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# Non-continuation of Solutions of Differential Equations

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We prove a general theorem on non-continuation of solutions of differential equations when there is a Liapunov function which fails to be mildly unbounded. The theorem is applied to a variety of second order equations. Necessary and sufficient conditions for continuation of solutions of a certain equation $x''+a(t)h(x')\cdot f(t, x)=0$ are given. Finally, we prove a general continuation theorem for $x''+g(t, x')f(t, x)=0$.

## 1. Introduction.

We consider an equation

$$X'=F(t, X) \qquad ('=d/dt) \tag{1}$$

in which $F: [0, \infty)\times R^{n}\rightarrow R^{n}$ and $F$ is continuous. Thus, for each $(t_0, X_0)$ in $[0, \infty)\times R^{n}$ there is a solution $X(t, t_0, X_0)$ defined on an interval $[t_0, t_0+\beta)$ for some $\beta>0$. Either $\beta=+\infty$ or there is some $T>0$ with the solution defined on $[t_0, T)$ and $\lim_{t\rightarrow T-}|X(t, t_0, X_0)|=+\infty$.

In case $F$ is sufficiently smooth ($F$ locally Lipschitzian) then every solution of (1) can be continued on $[t_0, \infty)$ if and only there is a mildly unbounded Liapunov function for (1) [8; p. 306]. For reference:

**Definition 1.** A function $V: [0, \infty)\times R^{n}\rightarrow[0, \infty)$ is a Liapunov function for (1) if $V(t, X)\geq 0$, $V$ is continuous, locally Lipschitzian in $X$, $V(t, 0)=0$, and

$$V'(t, X)=\limsup_{h\rightarrow 0+}[V(t+h, X+hF(t, X))-V(t, X)]/h\leq 0.$$

**Definition 2.** The function $V$ is mildly unbounded if for every $T>0$, $V(t, X)\rightarrow\infty$ as $|X|\rightarrow\infty$ uniformly in $t$ for $0\leq t\leq T$.

In case $V$ has continuous first partial derivatives, then

$$V'(t, X)=\text{grad } V\cdot F+\partial V/\partial t.$$

For a general $F$ the function $V$ can be constructed only if the solutions of (1) are known. However, for special $F$ many Liapunov functions are known.

The result of Kato and Strauss has been used extensively to prove continuation results, but we are unaware of its use in showing non-continuation.

In this note we prove a general non-continuation result and apply it to three second order equations. We also construct a Liapunov function for the equation

$$x''+f(t,x)g(t,x')=0 \tag{2}$$

which yields continuation of all solutions.

In recent years a number of authors have considered (2) and proved results regarding boundedness, oscillation, and boundary value problems. These results often fall in one of the following three categories:

a) An excessively stringent condition is placed on $f$ and $g$ implying continuation (cf., e.g., [5]).

b) The express assumption is made that all solutions are extendable (cf., e.g., [4]) or the result concludes that all extendable solutions have a certain property.

c) The tacit assumption is made that all solutions are extendable, thereby rendering the results questionable (cf., e.g., [7]).

## 2. Non-continuation.

Under fairly general conditions, one can view the geometry of Liapunov functions as follows. If there is a mildly unbounded Liapunov function for (1), then for each $c>0$ and $t_0\geq 0$, the set of points $X$ satisfying $V(t_0,X)=C$ is a bounded set. The work of Leighton [10] then gives a complete geometry when $t_0$ is kept fixed. If $V$ is a Liapunov function which fails to be mildly unbounded and which is unbounded, then there exist $t_0\geq 0$ and $c_1>0$ for which $V(t_0,X)=c_1$ is satisfied for all $X$ in some unbounded set $\bar{S}$. Suppose that $V'$ is negative and there are unbounded sets $S$ near $\bar{S}$ defined by $V(t_0,X)=c_2$ with $c_2>c_1$. If solutions starting on $S$ are defined in the future, then $V'$ negative suggests that these solutions may approach $\bar{S}$ at $t$ increases. However, if $|F(t,X)|$ is very large near $S$ and if $V'(t,X)$ is bounded away from $-\infty$ near $S$, then one might show that solutions starting on $S$ tend to infinity before reaching $\bar{S}$. That is the idea behind the next result, and it serves as a guide in determining where to look for non-continuable solutions.

We assume that there exist disjoint closed sets $S$ and $\bar{S}$ in $R^n$, a set $M$ in $R^n$, a sequence $\{X_n\}$ in $S$ with $|X_n|\to\infty$ as $n\to\infty$, a sequence $\{t_n\}$ in an interval $(t_0,t_0+\alpha)$ with $t_n\to t_0^+$, a differentiable function $V\colon [t_0,t_0+\alpha]\times M\to[0,\infty)$, and a continuous function $\mu\colon [t_0,t_0+\alpha]\to(-\infty,\infty)$ such that:

(A$_1$) The sequence $\{X_n\}$ is selected such that if $X(t,t_0,X_n)$ is defined on $[t_0,t_n]$, then $X(t_n,t_0,X_n)$ is in $\bar{S}$ with $|X(t_n,t_0,X_n)|\to\infty$ as $n\to\infty$ and $X(t,t_0,X_n)$ is in $M$ for $t_0\leq t\leq t_n$.

(A$_2$) There exists $\delta>0$ and $H>0$ such that if $\{Y_n\}$ is any sequence in $\bar{S}$ having the property that $|Y_n|\to\infty$, if $\{s_n\}$ is any sequence in $[t_0, t_0+\alpha)$ with $s_n\to t_0^+$, if $n$ is sufficiently large, and if $\{X_n\}$ is the sequence in $(A_1)$, then $V(s_n, Y_n)\leq H$ and $V(t_0, X_n)\geq V(s_n, Y_n)+\delta$.

$(A_3)$ If $(t, X)\in[t_0, t_0+\alpha]\times M$, then $V'(t, X)\geq\mu(t)V(t, X)$.

**Theorem 1.** *If* $(A_1)$ *through* $(A_3)$ *hold, then there are solutions of* (1) *with finite escape time.*

*Proof.* Suppose that for each $n$ we have $X(t, t_0, X_n)$ defined on $t_0\leq t\leq t_n$. Then by $(A_1)$ and $(A_3)$ we have

$$V'(t, X(t, t_0, X_n))\geq\mu(t)V(t, X(t, t_0, X_n))$$

on $t_0\leq t\leq t_n$. Thus, from this and $(A_2)$ we have

$$\begin{aligned}V(t_n, X(t_n, t_0, X_n))&\geq V(t_0, X_n)\exp\int_{t_0}^{t_n}\mu(t)dt\\&\geq[V(t_n, X(t_n, t_0, X_n))+\delta]\exp\int_{t_0}^{t_n}\mu(t)dt\end{aligned}$$

where we have taken $Y_n=X(t_n, t_0, X_n)$, $s_n=t_n$, and $n$ large.

Now $n\to\infty$, $t_n\to t_0$, and $\mu$ is continuous so

$$V(t_n, X(t_n, t_0, X_n))\geq[V(t_n, X(t_n, t_0, X_n))+\delta]\exp\int_{t_0}^{t_n}\mu(t)dt$$

is a contradiction for large $n$ since $V(t_n, X(t_n, t_0, X_n))\leq H$. This completes the proof.

*Example* 1. Consider the system form of (2) which is

$$\begin{aligned}x'&=y\\y'&=-f(t, x)g(t, y).\end{aligned}\tag{3}$$

Let $y_0\leq 0$, $x_0>0$ and define

$$\begin{aligned}S&=\{(x, y): x=x_0, y_0\geq y>-\infty\},\\\bar{S}&=\{(x, y): x=0, y_0\geq y>-\infty\},\end{aligned}$$

and

$$M=\{(x, y): 0\leq x\leq x_0, y_0\geq y>-\infty\}.$$

Suppose that there exist $t_0\geq 0$ and $\alpha>0$ such that on $[t_0, t_0+\alpha]\times M$ we have:

a) $f(t, x)>0$ if $x\neq 0$, $g(t, y)>0$,

b) $f, g, \partial f/\partial t$, and $\partial g/\partial t$ continuous,

c) $\partial f/\partial t \geq \mu(t)f(t, x)$ and $\partial g/\partial t \leq -\mu(t)g(t, y)$ for some continuous function $\mu$: $[t_0, t_0+\alpha] \to (-\infty, \infty)$,

d) if $s_n \to t_0^+$ and if $s$ and $n$ are large, then

$$\left| \int_0^{-s} [vdv/g(s_n, v)] - \int_0^{-n} [vdv/g(t_0, v)] \right| < \delta$$

where $2\delta = \int_0^{x_0} f(t_0, s)ds$, and

e) $\int_0^y [sds/g(t_n, s)] \leq H$ for some $H>0$, any sequence $t_n \to t_0^+$, and for all $y \leq y_0$.

Under these conditions, there are solutions of (3) having finite escape time.

To see this, let $X_n = (x_0, -n)$ for $-n < y_0$ and let $V = \int_0^y [sds/g(t, s)] + \int_0^x f(t, s)ds$. We note that a) will imply ($A_1$) since $x' = y \leq 0$, $y' = -f(t, x)g(t, y) \leq 0$ in $M$, assuring the existence of $\{t_n\}$ if solutions are defined in the future; more explicitly, as long as $X(t, t_0, X_n) \overset{\text{def}}{=} (x(t), y(t))$ is defined and in $M$, then we haev $x'(t) = y(t) \leq -n$ so $x(t) \leq x_0 - n(t - t_0)$ which implies that $x(t)$ reaches zero before $t - t_0$ reaches $x_0/n$. Now for ($A_2$), let $Y_n = (0, -u_n)$ where $u_n \to \infty$ and let $n$ be large. Use d) to obtain $\int_0^{-u_n} [vdv/g(s_n, v)] - \int_0^{-n} [sds/g(t_0, s)] \leq \left| \int_0^{-u_n} [vdv/g(s_n, v)] - \int_0^{-n} [sds/g(t_0, s)] \right| < (1/2) \int_0^{x_0} f(t_0, s)ds$. Using the first and last terms, we subtract $\delta$ from both sides and rearrange to obtain $\int_0^{-n} [sds/g(t_0, s)] + \int_0^{x_0} f(t_0, s)ds > \int_0^{-u_n} [vdv/g(s_n, v)] + (1/2)\int_0^{x_0} f(t_0, s)ds$ or $V(t_0, X_n) > V(s_n, Y_n) + \delta$, yielding ($A_2$) as e) supplies the $H$ bound for $V(s_n, Y_n)$. We now obtain ($A_3$) from c) by calculating

$$\begin{aligned} V' = &-\int_0^y [s(\partial g(t, s)/\partial t)/g^2(t, s)]ds \\ &+\int_0^x [\partial f(t, s)/\partial t]ds \geq \mu(t)V(t, X). \end{aligned}$$

A similar presentation could be made in Quadrant II if $\int_0^y [sds/g(t, s)]$ is bounded for $y > 0$.

If $g(t, y)$ can be expressed as the product of a function of $t$ and a function of $y$, then the example is considerably simplified.

*Example* 2. Let $t_0, \alpha, S, \bar{S}$, and $M$ be defined as in Example 1 and suppose that on $[t_0, t_0+\alpha] \times M$ there are positive continuous functions $a(t)$ and $h(y)$ with

$g(t, y)=a(t)h(y)$ where $a(t)$ has a continuous derivative and:

a)′ $f(t, x)>0$ if $x\neq 0$,

b)′ $\partial f/\partial t$ is continuous,

c)′ $\partial f/\partial t\geq \mu(t)f(t, x)$ for some continuous function $\mu$: $[t_0, t_0+\alpha)\to(-\infty, \infty)$,

and

d)′ $\int_0^{-\infty}[vdv/h(v)]$ converges.

Under these conditions, there are solutions of (3) with finite escape time.

We next show that the integral condition d)′ in Example 2 leads to the necessary and sufficient condition for non-continuation.

**Theorem 2.** *Let* $f$: $[0, \infty)\times(-\infty, \infty)\to(-\infty, \infty)$ *with* $xf(t, x)>0$ *if* $x\neq 0$, $g(t, y)=a(t)h(y)$ *where* $a$: $[0, \infty)\to(0, \infty)$, $h$: $(-\infty, \infty)\to(0, \infty)$, $|\partial f/\partial t|\leq b(t)|f(t, x)|$ *where* $b$: $[0, \infty)\to[0, \infty)$, *while* $f, \partial f/\partial t, a', b$, *and* $h$ *are continuous. Then all solutions of* (3) *can be continued for all future time if and only if* $\int_0^y[s/h(s)]ds\to\infty$ *as* $|y|\to\infty$.

*Proof.* The necessity was shown in Example 2 when $\int_0^{-\infty}[s/h(s)]ds<\infty$. If $\int_0^{\infty}[s/h(s)]ds<\infty$, then a similar argument holds in Quadrant II.

Suppose the integrals diverge. Then along solutions of (3) we have that

$$V'=-[a'(t)/a(t)]\int_0^y[s/a(t)h(s)]ds+\int_0^x[\partial f(t, s)/\partial t]ds$$
$$\leq[|a'(t)/a(t)|+|b(t)|]V$$

and so along any solution $X(t)$ of (3) we have

$$V(t, X(t))\leq V(t_0, X(t_0))\exp\int_{t_0}^t[|a'(s)/a(s)|+|b(s)|]ds.$$

Thus, if the solution $X(t)=(x(t), y(t))$ is defined on an interval $[t_0, T)$, then $|y(t)|\leq P$ for some $P$. As $x'=y$, it follows that $x(t)$ is also bounded on $[t_0, T)$. Therefore, it can not be that $x^2(t)+y^2(t)\to\infty$ as $t\to T^-$. This completes the proof.

*Example* 3. We consider a generalized van der Pol equation,

$$x''+m(x)n(x')x'+r(x)=0 \tag{4}$$

or

$$\begin{aligned} x'&=y \\ y'&=-m(x)n(y)y-r(x) \end{aligned} \tag{5}$$

in which $m(0)<0$, $n(y)>0$, and $xr(x)>0$ if $x\neq 0$. If $n(y)=1$, $m(x)=\varepsilon(x^2-1)$, and $r(x)=x$, then this is the van der Pol equation. It is known that the van der Pol equation (with $\varepsilon>0$) has a nontrivial periodic solution to which all nontrivial solutions converge. The introduction of a large function $n(x')$ destroys the behavior. The reader may verify that

$$V(x, y)=\int_0^y [ds/n(s)]+\int_0^x r(s)ds$$

and $M=\{(x, y): -a\leq x\leq 0, y\geq 0\}$ where $m(x)<0$ on $-a\leq x\leq 0$ can be used in Theorem 1 to show finite escape time in case $\int_0^y [ds/h(s)]$ is bounded above for $y>0$.

*Remark.* Before we give our final result it seems desirable to indicate the proper place for our second order results in the literature. To do this, we consider the equation

$$x''+a(t)\alpha(x)\beta(x')=0 \tag{6}$$

with $\beta(y)>0$ and $x\alpha(x)>0$ if $x\neq 0$. If $a(t)$ is positive and differentiable, then we can write $a(t)=b(t)c(t)$ where $b(t)>0$ and decreasing while $c(t)$ is positive and increasing so that for the system form

$$\begin{aligned} x'&=y \\ y'&=-b(t)c(t)\alpha(x)\beta(y) \end{aligned} \tag{7}$$

if $V=[1/c(t)]\int_0^y [s/\beta(s)]ds+b(t)\int_0^x \alpha(s)ds$ then $V'\leq 0$ and one can show that all solutions are continuable if and only if $\int_0^y [s/\beta(s)]ds\to\infty$ as $|y|\to\infty$. If the condition on $a(t)$ fail to hold, then there are essentially three ways in which solutions of (7) can fail to be extendable.

1) If $a(t)$ fails to be of bounded variation, then solutions may oscillate unboundedly (cf. [3] and [6]).

2) If $a(t)$ is continuous and negative at one point, then both $x(t)$ and $x'(t)$ may tend monotonically to infinity in finite time (cf. [2] and [11]).

3) If $\beta(y)$ becomes too large in some sense, then $x(t)$ may approach a finite limit and $|x'(t)|$ may approach infinity in finite time (cf. [1], [9], [11]), as we have already seen.

There seems be no known method to detect the general behavior described in 1). Coffman-Ulrich and Hastings have given examples of such behavior. Possibly a Prüfer transformation could be used to obtain a polar coordinate system for which a Liapunov function could be constructed satisfying $V'(\theta)\geq[V(\theta)]^r$ with $r>1$. Our efforts in that direction have failed.

The author and Grimmer [2] constructed a simple technique for detecting the behavior in 2).

## 3. Continuation.

The work in Example 1 can be used for a continuation result. Consider again

$$\begin{aligned} x' &= y \\ y' &= -f(t,x)g(t,y) \end{aligned} \tag{3}$$

with $f$: $[0,\infty)\times(-\infty,\infty)\to(-\infty,\infty)$, $g$: $[0,\infty)\times(-\infty,\infty)\to(0,\infty)$, $xf(t,x)>0$ if $x\neq 0$, $\partial f/\partial t$ and $\partial[1/g(t,y)]/\partial t$ continuous. We assume that the following holds.

($A_4$) There exist continuous functions $h$: $[0,\infty)\to(0,\infty)$, $k$: $[0,\infty)\to(0,\infty)$, $\phi$: $(-\infty,\infty)\to(0,\infty)$, and $\eta$: $(-\infty,\infty)\to(0,\infty)$ such that

$$\int_0^x [\partial f(t,s)/\partial t]ds\leq h(t)\left[\phi(x)+\int_0^x f(t,s)ds\right],$$

$$\int_0^y \{\partial[s/g(t,s)]/\partial t\}ds\leq h(t)\left[\eta(y)+\int_0^y [s/g(t,s)]ds\right],$$

and

$$k(t)\left[\int_0^y [s/g(t,s)]ds+\int_0^x f(t,s)ds\right]\geq(\phi(x)+\eta(y))$$

if $x^2+y^2$ is large.

**Theorem 3.** *Let* ($A_4$) *hold. If for each* $T>0$, $\int_0^y [s/g(t,s)]ds\to\infty$ as $|y|\to\infty$, *uniformly for* $0\leq t\leq T$, *then all solutions of* (3) *are continuable for all future time.*

*Proof.* Define

$$V(t,x,y)=\left[\int_0^y [s/g(t,s)]ds+\int_0^x f(t,s)ds\right]\exp-\int_0^t h(s)ds$$

so that along solutions of (3) we have

$$\begin{aligned} V'(t,x,y) &\leq h(t)[\phi(x)+\eta(y)]\exp-\int_0^t h(s)ds \\ &\leq h(t)k(t)V(t,s,y) \end{aligned}$$

if $x^2+y^2\geq M$ for some $M$. If there is a solution $(x(t),y(t))$ on an interval $[t_0,T)$

with $x^2(t)+y^2(t)\to\infty$ as $t\to T^-$, then there is a $t_1$ in $[t_0, T)$ with $V'(t, x(t), y(t))\leq h(t)k(t)V(t, x(t), y(t))$ on $[t_1, T)$ so

$$V(t, x(t), y(t))\leq V(t_1, x(t_1), y(t_1)) \exp \int_{t_1}^{T} h(t)k(t)dt.$$

Thus, $V$ is bounded, so $y(t)$ is bounded. But then $x'(t)=y(t)$ so $x(t)$ is bounded on $[t_1, T)$. This contradiction completes the proof.

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