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Linear Functional Differential Equations with L^2 Initial Functions

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1. Introduction.

Our goal is to investigate the stability properties of abstract autonomous linear functional differential equations with L^2 initial functions. By abstract equations we mean the solutions have values in a Hilbert space and, consequently, have applications for partial differential equations with deviating arguments. By equations with L^2 initial functions we mean the solutions are equal to given L^2 functions on a prescribed interval. Our work continues the investigations of [13] where an existence theory for such equations was developed. Our approach employs the theory of semigroups of linear operators and follows closely the development of J. Hale [3] in which a similar analysis was made for functional differential equations with continuous initial functions. Related studies of linear functional differential equations with initial functions in L^2 may be found in J. Hale [5] and R. Vinter [10].

Let H be a complex Hilbert space with norm $|\ |$ and inner-product $(\ ,\ )$. Let $r>0$ and let $\eta: [-r, 0]\rightarrow BL(H, H)$ such that $\eta(-r)=0$, η is of bounded variation on $[-r, 0]$, and $\lim_{\theta\rightarrow -r}\eta(\theta)\neq 0$. Define $\tau: [-r, 0]\rightarrow R$ by $\tau(\theta)=\int_{-r}^{\theta}|d\eta|$, the total variation of η between $-r$ and θ. We will study our problem in the Hilbert space $X=L^2(-r, 0; H; \mu)\times H$ where $d\mu(\theta)=\tau(\theta)d\theta$ and the norm and inner-product of X are given by

$$\|\{\phi, h\}\|=\left(\int_{-r}^{0}|\phi(\theta)|^2\tau(\theta)d\theta+|h|^2\right)^{\frac{1}{2}} \tag{1.1}$$

$$\langle\{\phi, h\}, \{\psi, k\}\rangle=\int_{-r}^{0}(\phi(\theta), \psi(\theta))\tau(\theta)d\theta+(h, k). \tag{1.2}$$

Let $F: C(-r, 0; H)\rightarrow H$ by $F(\phi)=\int_{-r}^{0}d\eta(\theta)\phi(\theta)$. Let $B: H\rightarrow H$ such that B is linear, $D(B)$ is dense, $\operatorname{Re}(Bx, x)\leq\alpha|x|^2$ for all $x\in D(B)$ for some constant $\alpha\in \boldsymbol{R}$, and $R(I-\lambda B)=H$ for all sufficiently small $\lambda>0$. We observe that the conditions on B are equivalent to requiring that B is the infinitesimal generator of a strongly continuous linear semigroup $S(t)$, $t\geq 0$ in H satisfying $|S(t)|\leq e^{\alpha t}$, $t\geq 0$.

We shall study the stability properties of the linear autonomous functional differential equation

$$(1.3)\qquad \begin{aligned} &\dot{x}(\phi, h)(t)=Bx(\phi, h)(t)+Fx_t(\phi, h), \qquad t\geq 0 \\ &x_0(\phi, h)=\phi, \qquad x(\phi, h)(0)=h. \end{aligned}$$

The notation of (1.3) means $\{\phi, h\} \in X$, $x(\phi, h): [-r, \infty)\to H$, and $x_t(\phi, h) \in L^2(-r, 0; H; \mu)$ is defined by $x_t(\phi, h)(\theta)=x(\phi, h)(t+\theta)$ for a.e. $\theta \in [-r, 0]$. The solutions will be studied as a semigroup of bounded linear operators in X. Define the linear operator $A: X\to X$ by

$$(1.4)\qquad \begin{aligned} &D(A)=\{\{\phi, h\} \in X: \phi \text{ is absolutely continuous}, \\ &\qquad \phi' \in L^2(-r, 0; H; \mu),\ h=\phi(0) \in D(B)\}, \\ &A\{\phi, h\}=\{-\phi', -Bh-F\phi\}. \end{aligned}$$

The following proposition is proved in [13], Propositions 4.1 and 4.2.

Proposition 1.1. *$D(A)$ is dense,* $\operatorname{Re}\langle -A\{\phi, h\}, \{\phi, h\}\rangle \leq \gamma \|\{\phi, h\}\|^2$ *for all* $\{\phi, h\} \in D(A)$ *where* $\gamma=\max\{0, \tau(0)+\alpha\}$, *and* $(I+\lambda A)^{-1} \in BL(X, X)$ *for all sufficiently small* $\lambda>0$. *Consequently,* $-A$ *is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators* $T(t)$, $t\geq 0$ *in* X *satisfying* $|T(t)| \leq e^{\gamma t}$, $t\geq 0$.

The connection of the semigroup $T(t)$, $t\geq 0$ to the solutions of (1.3) is treated in [13]. Denote by Π_1, Π_2 the projections $\Pi_1\{\phi, h\}=\phi$, $\Pi_2\{\phi, h\}=h$. In [13], Proposition 5.3, a result by H. Flaschka and M. J. Leitman [3] is used to establish the fact that for $\{\phi, h\} \in X$, $t\geq 0$, $-r\leq\theta\leq 0$,

$$(1.5)\qquad \begin{aligned} &(\Pi_1 T(t)\{\phi, h\})(\theta)=\phi(t+\theta), \qquad t+\theta<0 \\ &(\Pi_1 T(t)\{\phi, h\})(\theta)=\Pi_2 T(t+\theta)\{\phi, h\}, \qquad t+\theta\geq 0. \end{aligned}$$

This fact is then used in [13], Proposition 5.8, to show that the solution of (1.3) is given by

$$(1.6)\qquad x(\phi, h)(t)=\Pi_2 T(t)\{\phi, h\},\ x_t(\phi, h)=\Pi_1 T(t)\{\phi, h\}, \qquad t\geq 0$$

in the sense that $x(\phi, h)(t)$ satisfies (1.3) for a.e. $t\geq 0$ provided $\{\phi, h\} \in D(A)$. For arbitrary $\{\phi, h\} \in X$ we may associate with equation (1.3) the function $x(\phi, h)(t)$ defined by (1.6) and consider $x(\phi, h)(t)$ as being a solution of equation (1.3) in a generalized sense. Because we allow B to be unbounded we cannot expect that $x(\phi, h)(t)$ will be a solution to (1.3) for $\{\phi, h\} \notin D(A)$. In fact the right-hand side of (1.3) makes sense only when $h \in D(B)$ and $\phi \in C(-r, 0; H)$. If B is bounded, however, the function $x(\phi, h)(t)$ defined by (1.6) is a solution of the Stieltjes integral equation

$$x(\phi, h)(t)=h+\int_0^t By(\phi, h)(s)ds+\int_{-r}^0 d\eta(\theta)\left(\int_0^t x(\phi, h)(s+\theta)ds\right) \tag{1.7}$$

for arbitrary $\{\phi, h\} \in X$, $t\geq 0$ ([13], Corollary 5.13).

We remark that the norm of X depends on A through the function τ. The need to introduce the weight function τ arises in order to obtain the accretiveness condition on A in Proposition 1.1. If this weight function is absent, then the solution semigroup $T(t)$, $t\geq 0$ will, in general, satisfy a condition of the form $|T(t)| \leq Me^{\omega t}$ where $M\geq 1$.

2. Properties of the infinitesimal generator.

For each $\lambda \in C$ define the linear operator $\Delta(\lambda): D(B)\to H$ by

$$\Delta(\lambda)h=-Bh+\lambda h-F(e^{\lambda\theta}h), \qquad h \in D(B). \tag{2.1}$$

If $\Delta(\lambda)h=0$ for some $h\neq 0$, we say that λ is a characteristic value. Many of the properties of A can be discerned from the corresponding properties of $\Delta(\lambda)$.

Proposition 2.1. *Let* $\lambda \in C$. *The following hold*:

(2.2) $\{\phi, h\} \in N(A+\lambda I)$ *iff* $\phi(\theta)=e^{\lambda\theta}h$ *and* $\Delta(\lambda)h=0$.

(2.3) λ *is an eigenvalue of* $-A$ *iff* $\Delta(\lambda)h=0$ *for some* $h\neq 0$.

(2.4) $(A+\lambda I)^{-1} \in BL(X,X)$ *iff* $\Delta(\lambda)^{-1} \in BL(H,H)$.

(2.5) *If H is finite dimensional and* $(A+\lambda I)^{-1} \in BL(X,X)$, *then* $(A+\lambda I)^{-1}$ *is compact.*

(2.6) $(A+\lambda I)^{-1}$ *has a pole of order n at* λ_0 *iff* $\Delta(\lambda)^{-1}$ *has a pole of order n at* λ_0.

Proof. $(A+\lambda I)\{\phi, h\}=\{\psi, k\}$ iff $-\phi'+\lambda\phi=\psi$ and $-Bh-F\phi+\lambda h=k$ iff $\phi(\theta) =e^{\lambda\theta}h-\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi$ and $-B+\lambda h-\int_{-r}^0 e^{\lambda\theta}d\eta(\theta)h+\int_{-r}^0\int_0^\theta e^{\lambda(\theta-\xi)}d\eta(\theta)\psi(\xi)d\xi=k$. Define $\gamma: C\times X\to H$ by

$$\gamma(\lambda, \{\psi, k\})\overset{\text{def}}{=}k-\int_{-r}^0\int_0^\theta e^{\lambda(\theta-\xi)}d\eta(\theta)\psi(\xi)d\xi. \tag{2.7}$$

Then,

$$\begin{aligned} &(A+\lambda I)\{\phi, h\}=\{\psi, k\} \\ &\text{iff } \phi(\theta)=e^{\lambda\theta}h-\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi \text{ and } \Delta(\lambda)h=\gamma(\lambda, \{\psi, k\}), \end{aligned} \tag{2.8}$$

and (2.2) and (2.3) follow immediately. Also, (2.8) implies that $A+\lambda I$ is one to one and onto iff $\Delta(\lambda)$ is and, consequently,

$$\begin{aligned}(A+\lambda I)^{-1}\{\psi, k\} \\ =\left\{e^{\lambda\theta}\Delta(\lambda)^{-1}\gamma(\lambda,\{\psi,k\})-\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi,\ \Delta(\lambda)^{-1}\gamma(\lambda,\{\psi,k\})\right\}.\end{aligned}\tag{2.9}$$

Then, (2.4) follows observing that both A and $\Delta(\lambda)$ are closed so that we may apply the closed graph theorem. Also, (2.5) follows observing that $\gamma: X\to H$ is continuous and $\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi: L^2(-r, 0; H; \mu)\to L^2(-r, 0; H; \mu)$ is compact provided H is finite dimensional. Finally, we prove (2.6). First, consider $\Pi_1(A+\lambda I)^{-1}: \mathbf{C}\to BL(X, L^2)$ where $L^2=L^2(-r, 0; H; \mu)$. We write $\Pi_1(A+\lambda I)^{-1}=k(\lambda)\Delta(\lambda)^{-1}g(\lambda)-f(\lambda)$, where $f(\lambda): \mathbf{C}\to BL(L^2, L^2)$ by $f(\lambda)\psi=\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi$, $g(\lambda): \mathbf{C}\to BL(X, H)$ by $g(\lambda)\{\psi, k\}=\gamma(\lambda,\{\psi, k\})$, and $k(\lambda): \mathbf{C}\to BL(H, L^2)$ by $k(\lambda)h=e^{\lambda\theta}h$. Then, $f(\lambda)=\sum_{n=0}^\infty(\lambda-\lambda_0)^n a_n$ where $a_n\in BL(L^2, L^2)$ and $a_0\neq 0$. The same claim can be made for $g(\lambda)$ and $k(\lambda)$ and, consequently, $\Pi_1(A+\lambda I)^{-1}$ has a pole at λ_0 of order m if $\Delta(\lambda)^{-1}$ does. A similar argument can be made for $\Pi_2(A+\lambda I)^{-1}$ and the converse implication, thus completing the proof.

In analyzing the stability properties of (1.3) the adjoint of A plays an important role and one advantage of studying the equation in X is that the adjoint theory is very simple and natural.

Proposition 2.2. *The adjoint of A is given by*

$$\begin{aligned}D(A^*)=\{\{\psi, k\}: \tau\psi-\eta^*k \text{ is absolutely continuous, } (1/\tau)(\tau\psi-\eta^*k)'\in \\ L^2(-r, 0; H; \mu), \text{ and } k\in D(B^*)\} \\ A^*\{\psi, k\}=\{(1/\tau)(\tau\psi-\eta^*k)',\ -B^*k-\tau(0)\psi(0)\}.\end{aligned}\tag{2.10}$$

Proof. Suppose $\{\psi, k\}$, $\{\alpha, j\}\in X$ such that $\langle A\{\phi, h\}, \{\psi, k\}\rangle=\langle\{\phi, h\}, \{\alpha, j\}\rangle$ for all $\{\phi, h\}\in D(A)$. Then, for all $\{\phi, h\}\in D(A)$

$$-\int_{-r}^0(\phi', \psi)\tau-\left(Bh+\int_{-r}^0 d\eta\phi, k\right)=\int_{-r}^0(\phi, \alpha)\tau+(h, j).\tag{2.11}$$

First, define $\beta(\theta)=\int_{-r}^\theta\tau\alpha$ so that

$$\int_{-r}^0(\phi, \alpha)\tau=\int_{-r}^0(\phi, \tau\alpha)=\int_{-r}^0(\phi, \beta')=-\int_{-r}^0(\phi', \beta)+(h, \beta(0)).$$

Next,

$$\left(\int_{-r}^0 d\eta\phi, k\right)=\int_{-r}^0(\phi, d\eta^*k)=-\int_{-r}^0(\phi', \eta^*k)+(\phi(0), \eta^*(0)k),$$

where we have used Lemma 4.1 of [13] and the fact that $\eta(-r)=0$. Then, (2.11) implies that for all $\{\phi, h\}\in D(A)$

$$(2.12)\qquad \int_{-r}^{0} (\phi', -\tau\psi+\eta^*k+\beta)-(Bh, k)=(h, \eta^*(0)k+\beta(0)+j).$$

Since for $h \in D(B)$ and ϕ identically h, $\{\phi, h\} \in D(A)$ and ϕ' is identically 0, (2.12) implies that $k \in D(B^*)$ and $j=-B^*k-\eta^*(0)k-\beta(0)$. Since ϕ' are dense in $L^2(-r, 0; H)$ for $\{\phi, h\} \in D(A)$, (2.12) implies that $\beta=\tau\psi-\eta^*k$. Thus, $\int_{-r}^{\theta} \tau\alpha= \tau(\theta)\psi(\theta)-\eta^*(\theta)k$, $\tau\psi-\eta^*k$ is absolutely continuous, $\tau\alpha=(\tau\psi-\eta^*k)'$, and $\alpha= (1/\tau)(\tau\psi-\eta^*k)' \in L^2(-r, 0; H; \mu)$. Also, $\beta(0)=\int_{-r}^{0} \tau\alpha=(\tau(0)\psi(0)-\eta^*(0)k)- (\tau(-r)\psi(-r)-\eta^*(-r)k)$. Recalling that $0=\eta(-r)=\eta^*(-r)=\tau(-r)$, we have that $j=-B^*k-\tau(0)\psi(0)$ and the proof is complete.

3. Decomposition of X with the spectrum of $-A$.

By using the spectrum of $-A$ we may decompose X into subspaces in which the solutions of (1.3) exhibit different stability properties. One advantage of using the space X is that this decomposition has a simple and familiar structure. We collect some well-known facts into the following lemma.

Lemma 3.1. *Let $T(t)$, $t \geq 0$ be a strongly continuous semigroup of bounded linear operators in a Banach space X with infinitesimal generator $-A$. Suppose $\Lambda=\{\lambda_1, \cdots, \lambda_k\}$ is a set of isolated points of the spectrum $\sigma(-A)$ of $-A$. The following hold:*

(3.1) *For $1 \leq j \leq k$ define $P_j=(1/2\Pi i)\int_{\Gamma} (A+\lambda I)^{-1}d\lambda$, where Γ is a closed curve in $\mathbf{C}$ enclosing λ_j but no other point of $\sigma(-A)$, and define $M_j= P_jX$. Then, $P_j \in BL(X, X)$, $P_iP_j=\delta_{ij}P_j$, and $-A$ restricted to M_j is bounded with spectrum consisting of the single point λ_j.*

(3.2) *Define $P_0=I-\sum_{j=1}^{k} P_j$ and $M_0=P_0X$. Then, $-A$ restricted to M_0 has spectrum $\sigma(-A)-\Lambda, A$ commutes with P_j and leaves M_j invariant for $0 \leq j \leq k$, and $X=M \oplus M_0$, where $M=M_1 \oplus \cdots \oplus M_k$ and $\oplus$ means direct sum as in* [7], p. 4 *(but the decomposition is not necessarily orthogonal).*

(3.3) *$T(t)$, $t \geq 0$ commutes with P_j and leaves M_j invariant for $0 \leq j \leq k$.*

(3.4) *If λ_j is a pole of $(A+\lambda I)^{-1}$ of order m, then $M_j=N(A+\lambda_j I)^m$ and $R(I-P_j)=R(A+\lambda_j I)^m$.*

(3.5) *$X=M_1^* \oplus \cdots \oplus M_k^* \oplus M_0^*$, where $M_j^*=P_j^*X$ and* $\dim M_j=\dim M_j^*, 0 \leq j \leq k$.

(3.6) *If λ_j is a pole of $(A+\lambda I)^{-1}$ of order m, then $\bar{\lambda}_j$ is a pole of $(A^*+\lambda I)^{-1}$ of order m and $M_j^*=N(A^*+\bar{\lambda}_j I)^m$.*

(3.7) $M_0=(M_1^*\oplus\cdots\oplus M_k^*)^\perp$, *where* $\perp$ *means orthogonal complement.*

Proof. (3.1) and (3.2) are proved in [7], pp. 178–181. Since $T(t)$ commutes with $(A+\lambda I)^{-1}$, the definition of P_j implies that $T(t)$ commutes with P_j and, consequently, leaves M_j invariant, thus proving (3.3). (3.4) is proved by Theorem 3, p. 229 of [14]. (3.5) is proved in [7], p. 184. If λ_j is a pole of order m of $(A+\lambda I)^{-1}$, then $\bar{\lambda}_j$ is a pole of order m of $(A^*+\lambda I)^{-1}$ by virtue of formulas (6.32), p. 180 and (6.54), p. 184 of [7]. Moreover, $M_j^*=R(P_j^*)=N(A^*+\bar{\lambda}_j I)^m$ by virtue of formula (6.52), p. 184 of [7] and Theorem 3, p. 229 of [14], thus proving (3.6). Finally, (3.7) follows from $M_0=R(P_0)=R(I-\sum_{j=1}^k P_j)=N(\sum_{j=1}^k P_j)$ (since $P_iP_j=\delta_{ij}P_i$) $=(R(\sum_{j=1}^k P_j^*))^\perp$ (see [7], p. 267) $=(\sum_{j=1}^k R(P_j^*))^\perp$ (since $P_i^*P_j^*=\delta_{ij}P_i^*$) $=(\sum_{j=1}^k R(P_j^*)\oplus)^\perp$ (since $P_i^*P_j^*=\delta_{ij}P_i^*$).

Proposition 3.2. *Suppose* $\Lambda=\{\lambda_1,\cdots,\lambda_k\}$ *is a set of isolated points of* $\sigma(-A)$ *and the algebraic eigenspaces* M_j, $1\leq j\leq k$ *are finite dimensional. The following hold:*

(3.8) $-A$ *restricted to* M_j *has* λ_j *as an eigenvalue with no other spectrum,* λ_j *is an eigenvalue of* $-A$, *and* λ_j *is a pole of* $(A+\lambda I)^{-1}$.

(3.9) *There is a row vector* U_j, *whose elements are a basis for* M_j, *so that* $M_j=\{\{\phi,h\}\in X:\{\phi,h\}=U_jb$ *where* b *is a column vector of the same dimension as* $U_j\}$ *and there exists a square matrix* A_j *of the same dimension as* U_j *such that* $AU_jb=U_jA_jb$ *for all column vectors* b *of the same dimension as* U_j.

(3.10) *For* $U_jb\in M_j$, $T(t)U_jb=U_je^{-A_jt}b$, $t\geq 0$, $(\Pi_1T(t)U_jb)(\theta)=\Pi_2U_je^{-A_j(t+\theta)}b$, $-r\leq\theta\leq 0$, $t\geq 0$, *and* $T(t)$ *can be defined for all* $t\in(-\infty,\infty)$ *on* M_j.

(3.11) *Let* $U_\Lambda=\mathrm{row}\,(U_1,\cdots,U_k)$ *and let* $A_\Lambda=\mathrm{diag}\,(A_1,\cdots,A_k)$. *Then* $M=\{\{\phi,h\}\in X:\{\phi,h\}=U_\Lambda b$ *where* b *is a column vector of the same dimension as* $U_\Lambda\}$. *If* $U_\Lambda b\in M$, *then* $T(t)U_\Lambda b=U_\Lambda e^{-A_\Lambda t}b$, $t\geq 0$, $(\Pi_1T(t)U_\Lambda b)(\theta)=\Pi_2U_\Lambda e^{-A_\Lambda(t+\theta)}b$, $-r\leq\theta\leq 0$, $t\geq 0$, *and* $T(t)$ *can be defined for all* $t\in(-\infty,\infty)$ *on* M.

(3.12) *There exists a column vector* V_j, *whose elements are a basis for* M_j^*, *and if* $V_\Lambda=\mathrm{col}\,(V_1,\cdots,V_k)$, *then* $M_0=\{\{\phi,h\}\in X:\langle V_\Lambda,\{\phi,h\}\rangle=0\}$.

(3.13) *Let* $P=\sum_{j=1}^k P_j$. *If* $\{\phi,h\}\in X$, *then* $P\{\phi,h\}=U_\Lambda b$, *where* b *is the column vector given by* $\langle V_\Lambda,U_\Lambda\rangle^{-1}\langle V_\Lambda,\{\phi,h\}\rangle$.

Proof. (3.8) is proved in [7], p. 181. (3.9) follows directly from Lemma 3.1. To establish (3.10) we first observe that $T(t)U_jb$ satisfies uniquely the equation

$(d/dt)T(t)U_jb=-AT(t)U_jb$, $t\geq 0$, $T(0)U_jb=U_jb$. Since $(d/dt)U_je^{-A_jt}b=-U_jA_je^{-A_jt}b$, $-AU_je^{-A_jt}b=-U_jA_je^{-A_jt}b$, and $U_je^{-A_j0}b=U_jb$, we conclude that $T(t)U_jb=U_je^{-A_jt}b$, $t\geq 0$. The definition of A in (1.4) and the fact that $AU_jb=U_jA_jb$ mean that $-(d/d\theta)(\Pi_1U_jb)(\theta)=(\Pi_1U_jA_jb)(\theta)$, $-r\leq\theta\leq 0$, and $(\Pi_1U_jb)(0)=\Pi_2U_jb$. By uniqueness of solutions to this differential equation we must have $(\Pi_1U_jb)(\theta)=\Pi_2U_je^{-A_j\theta}b$, $-r\leq\theta\leq 0$. Since we know from the above $(\Pi_1T(t)U_jb)(\theta)=(\Pi_1U_je^{-A_jt}b)(\theta)$, $-r\leq\theta\leq 0$, $t\geq 0$, we conclude that $(\Pi_1T(t)U_jb)(\theta)=\Pi_2U_je^{-A_j\theta}\cdot e^{-A_jt}b$, $-r\leq\theta\leq 0$, $t\geq 0$, thus proving (3.10). (3.11) follows from Lemma 3.1 and (3.9) and (3.10). (3.12) follows from Lemma 3.1. Finally we prove (3.13). For $\{\phi,h\}\in X$ there exists a column vector b such that $P\{\phi,h\}=U_\Lambda b$. Then, $\langle V_\Lambda,\{\phi,h\}\rangle=\langle V_\Lambda,P\{\phi,h\}+P_0\{\phi,h\}\rangle=\langle V_\Lambda,U_\Lambda b\rangle=\langle V_\Lambda,U_\Lambda\rangle b$. It remains to show $\langle V_\Lambda,U_\Lambda\rangle$ is invertible. Suppose there is some column vector c such that $\langle V_\Lambda,U_\Lambda\rangle c=\langle V_\Lambda,U_\Lambda c\rangle=0$. Then, $U_\Lambda c\in M\cap M_0$, which means $U_\Lambda c=0$. Since U_Λ is a basis, $c=0$ and this proves (3.13).

In order to determine the stability properties of the subspace M_0 in the decomposition of Proposition 3.2 we need to impose the additional requirement that $T(t)$ is compact for some $t>0$. We first state two lemmas, the first of which is proved in [4], Lemma 22.2, p. 112, and the second of which is proved in [8], Proposition 4.7 (see also Corollary 1, p. 241 of [14]).

Lemma 3.3. *Let $T(t)$, $t\geq 0$ be a strongly continuous semigroup of bounded linear operators in a Banach space X and for some $t_1>0$ let $\rho=$spectral radius of $T(t_1)$, where $\rho\neq 0$. For any $\alpha>0$ there is a constant $K(\alpha)\geq 1$ such that*

$$|T(t)|\leq K(\alpha)e^{(\beta+\alpha)t},\quad t\geq 0\qquad \textit{where } \beta=(\log\rho)/t_1. \tag{3.14}$$

Lemma 3.4. *Let $T(t)$, $t\geq 0$ be a strongly continuous semigroup of bounded linear operators in a Banach space X with infinitesimal generator $-A$ and suppose $T(t)$ is compact for some $t>0$. If $\beta\in\boldsymbol{R}$, then $\Lambda=\{\lambda\in p\sigma(-A):\mathrm{Re}\,\lambda\geq\beta\}$ is finite, where $p\sigma(-A)$ means the point spectrum of $-A$.*

Proposition 3.5. *Suppose $\Lambda=\{\lambda_1,\cdots,\lambda_k\}$ is a set of isolated points of $\sigma(-A)$ and the algebraic eigenspaces M_j, $1\leq j\leq k$ are finite dimensional. Suppose also $T(t)$ is compact for some $t>0$. Let $X=M\oplus M_0$ as in Proposition 3.2, let $\beta=\inf\{\mathrm{Re}\,\lambda:\lambda\in\Lambda\}$, and let $\beta_0=\sup\{\mathrm{Re}\,\lambda:\lambda\in p\sigma(-A)-\Lambda\}$. For each $\alpha>0$ there exists $K(\alpha)\geq 1$ such that for all $\{\phi,h\}\in X$*

$$\|PT(t)\{\phi,h\}\|\leq K(\alpha)e^{(\beta-\alpha)t}\|P\{\phi,h\}\|,\qquad t\leq 0 \tag{3.15}$$

$$\|P_0T(t)\{\phi,h\}\|\leq K(\alpha)e^{(\beta_0+\alpha)t}\|P_0\{\phi,h\}\|,\qquad t\geq 0. \tag{3.16}$$

Proof. To prove (3.15) recall that $T(-t)U_\Lambda b=U_\Lambda e^{A_\Lambda t}b$, $t\geq 0$, for $U_\Lambda b\in M$ by

(3.11). By (3.8) $\sigma(A_\Lambda)$ consists of the eigenvalues $-\lambda_1, \cdots, -\lambda_k$, whose real parts are bounded above by $-\beta$. By Corollary 1, p. 227, [14] $\sigma(e^{A_\Lambda t}) = e^{t\sigma(A_\Lambda)}$. Then Lemma 3.3 yields (3.15). To prove (3.16) we first observe that $T(t_1)$ is compact for some $t_1 > 0$ and thus $p\sigma(T(t_1)) = \sigma(T(t_1))$ except possibly for $\{0\}$ (see Theorem 6.26, p. 185, [7]). By Theorem 16.7.2, p. 467, [6], $e^{t_1 p\sigma(-A_0)} = p\sigma(T(t_1)P_0)$ plus possibly $\{0\}$ (where A_0 is the restriction of A to M_0) in the sense that if $\mu \neq 0 \in p\sigma(T(t_1)P_0)$ then there exists $\lambda \in p\sigma(-A_0)$ such that $e^{t_1\lambda} = \mu$. But such a λ belongs to $p\sigma(-A) - \Lambda$ by (3.2) and thus has real part bounded above by β_0. Then, (3.16) follows by Lemma 3.3 and the proof is complete.

Remark 3.6. The development of this section is somewhat complicated because we allow for H to be infinite dimensional. In the case that H is finite dimensional, however, Propositions 3.2 and 3.5 are always applicable. By (2.5) $-A$ has compact resolvent and therefore $\sigma(-A)$ consists entirely of isolated eigenvalues with corresponding algebraic eigenspaces finite dimensional (see Theorem 6.29, p. 187, [7]). In addition H finite dimensional implies that $T(t)$ is compact for $t \geq 2r$ (see Proposition 5.6 of [13] and Lemma 19.1 (iii) of [4]). There is another useful case in which the compactness condition of Proposition 3.4 is satisfied. That is, $T(t)$ is compact for $t > 2r$ if the semigroup $S(t)$, $t \geq 0$ with infinitesimal generator B satisfies $S(t)$ is compact for $t > 0$ (see Proposition 5.6 of [13] and Proposition 2.4 of [8]).

Remark 3.7. In [4] the idea of an adjoint equation to (1.3) (considered in $C(-r, 0; R^n)$) is exploited by using a certain bilinear form. In [4], however, the infinitesimal generator associated with the solution semigroup of the adjoint equation is not the operator adjoint of the infinitesimal generator of the original solution semigroup. In our setting in X we obtain the adjoint semigroup $T^*(t)$, $t \geq 0$ with infinitesimal generator $-A^*$ from the general theory of semigroups in Hilbert space (see [1], Section 1.4). In general, however, we do not know of a functional differential equation analogous to (1.3) corresponding to the adjoint semigroup $T^*(t)$, $t \geq 0$, and the existence of such an equation remains an interesting unanswered problem.

4. The decomposition for the nonhomogeneous equation.

In this section we investigate the decomposition of Section 3 in the variation of constants formula for the nonhomogeneous equation

$$(4.1) \qquad \dot{y}(t) = By(t) + Fy_t + f(t), \qquad t \geq 0,\ y_0 = \phi,\ y(0) = h.$$

If $f \in C^1(0, \infty; H)$, then $\{0, f\} \in C^1(0, \infty; X)$, and the solution of

$$(4.2) \qquad (d/dt)u(t) = -Au(t) + \{0, f(t)\}, \quad t \geq 0, \quad u(0) = \{\phi, h\} \in D(A)$$

is given uniquely by

$$u(t)=T(t)\{\phi,h\}+\int_0^t T(t-s)\{0,f(s)\}ds, \qquad t\geq 0 \tag{4.3}$$

(see [7], Theorem 1.19, p. 486). We claim that the solution of (4.1) is given uniquely by $y(t)\overset{\mathrm{def}}{=}\Pi_2 u(t)$. To justify this claim it suffices to apply Π_2 to both sides of (4.2), invoke the definition of A, and establish the fact that for $\{\phi,h\}\in X$, $t\geq 0$, $-r\leq\theta\leq 0$,

$$\begin{aligned}(\Pi_1 u(t))(\theta)&=\phi(t+\theta) && \text{if } t+\theta\leq 0\\ (\Pi_1 u(t))(\theta)&=\Pi_2 u(t+\theta) && \text{if } t+\theta\geq 0.\end{aligned} \tag{4.4}$$

But (4.4) follows from (4.3) and (1.5) with the observation that $t-s+\theta\leq 0$ implies $(\Pi_1 T(t-s)\{0,f(s)\})(\theta)=0$. More generally, we investigate (4.1) by means of the variation of constants formula (4.3) where $f\in L^1(0,\infty;H)$ and $\{\phi,h\}\in X$.

Proposition 4.1. *Suppose* $\Lambda=\{\lambda_1,\cdots,\lambda_k\}$ *is a set of isolated points of* $\sigma(-A)$ *and the algebraic eigenspaces* M_j, $1\leq j\leq k$ *are finite dimensional. Suppose* $f\in L^1(0,\infty;H)$ *and* $\{\phi,h\}\in X$. *Then* $u(t)\colon[0,\infty)\to X$ *satisfies* (4.3) *iff*

$$P_0u(t)=T(t)P_0\{\phi,h\}+\int_0^t T(t-s)P_0\{0,f(s)\}ds, \qquad t\geq 0, \tag{4.5}$$

and

$$Pu(t)=U_\Lambda z(t) \tag{4.6}$$

where

$$\begin{aligned}\dot z(t)&=A_\Lambda z(t)+\langle V_\Lambda,U_\Lambda\rangle^{-1}\langle V_\Lambda,\{0,f(t)\}\rangle, \qquad t\geq 0,\\ z(0)&=\langle V_\Lambda,U_\Lambda\rangle^{-1}\langle V_\Lambda,\{\phi,h\}\rangle.\end{aligned}$$

Proof. Set $J=\langle V_\Lambda,U_\Lambda\rangle^{-1}$, $K=\langle V_\Lambda,\{\phi,h\}\rangle$, $L(t)=\langle V_\Lambda,\{0,f(t)\}\rangle$. From (3.12) and (3.13)

$$\dot z(t)=A_\Lambda z(t)+JL(t), \qquad t\geq 0,\ z(0)=JK$$

iff for $t\geq 0$

$$\begin{aligned}z(t)&=e^{A_\Lambda t}JK+\int_0^t e^{A_\Lambda(t-s)}JL(s)ds\\ &=J\left\langle V_\Lambda, Ue^{A_\Lambda t}JK+\int_0^t U_\Lambda e^{A_\Lambda(t-s)}JL(s)\,ds\right\rangle\\ &=J\left\langle V_\Lambda, T(t)\{\phi,h\}+\int_0^t T(t-s)\{0,f(s)\}ds\right\rangle.\end{aligned}$$

The claimed equivalence follows immediately.

The framework we have developed can be utilized in the stability analysis of forced linear systems and the existence of saddle point properties for nonlinear equations in a manner very similar to that in [4], Chapters 25 and 26. As another illustration of how one uses the decomposition of X we investigate the asymptotic equivalence of the homogeneous and nonhomogeneous equations $\dot{x}(t)=Bx(t)+Fx_t$ and $\dot{y}(t)=By(t)+Fy_t+f(t)$.

Proposition 4.2. *Suppose* $\Lambda=\{\lambda \in \sigma(-A) : \operatorname{Re}\lambda>0\}$ *is finite and the algebraic eigenspace* M_j *corresponding to each* $\lambda_j \in \Lambda$ *is finite dimensional. Suppose* $T(t)$ *is compact for some* $t>0$ *and there exist no* $\lambda \in \sigma(-A)$ *such that* $\operatorname{Re}\lambda=0$. *Suppose also* $f \in L^1(0, \infty ; H)$ *and*

$$\lim_{t\to\infty} |f(t)|=0 \quad \textit{and} \quad \int_0^\infty e^{\xi s}|f(s)|\,ds<\infty \quad \textit{for some } \xi>0. \tag{4.7}$$

If $\{\phi, h\} \in X$ *there exists* $\{\psi, k\} \in X$ *such that for* $u(t)\overset{\text{def}}{=}U(t)\{\phi, h\}$ *as in* (4.3), *then*

$$\lim_{t\to\infty} \|U(t)\{\phi, h\}-T(t)\{\psi, k\}\|=0. \tag{4.8}$$

Moreover, the projection of $\{\psi, k\}$ *into* M *is unique.*

Proof. By Lemma 3.4 we can choose $\beta>0$ and $\beta_0<0$ in Proposition 3.5, and thus there exist $0<\alpha<\xi$ and $K\geq 1$ such that

$$\|PT(t)\{\phi, h\}\|\leq Ke^{\alpha t}\|P\{\phi, h\}\|, \qquad \{\phi, h\} \in X,\ t\leq 0, \tag{4.9}$$

$$\|P_0T(t)\{\phi, h\}\|\leq Ke^{-\alpha t}\|P_0\{\phi, h\}\|, \qquad \{\phi, h\} \in X,\ t\geq 0. \tag{4.10}$$

Observe that

$$\begin{aligned} U(t)\{\phi, h\}&=T(t)P\{\phi, h\}+T(t)P_0\{\phi, h\}\\ &\quad+\int_0^t T(t-s)P\{0, f(s)\}ds+\int_0^t T(t-s)P_0\{0, f(s)\}ds. \end{aligned} \tag{4.11}$$

First, $\lim_{t\to\infty} T(t)P_0\{\phi, h\}=0$ from (4.10). Next,

$$\int_0^\infty T(-s)P\{0 ; f(s)\}ds$$

exists, since for $s\geq 0$, $T(-s)$ is defined on M and (4.9) implies

$$\left\|\int_0^\infty T(-s)P\{0, f(s)\}ds\right\|\leq(K/\alpha)\,|P|\,|f|_{L^1(0,\infty;H)}.$$

Lastly, by (4.7)

$$\lim_{t\to\infty}\left\|\int_t^\infty T(t-s)P\{0, f(s)\}ds\right\| \leq \lim_{t\to\infty}\left(\sup_{s\geq t}|f(s)|\right)K|P|/\alpha=0,$$

$$\lim_{t\to\infty}\left\|\int_0^t T(t-s)P_0\{0, f(s)\}ds\right\| \leq \lim_{t\to\infty} Ke^{-\alpha t}\int_0^\infty e^{\alpha s}|P_0|\,|f(s)|\,ds=0.$$

Define

$$\{\psi, k\}=P\{\phi, h\}+\int_0^\infty T(-s)P\{0, f(s)\}ds$$

and the calculations above yield (4.8). The uniqueness claim is immediately seen, completing the proof.

Remark 4.3. The conclusion of Proposition 4.2 holds true even if there exist characteristic values on the imaginary axis provided that they are simple eigenvalues. The argument carries over easily using (3.4) and the fact that $\lambda\in \boldsymbol{R}$, $A\{\phi, h\}=-i\lambda\{\phi, h\}$ implies $T(t)\{\phi, h\}=e^{i\lambda t}\{\phi, h\}$.

5. An example.

We illustrate the theory we have developed for the delay partial differential equation

$$\begin{aligned} &w_t(x, t)=w_{xx}(x, t)+w(x, t)-(\Pi/2)w(x, t-1),\ 0\leq x\leq \Pi,\ t\geq 0,\\ (5.1)\qquad &w(0, t)=w(\Pi, t)=0,\ t\geq 0,\\ &w(x, t)=\phi(x, t),\ 0\leq x\leq \Pi,\ -1\leq t<0,\ w(x, 0)=h(x),\ 0\leq x\leq \Pi. \end{aligned}$$

The equation (5.1) may be studied abstractly in the formulation (1.3). We let $H=L^2(0, \Pi; C)$, $B: H\to H$ by $Bh=\ddot{h}+h$, $D(B)=\{h\in H: h$ and $\dot{h}$ are absolutely continuous, $\ddot{h}\in H$, $h(0)=h(\Pi)=0$, and $F: C(-1, 0; H)\to H$ by $F\phi=-(\Pi/2)\phi(-1)$. The conditions of Section 1 are met with $\alpha=0$, $\eta(-1)=0$, $\eta(\theta)=-(\Pi/2)I$ for $-1<\theta\leq 0$, $\tau(-1)=0$, and $\tau(\theta)=\Pi/2$ for $-1<\theta\leq 0$.

From (2.1) we see that $\Delta(\lambda)h=-\ddot{h}-h+\lambda h+(\Pi/2)e^{-\lambda}h$, so that λ is a characteristic value iff $\lambda+(\Pi/2)e^{-\lambda}$ is an eigenvalue of B. It is well-known that $\sigma(B)=1-n^2$, $n=1, 2, \cdots$, and each $1-n^2\in\sigma(B)$ is an eigenvalue of B with corresponding eigenvector $\sin nx$. By virtue of Proposition 2.1 $\lambda\in\sigma(-A)$ iff $\lambda+(\Pi/2)e^{-\lambda}=1-n^2$, $n=1, 2, \cdots$, and each such λ is an eigenvalue of $-A$ with corresponding eigenvector $e^{\lambda\theta}\sin nx$. To compute the adjoint of A one verifies that $\tau\psi-\eta^*k=\tau\psi-\eta k$ is absolutely continuous iff ψ is absolutely continuous and $k=-\psi(-1)$. Since B is self-adjoint, Proposition 2.2 yields

$$\begin{aligned} (5.2)\qquad &D(A^*)=\{\{\psi, k\}: \psi \text{ is absolutely continuous, } \psi'\in L^2(-1, 0; H; \mu), \text{ and } k=\\ &\quad -\psi(-1)\in D(B)\}\\ &A^*\{\psi, k\}=\{\psi', -\ddot{k}-k-(\Pi/2)\psi(0)\}, \end{aligned}$$

The decomposition of Section 3 has a simple application for this example. The equation $\lambda+(\Pi/2)e^{-\lambda}=1-n^2$ is equivalent to

$$e^{\mathrm{Re}\,\lambda}\,\mathrm{Re}\,\lambda=-(\Pi/2)\cos\mathrm{Im}\,\lambda+(1-n^2)e^{\mathrm{Re}\,\lambda}, \tag{5.3}$$

and

$$e^{\mathrm{Re}\,\lambda}\,\mathrm{Im}\,\lambda=(\Pi/2)\sin\mathrm{Im}\,\lambda. \tag{5.4}$$

Obviously, $\lambda=\pm i\Pi/2$ solves (5.3) and (5.4) for $n=1$ and these are the only solutions for $\mathrm{Re}\,\lambda=0$. We claim the remaining solutions have real parts negative. Assume that λ solves (5.3) and (5.4) for $\mathrm{Re}\,\lambda>0$. If $|\mathrm{Im}\,\lambda|\leq\Pi/2$, then the right-hand side of (5.3) is positive but the left-hand side is not. If $\Pi/2<|\mathrm{Im}\,\lambda|$, then the right-hand side of (5.4) has absolute value greater than $\Pi/2$ but the left-hand side does not.

We next claim that $T(t)$ is compact for $t>2r$. But this follows immediately from Remark 3.6 and the well-known fact that the semigroup $S(t)$, $t\geq 0$ generated by B is compact for $t>0$ (see Examples 5.2 and 5.4 of [8]). Thus, by Lemma 3.4, $\sigma(-A)$, which is exactly $p\sigma(-A)$, consists of isolated points. Next, we claim that the algebraic eigenspace M_j corresponding to a characteristic value λ_j is finite dimensional. It is shown in [8], Lemmas 5.6, 5.7, and 5.8, that $\Delta(\lambda)^{-1}$ has simple poles at each characteristic value λ_j. By (2.6) $(A+\lambda I)^{-1}$ has simple poles at these λ_j's. By (3.4) $M_j=N(A+\lambda_jI)$. By (2.2) $\{\phi, h\}\in N(A+\lambda_jI)$ iff $\phi(\theta)=e^{\lambda}j^{\theta}h$ where $\Delta(\lambda_j)h=0$. But $h(x)=c\sin nx$ for some $n=1, 2, \cdots$, where c is constant, and therefore, M_j is one-dimensional.

Now we set $\Lambda=\{\lambda_1, \lambda_2\}$, $\lambda_1=i\Pi/2$, $\lambda_2=-i\Pi/2$, and apply Proposition 3.2. We may take as a basis for M, $U_\Lambda=(\{\sin(\Pi\theta/2)\sin x, 0\}, \{\cos(\Pi\theta/2)\sin x, \sin x\})$. The matrix A_Λ of (3.11) is given by

$$A_\Lambda=\begin{bmatrix} 0 & \Pi/2 \\ -\Pi/2 & 0 \end{bmatrix}$$

as may be seen by solving the system of equations $AU_\Lambda=U_\Lambda A_\Lambda$. We wish also to find a basis V_Λ as in (3.12). By virtue of (3.6), it suffices to find eigenvectors of $-A^*$ corresponding to the eigenvalues $\pm i\Pi/2$. From (5.2) we have that $A^*\{\psi, k\}=-\lambda\{\psi, k\}$ iff

$$\psi(\theta)=e^{-\lambda\theta}\psi(0) \quad\text{and}\quad (B-(\lambda+(\Pi/2)e^{-\lambda})I)\psi(-1)=0. \tag{5.5}$$

For $\lambda=\pm i\Pi/2$ (5.5) holds with $(\psi(-1))(x)=\sin x$ and thus we may choose $V_\Lambda=U_\Lambda$. In this case the matrix $\langle V_\Lambda, U_\Lambda\rangle^{-1}$ of (3.13) is given by

$$\begin{bmatrix} \Pi^2/2 & -\Pi/4 \\ -\Pi/4 & \Pi^2/8+\Pi/2 \end{bmatrix}$$

Also, we see that in applying Proposition 3.5 with this choice of Λ we may take $\beta_0 < 0$. Consequently, (3.16) yields that $\lim_{t\to\infty} \|P_0 T(t)\{\phi, h\}\| = 0$ exponentially for all $\{\phi, h\} \in X$. Finally, we note that Remark 4.3 applies to this example.

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