}{[\alpha+n+1]} - 1.$$

Hence,

$$\frac{h_n}{h_{n-1}} = \frac{[n]q^{-2}}{[n+\alpha]},$$

after further manipulation, so that

$$(5.12) \qquad h_n = \frac{[n]!q^{-2n}}{[1+\alpha]_n}h_0$$

where

$$(5.13) \qquad h_0 = \int_0^{\infty} (qx)^{\alpha}E(-x)d(qx) = q^{-\frac{1}{2}\alpha(\alpha-1)}\Gamma_q(\alpha+1), \qquad |q|<1$$
$$= q^{\alpha}\Gamma_q(\alpha+1), \qquad |q|>1\,;$$

the basic gamma function is given by

$$(5.14) \qquad \Gamma_q(\alpha+1) = \lim_{k\to 0} \frac{[k]!}{[\alpha+1][\alpha+2]\cdots[\alpha+k]}[k]^{\alpha}, \qquad |q|<1$$

or by

$$\Gamma_q(\alpha+1)=q^{\frac{1}{2}\alpha(\alpha+1)} \lim_{k\to 0} \frac{[k]!}{[\alpha+1][\alpha+2]\cdots[\alpha+k]}[k]^{\alpha}, \qquad |q|>1, \tag{5.15}$$

see [3], p. 64.

It is evident that if $f(x)$ may be expanded in the form of a convergent series $f(x)=\sum_{r=0}^{\infty} b_r r^r$, then, in this case, the formal expansion (5.1) is certainly valid.

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