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Description of a Class of Hereditary Differential-integral Equations with Non-converging Successive Approximations

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§ 1. Introduction.

Many authors ([2]–[5], [8], [9], [12]) have been discussed the problem of the convergence of successive approximations to a solution of a differential equation, when the latter is unique in virtue of a unique criterion. A well known example due to Müller ([11]) (see also [4], p. 53) shows that the continuity of the right-hand side of a given equation and uniqueness of its solutions are not sufficient to guarantee the convergence of its successive approximations. In [14] it has been proved that "convergence of successive approximations" is a generic property, i.e., that there is a dense second category set M in the space of continuous functions such that for all $f \in M$, the successive approximations converge. This result has a greater generality since it shows that for $f \in M$ the successive approximation have a unique limit for every starting point. Thus it follows as corollary that "uniqueness of solutions" is a generic property, as proved earlier by Orlicz [13] for f's defined on strips $I \times R$ and by Lasota-Yorke [10] for f's defined on open subsets of any Banach space. We show here that the class of all $(\mathscr{S}, f, g)$ for which the successive approximations of the equation

$$(1)\qquad \begin{cases} y(t)=\mathscr{S}(t) & \text{for } t \leq t_0 \\ y'(t)=\displaystyle\int_0^\infty f(t, y(t-s))d_S r(t, s)-g(t) & \text{for a.e. } t \in [t_0, T], \end{cases}$$

where integration is of a Riemann-Stieltjes type with respect to $s \geq 0$, do not converge, is of Baire's first category in any complet metric space. In virtue of Baire's theorem our result generalise the result given in [14].

Let R denote the real line, and let R^n be an n-dimensional linear vector space with the norm $\|x\|=\max(|x_1|, |x_2|, \cdots, |x_n|)$ for $x=(x_1, x_2, \cdots, x_n)$. Let P denote the set in R^n defined by $P=\{(t, y): t_0 \leq t \leq T, y \in R^n\}$ and let $Q=\{(t, y) \in P: \|y-\eta\| \leq a\}$, where $\eta \in R^n$ and $a>0$. Let us denote by G the Banach space of all Lebesgue-integrable functions $g: [t_0, T] \to R^n$ with the norm $\|g\|_G=\int_{t_0}^T \|g(t)\|\, dt$ and let $\overset{\infty}{\underset{s=0}{\vee}} r(t, s)$ denote the variation of $r(t, s)$ with respect to $s \geqslant 0$.

In this paper we shall take into account functions $f: P \to R^n$ satisfying the following Carathéodory hypotheses:

(C_1) $f(\cdot, y): [t_0, T] \to R$ is measurable for all $y \in R^n$,

(C_2) $f(t, \cdot): R^n \to R^n$ is continuous for a.e. $t \in [t_0, T]$,

(C_3) there exists $m_f \in \mathscr{L}^1([t_0, T], R^+)$, where $R^+ = [0, \infty)$, such that $\|f(t, y)\| \leqslant m_f(t)$ for a.e. $t \in [t_0, T]$ and $y \in R^n$.

About a function $r: [t_0, T] \times [0, \infty) \to R$ we shall assume:

(I) $r(t, 0) = 0$ for $t \in [t_0, T]$,

(II) there exists a number $V > 0$ such that $\overset{\infty}{\underset{s=0}{\vee}} r(t, s) \leqslant V$ for $t \in [t_0, T]$,

(III) for every $\varepsilon > 0$ there exists a number $K > 0$ such that $\overset{\infty}{\underset{s=K}{\vee}} r(t, s) < \varepsilon$ for $t \in [t_0, T]$,

(IV) for every $\alpha > 0$ and $u \in [t, T]$ $\lim_{u \to t} \int_0^\alpha |r(t, s) - r(u, s)| \, ds = 0$.

Let us denote by Φ the space of all continuous and bounded functions $\mathscr{S}: (-\infty, t_0] \to R^n$ with the norm $\|\mathscr{S}\|_\Phi = \sup_{t \leqslant t_0} \|\mathscr{S}(t)\|$.

§ 2. Fundamental metric space and basic theorems.

It was proved in [1] the following approximation theorem:

Theorem 1. *If $f: Q \to R^n$ satisfies Carathéodory hypotheses, then there exist continuous functions $f_n: Q \to R^n$ such that*

(i) *$\|f_n(t, y)\| \leqslant m_f(t)$ for a.e. $t \in [t_0, T]$ and $y \in R^n$,*

(ii) *$\lim_{n \to \infty} \max_y \{\|f_n(t, y) - f(t, y)\| : (t, y) \in Q\}$ for almost every $t \in [t_0, T]$.*

Hence it is not difficult to obtain ([6])

Theorem 2. *Suppose $f: Q \to R^n$ satisfies Carathéodory hypotheses. Then for every $\varepsilon > 0$ there exists a continuous function $f^\varepsilon: Q \to R^n$ such that*

(i) *$\|f^\varepsilon(t, y)\| \leqslant m_f(t)$ for a.e. $t \in [t_0, T]$ and all $y \in R^n$ such that $(t, y) \in Q$,*

(ii) *$\lim_{\varepsilon \to 0} \max_y \{\|f^\varepsilon(t, y) - f(t, y)\| : (t, y) \in Q\} = 0$ for a.e. $t \in [t_0, T]$,*

(iii) *$f^\varepsilon(t, y)$ has continuous partial derivatives of all orders with respect to $y_1, y_2, \cdots, y_n$.*

Let us denote by $F(P)$ the set of all functions $f: P \to R$ satisfying Carathéodory hypotheses and let $\tilde{f}$ be the class of all functions of $F(P)$ which are different only on a set of measure zero for fixed $y \in R^n$. Let us denote by $\mathscr{F}(P)$ the set of all classes

$\tilde{f}$. In the next paragraph we shall consider the metric space $(\mathscr{F}(P), \rho_{\mathscr{F}})$ with $\rho_{\mathscr{F}}(\tilde{f}_1, \tilde{f}_2)=\|\tilde{f}_1-\tilde{f}_2\|_{\mathscr{F}}$, where $\|\tilde{f}\|_{\mathscr{F}}=\int_{t_0}^{T}\sup_y\{\|f(t,y)\|: (t,y)\in P\}dt$ for $\tilde{f}, \tilde{f}_1, \tilde{f}_2 \in \mathscr{F}(P)$ and $f\in\tilde{f}$. It was proved in [7] that $(\mathscr{F}(P), \rho_{\mathscr{F}})$ is a complete metric space. Therefore $(\mathscr{H}, \rho)$, where $\mathscr{H}=\Phi\times\mathscr{F}(P)\times G$ and $\rho=\max(\rho_{\Phi}, \rho_{\mathscr{F}}, \rho_G)$ is a complete metric space too.

For a given $(\mathscr{S}, f, g)\in\mathscr{H}$ and a r satisfying (I)–(IV) by a solution of (1) we mean a function $y\in C((-\infty, T], R^n)\cap AC([t_0, T], R^n)$ satisfying (1) for $t\leqslant T$. Throughout $C(I, R^n)$ and $AC(I, R^n)$ denote the set of all continuous and absolutely continuous functions from I to R^n respectively.

It is well known that for every $(\mathscr{S}, f, g)$ and r satisfying (I)–(IV) there exists at least one solution of (1). It is easy to see in this case that every solution y of (1) satisfies $\max_{t_0\leqslant t\leqslant T}\|y(t)\|\leqslant b$ where $b=\|\mathscr{S}\|_{\Phi}+V\cdot\|m_f\|_G+\|g\|_G$.

Now for $h=(\mathscr{S}, \tilde{f}, g)\in\mathscr{H}$ and r satisfying (I)–(IV) let us consider the sequence $\{y_n^h\}$ of successive approximations defined by

$$
(2)\qquad
\begin{aligned}
y_0^h(t)&=\begin{cases}\mathscr{S}(t) & \text{for } t\leqslant t_0\\ \mathscr{S}(t_0) & \text{for } t\in[t_0, T],\end{cases}\\
y_n^h(t)&=\begin{cases}\mathscr{S}(t) & \text{for } t\leqslant t_0\\ \mathscr{S}(t_0)+\int_{t_0}^{t}\left\{\int_0^{\infty} f(u, y_{n-1}^h(u-s))d_S r(u,s)+g(u)\right\}du & \\ \qquad\text{for } t\in[t_0, T] \text{ and } n=1,2,\cdots. \end{cases}
\end{aligned}
$$

In similar way as in the theory of ordinary differential equations it is easy to verify if $\tilde{f}\in\mathscr{F}(P)$ is Lipschitz continuous with respect to $y\in R$, i.e. if there exists a $k\in\mathscr{L}^1([t_0, T], R^+)$ such that $\|f(t, y_1)-f(t, y_2)\|\leqslant k(t)\,\|y_1-y_2\|$ for a.e. $t\in[t_0, T]$ and all $y_1, y_2\in R^n$, then $\{y_n^h\}$ is uniformly convergent to the unique solution of (1). It is easy to see that for every $h=(\mathscr{S}, \tilde{f}, g)$ and $n=0, 1, 2, \cdots$ we have $\sup_{t\leqslant T}\|y_n^h(t)\|\leqslant b$, where b is defined as above.

§ 3. Non-convergence of successive approximations.

Now we shall prove that non-convergence of $\{y_n^h\}$ is in any sense a rare case. Exactly we shall show that the set $\mathscr{A}\subset\mathscr{H}$ of those $h=(\mathscr{S}, \tilde{f}, g)$ for which $\{y_n^h\}$ is not convergent is of Baire's first category in the space $(\mathscr{H}, \rho)$ for fixed r satisfying (I)–(IV).

Suppose r satisfies (I)–(IV). For fixed $h=(\mathscr{S}, \tilde{f}, g)$ and $t\in[t_0, T]$ let $\Delta(h, t)=\limsup_{n\to\infty}\{\operatorname{diam} E[y_n^h(t)]\}$, where $E[y_n^h(t)]=\{y_n^h(t), y_{n+1}^h(t), \cdots\}$ for $n=1, 2, \cdots$ and $\operatorname{diam} A$ denotes the diameter of a set $A\subset R^n$. Obviously $\Delta(h, t)=0$ for each $t\leqslant T$ is equivalent to the convergence of the sequence $\{y_n^h\}$. Then the sequence $\{y_n^h\}$ is not converging in $[t_0, T]$ iff there is a $\tilde{t} \in [t_0, T]$ such that $\Delta(h, \tilde{t})>0$. Let $\{t_\tau\}$ be a sequence of points of $[t_0, T]$ dense in $[t_0, T]$. Then let $\Omega_{MNpq\tau}$ be defined by $\Omega_{MNpq\tau}=\{(\mathscr{S}, \tilde{f}, g) \in \mathscr{H}: \|\mathscr{S}\|_\Phi \leqslant N, \|\tilde{f}\|_{\mathscr{F}} \leqslant M, \|g\|_G \leqslant q, \Delta(h, t_\tau) \geqslant 1/p\}$, where $h=(\mathscr{S}, \tilde{f}, g)$. In the proof of our main result we shall use following lemmas:

Lemma 3. *$\Omega_{MNpq\tau}$ are closed subsets of $\mathscr{H}$ for every $M, N, p, q, \tau=1, 2, \cdots$.*

Proof. Suppose $\{h_r\}$ is a sequence of $\mathscr{H}$ such that $h_r \in \Omega_{MNpq\tau}$ for $r=1, 2, \cdots$ and $\lim_{r\to\infty} \rho(h_r, h)=0$, where $h \in \mathscr{H}$. Let $h_r=(\mathscr{S}_r, \tilde{f}_r, g_r)$ and $h=(\mathscr{S}, \tilde{f}, g)$. It is easy to see that $\|\mathscr{S}\|_\Phi \leqslant N$, $\|f\|_{\mathscr{F}} \leqslant M$ and $\|g\|_G \leqslant q$. Furthermore there exists a subsequence $\{h_k\}$ of $\{h_r\}$ such that $\|\mathscr{S}_k(t)-\mathscr{S}(t)\| \rightrightarrows 0$ for $t \leqslant t_0$, $\|g_k(t)-g(t)\| \to 0$ and $\sup_y \{\|f_k(t, y)-f(t, y)\|: (t, y) \in P\} \to 0$ for a.e. $t \in [t_0, T]$ as $k \to \infty$. For each $k=1, 2, \cdots$ we have $\Delta(h_k, t_\tau) \geqslant 1/p$. Then $\sup_m \{\operatorname{diam} E[y^{h_k}_{n+m}(t_\tau)]\} \geqslant 1/p$ for $n, k=1, 2, \cdots$. Hence it follows that for every $l=1, 2, \cdots$ there exists m_l such that $\operatorname{diam} E[y^{h_k}_{n+m_l}(t_\tau)]>1/p-1/l$, i.e. $\sup_{(\mu,\nu)} \|y^{h_k}_{n+m_l+\mu}(t_\tau)-y^{h_k}_{n+m_l+\nu}(t_\tau)\|>1/p-1/l$. Then for every $j=1, 2, \cdots$ there is (μ_j, ν_j) such that

$$\|y^{h_k}_{n+m_l+\mu_j}(t_\tau)-y^{h_k}_{n+m_l+\nu_j}(t_\tau)\|>1/p-1/l-1/j \tag{3}$$

for every $n, k=1, 2, \cdots$. Let $x(k, n+m_l+\mu_j)=y^{h_k}_{n+m_l+\mu_j}$. It is easy to verify that the family $X \subset C([t_0, T], R^n)$ defined by

$$X=\{x(k, n+m_l+\mu_j)\}_{k,n,l,j=1,2,\cdots}$$

satisfy the hypotheses of Arzela's theorem. Then there is a subsequence $\{x(n_k, n+m_l+\mu_j)\}$ of $\{x(k, n+m_l+\mu_j)\}$ which is uniformly convergent on $[t_0, T]$. Suppose $\lim_{k\to\infty} x(n_k, n+m_l+\mu_j)(t)=x(n+m_l+\mu_j)(t)$ for fixed n, m_l and μ_j. For $t \in [t_0, T]$ we have

$$\begin{aligned} x(n+m_l+\mu_j)(t)-\mathscr{S}(t_0)-\int_{t_0}^t \Big\{\int_0^\infty & f(u, x(n+m_l+\mu_j-1)(u-s))d_S r(u, s) \\ & +g(u)\Big\}du=\sum_{i=1}^5 \Lambda_i(t), \end{aligned} \tag{4}$$

where

$$\Lambda_1(t)=x(n+m_l+\mu_j)(t)-x(n_k, n+m_l+\mu_j)(t),\ \Lambda_2(t)=\mathscr{S}_{n_k}(t_0)-\mathscr{S}(t_0),$$

$$\begin{aligned} \Lambda_3(t)=\int_{t_0}^t \Big\{\int_0^\infty & [f_{n_k}(u, x(n_k, n+m_l+\mu_j-1)(u-s)) \\ & -f(u, x(n_k, n+m_l+\mu_j-1)(u-s))]d_S r(u, s)\Big\}du, \end{aligned}$$

$$\Lambda_4(t)=\int_{t_0}^{t}\left\{\int_0^\infty [f(u,x(n_k,n+m_l+\mu_j-1)(u-s))\right.$$
$$\left.-f(u,x(n+m_l+\mu_j-1)(u-s))]d_S r(u,s)\right\}du,$$
$$\Lambda_5(t)=\int_{t_0}^{t}[g_{n_k}(u)-g(u)]du.$$

It is easy to see that for $t\in[t_0,T]$ we have $\|\Lambda_3(t)\|\leqslant V\|\tilde f_{n_k}-\tilde f\|_{\mathscr{F}}$ and $\|\Lambda_5(t)\|\leqslant\|g_{n_k}-g\|_G$. Let $W_{n_k}(t)=\|f(u,x(n_k,n+m_l+\mu_j-1)(u-s))-f(u,x(u+m_l+\mu_j-1)(u-s))\|$ for fixed n,m_l and μ_j. The functions W_{n_k} are measurable on $[t_0,T]$ and such that $\|W_{n_k}(t)\|\leqslant 2m_f(t)$. Since $\lim W_{n_k}(t)=0$ uniformly with respect to s,n,m_l and μ_j, then $\Lambda_4(t)\to 0$ as $k\to\infty$. Then passing to the limit in (4) as $k\to\infty$ we obtain

$$x(n+m_l+\mu_j)(t)=\mathscr{S}(t_0)+\int_{t_0}^{t}\left\{\int_0^\infty f(u,x(n+m_l+\mu_j-1)(u-s))d_S r(u,s)+g(u)\right\}du$$

for $t\in[t_0,T]$ and $n,l,j=1,2,\cdots$. Obviously for $t\leqslant t_0$ and $n,l,j=1,2,\cdots$ we have $x(n+m_l+\mu_j)(t)=\mathscr{S}(t)$. Therefore $x(n+m_l+\mu_j)=y^h_{n+m_l+\mu_j}$ for $n,l,j=1,2,\cdots$. For $k,n,l,j=1,2,\cdots$ we have $\|x(n_k,n+m_l+\mu_j)(t_\tau)-x(n_k,n+m_l+\nu_j)(t_\tau)\|>1/p-1/l-1/j$. Therefore $\|y^h_{n+m_l+\mu_j}(t_\tau)-y^h_{n+m_l+\nu_j}(t_\tau)\|>1/p-1/l-1/j$ for $n,l,j=1,2,\cdots$. Hence it is not difficult to see that $\Delta(h,t_\tau)\geqslant 1/p$. Then $h\in\Omega_{MNpq\tau}$.

Lemma 4. *$\Omega_{MNpq\tau}$ are non-dense in $\mathscr{H}$ for every $M,N,p,q,\tau=1,2,\cdots$.*

Proof. Suppose $\Omega_{\bar M\bar N\bar p\bar q\bar\tau}$ is dense in a sphere $S_\xi(h_0)$ with a center $h_0=(\mathscr{S}_0,\tilde f_0,g_0)\in\mathscr{H}$ and a radius $\xi>0$. Then $S_\xi(h_0)\subset\bar\Omega_{\bar M\bar N\bar p\bar q\bar\tau}$. Hence and Lemma 3 we obtain $S_\xi(h_0)\subset\Omega_{\bar M\bar N\bar p\bar q\bar\tau}$. Note that for every $h=(\mathscr{S},\tilde f,g)$ and $\eta\in \boldsymbol{R}^n$ there exists a number $a>0$ such that the sequence $\{y^h_n\}$ corresponding to h is just the same as $\{y^{h|Q}_n\}$ corresponding to $h|Q$, where $h|Q=(\mathscr{S},\tilde f|Q,g)$. Here $\tilde f|Q$ denotes the constraction of $\tilde f$ to a set $Q=\{(t,y)\in P:\|y-\eta\|\leqslant a\}$. In virtue of Theorem 2 for $f_0|Q$ and $\delta>0$ there is a function $f^\delta:Q\to R$ such that (i)–(iii) hold. Then $\max_y\{\|f^\delta(t,y)-f_0(t,y)\|:(t,y)\in Q\}<\delta$ for a.e. $t\in[t_0,T]$. Taking $\delta<\xi/(T-t_0)$ we obtain $\|\tilde f^\delta-\tilde f_0\|_{\mathscr{F}}<\xi$. Then $h^\delta=(\mathscr{S},\tilde f^\delta,g)\in S_\xi(h_0)$. But $\Delta(h^\delta,t)=0$ for every $t\in[t_0,T]$. Then $h^\delta\notin\Omega_{\bar M\bar N\bar p\bar q\bar\tau}$. This completes the proof.

Now we can prove the main result of this paper.

Theorem 5. *The set $\mathscr{A}$ of those $h=(\mathscr{S},\tilde f,g)\in\mathscr{H}$ for which succesive approximations $\{y^h_n\}$ are not converging is of Baire's first category in the space $(\mathscr{H},\rho)$.*

Proof. In virtue of Lemma 4 it is enough to show that

$$\mathscr{A}=\bigcup_{M=1}^\infty\bigcup_{N=1}^\infty\bigcup_{p=1}^\infty\bigcup_{q=1}^\infty\bigcup_{\tau=1}^\infty\Omega_{MNpq\tau}.$$

Since $\Omega_{MNpq\tau}\subset\mathscr{A}$ for $M,N,p,q,\tau=1,2,\cdots$ then it remain to show that

$\mathscr{A} \subset \bigcup_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcup_{\tau=1}^{\infty} \Omega_{MNpq\tau}$. Let us observe that $\mathscr{A} = \{h \in \mathscr{H}$: there is $t_h \in [t_0, T]$; $\Delta(h, t_n) > 0\}$. Suppose $h = (\mathscr{S}, \tilde{f}, g) \in \mathscr{A}$. Then there is a positive integer $\tilde{p}$ and a point $\tilde{t}_h \in [t_0, T]$ so that $\Delta(h, \tilde{t}_h) \geqslant 2/\tilde{p}$. In similar way as in the proof of Lemma 3 we can verify that for a given $\tilde{p}$ there is an element $\tilde{t}_\tau$ of the sequence $\{t_\tau\}$ such that $\Delta(h, \tilde{t}_\tau) \geqslant \Delta(h, \tilde{t}_h) - 1/\tilde{p}$. Therefore there exists a positive integer τ such that $\Delta(h, t_{\tilde{\tau}}) \geqslant 1/\tilde{p}$. Obviously we can finde positive integers $\tilde{M}, \tilde{N}$ and $\tilde{q}$ so that $\|\mathscr{S}\|_\phi \leqslant \tilde{N}$, $\|\tilde{f}\|_{\mathscr{F}} \leqslant \tilde{M}$ and $\|g\|_G \leqslant \tilde{q}$. Therefore $h \in \Omega_{\tilde{M}\tilde{N}\tilde{p}\tilde{q}\tilde{\tau}}$. Hence it follows $\mathscr{A} \subset \bigcup_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcup_{\tau=1}^{\infty} \Omega_{MNpq\tau}$.

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