Funkcialaj Ekvacioj, 18 (1975) 257-270

Asymptotic Simplification of Linear Hamiltonian Differential Equations with a Parameter

By Wolfgang Wasow

(University of Wisconsin, U. S. A. and University of Tokyo, Japan)

Abstract

If the n-by-n matrix $A(t, \varepsilon)$ has an asymptotic expansion in powers of ε, as $\varepsilon \rightarrow 0+$ and $A(t, 0)$ has at least two distinct eigenvalues, it is known that the system $\varepsilon\, dy/dt = A(t, \varepsilon) y$ can be split, in many ways, into systems of lower order by means of transformations with asymptotic power series in ε. It is shown that if $A(t, \varepsilon)$ is Hamiltonian the decomposition can be carried out in such a way that the resulting lower order systems are again Hamiltonian. One application of this result concerns the adiabatic invariants of linear Hamiltonian systems.

1. Introduction

A decisive first step in the asymptotic theory of homogeneous systems of ordinary linear differential equations is their decomposition into several similar systems of lower order in such a way that, at least asymptotically, distinct systems have distinct eigenvalues. (See, *e.g.* [3], and [6] §11 and 25.) The usual procedures for such decomposition often destroy important special properties of the original system. The aim of the present paper is to describe, for equations depending in a singular manner on a small parameter, a reduction scheme that preserves the *Hamiltonian* character of Hamiltonian systems.

This investigation was motivated by the article [2] by Leung and K. Meyer, which deals with the construction of adiabatic invariants for linear Hamiltonian systems. Because of such physical applications the emphasis is on a procedure that applies to the whole real time axis and not only in unspecified complex neighborhoods of a given point, as in [3] and [6]. In addition, I have attempted to replace the hypothesis of analyticity made there and in [1] by the more natural one of infinite differentiability.

In [7], I made an analogous analysis for a system with an anti-Hermitian coefficient matrix. The present paper has some features in common with that earlier one, but as the structure of Hamiltonian matrices is less simple than that of anti-Hermitian ones, additional difficulties have to be overcome.

The results of this paper yield, as a special case, an alternate proof of the formal reduction theorem contained in [2].

2. Hypotheses and a Preliminary Transformation

A real $2n$-by-$2n$ matrix M is called Hamiltonian if $J_{2n}M$ is symmetric, where

$$J_{2n}=\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

and I_n is the identity matrix of order n. Equivalently, M is Hamiltonian, if

$$J_{2n}M+M^T J_{2n}=0. \tag{2.1}$$

A linear system of differential equations is said to be Hamiltonian if its coefficient matrix is Hamiltonian.

It is an elementary property of Hamiltonian matrices that if λ is an eigenvalue, so is $-\lambda$. Therefore, the characteristic polynomial of a Hamiltonian matrix contains no odd powers.

We will be dealing with asymptotic properties, as $\varepsilon\to 0+$, of Hamiltonian differential equations of the from

$$\varepsilon\dot{y}=A(t,\varepsilon)y, \qquad \left(\dot{y}=\frac{dy}{dt}\right). \tag{2.2}$$

For a concise description of the regularity properties required of A we introduce the following terminology.

Definition 1.1. A complex valued function f of t will be called "gentle", if $f\in C^\infty[-\infty,\infty]$, and $df^k/dt^k\in L^1(-\infty,\infty)$ for all $k\geq 0$. A function whose first derivative is a gentle function is called "pre-gentle".

Hypothesis (H):

(i) $A(t,\varepsilon)$ is a real $2n\times 2n$ Hamiltonian matrix for all real t and for $0\leq\varepsilon\leq\varepsilon_0$;

(ii) $A(t,\varepsilon)$ is pre-gentle for $0\leq\varepsilon\leq\varepsilon_0$, and $A(t,\varepsilon)-A(t,0)$ is gentle;

(iii) A admits uniformly for $-\infty\leq t\leq\infty$, an asymptotic expansion

$$A(t,\varepsilon)\sim\sum_{r=0}^{\infty}A_r(t)\varepsilon^r \quad \text{as} \quad \varepsilon\to 0+$$

which can be indefinitely formally differentiated with respect to t.

(iv) The characteristic polynomial $\pi(\lambda^2,t)$ of $A_0(t)$ can be factored, for $-\infty\leq t\leq\infty$, into two non-constant, real, relatively prime polynomials,

$$\pi(\lambda^2,t)=\pi_1(\lambda^2,t)\pi_2(\lambda^2,t), \tag{2.3}$$

of degrees $2p$ and $2(n-p)$, respectively.

The aim of this paper is to transform the differential equation (2.2) so that the coefficient matrix becomes the direct sum of two Hamiltonian matrices of order $2p$, $2(n-p)$, respectively. The matrix

$$\hat{J}_{2n}=\begin{bmatrix} J_{2p} & 0 \\ 0 & J_{2(n-p)} \end{bmatrix} \tag{2.4}$$

will therefore appear frequently in the calculations. The subscript $2n$ will be omitted from now on, when no misunderstanding is to be feared.

A real $2n$-by-$2n$ matrix is commonly called *symplectic*, if

$$M^T J M = J. \tag{2.5}$$

One important——and easily proved——property of symplectic matrices is that a linear transformation with a symplectic matrix takes Hamiltonian differential equations again into Hamiltonian differential equations. This is even true if the transformation depends differentiably on t.

We will call a $2n\times 2n$ matrix M *block-Hamiltonian*, if

$$\hat{J}M + M^T\hat{J} = 0 \tag{2.6}$$

and *block-symplectic* if

$$M^T\hat{J}M = \hat{J}. \tag{2.7}$$

In the lemma below a few useful facts have been collected. The proofs are elementary exercises and have, therefore, been omitted.

Lemma 2.1.

(i) *There exists a* $2n\times 2n$ *permutation matrix* Π *such that*

$$\hat{J} = \Pi^{-1}J\Pi = \Pi^T J\Pi;$$

(ii) M *is Hamiltonian if and only if* $\Pi^T M\Pi$ *is block-Hamiltonian*;

(iii) M *is symplectic if and only if* $\Pi^T M\Pi$ *is block-symplectic*;

(iv) *A differentiable block-symplectic transformation takes a block-Hamiltonian system of differential equations (i.e., a system with a block-Hamiltonian coefficient matrix) again into a block-Hamiltonian system*;

(v) *The relation*

$$M^T J M = \hat{J} \tag{2.8}$$

holds if and only if $M\Pi^T$ *is symplectic*;

(vi) *A linear transformation with a differentiable matrix* M *that satisfies* (2.8) *takes a Hamiltonian system of differential equations into a block-Hamiltonian one.*

The proof the theorem below has been postponed to Section 5, so as not to interrupt the description of the reduction procedure.

Theorem 2.1. *Assume that Hypothesis* (H) *is satisfied. Then there exists a real, nonsingular pre-gentle matrix function* $M(t)$ *of dimension* $2n\times 2n$ *such that*

$$(2.9) \qquad M^{-1}(t)A_0(t)M(t) = \operatorname{diag}[B_0^{11}(t), B_0^{22}(t)],$$

where $B_0^{jj}(t)$, $j=1,2$ *are pre-gentle Hamiltonian matrix functions of order* $2p$, $2(n-p)$, *respectively that have* $\pi_j(\lambda^2, t)$ *as their characteristic polynomials. Moreover,* M *satisfies the relation* (2.8).

We now transform the given differential equation (2.2) by setting

$$(2.10) \qquad y = M(T)\tilde{y},$$

$M(t)$ being the matrix in (2.9). The resulting differential equation is block-Hamiltonian by Lemma 2.1 (v). It has the form

$$(2.11) \qquad \varepsilon\frac{d\tilde{y}}{dt} = \tilde{A}(t,\varepsilon)\tilde{y} = (B_0(t) + \tilde{A}_1(t,\varepsilon))\tilde{y}$$

with

$$(2.12) \qquad B_0(t) = \operatorname{diag}(B_0^{11}(t), B_0^{22}(t)).$$

Since $M(t)$ is pre-gentle, $\dot{M}$ is gentle and it follows from Hypothesis (H ii) that $\tilde{A}_1(t,\varepsilon)$ is gentle. Hypothesis (H iii) implies that $\tilde{A}_1(t,\varepsilon)$ possesses an asymptotic expansion,

$$(2.13) \qquad \tilde{A}_1(t,\varepsilon) \sim \sum_{r=1}^{\infty} \tilde{A}_r(t)\varepsilon^r, \qquad \varepsilon \to 0+,$$

uniformly valid in $-\infty \leq t \leq \infty$. This relation may be indefinitely differentiated with respect to t.

Thus equation (2.11) also satisfies Hypothesis (H) except that it is block-Hamiltonian instead of Hamiltonian and it has the additional advantage that the leading term of its coefficient matrix is block-diagonal.

3. Complete Formal Block-Diagonalization

Lemma 3.1. *A matrix* Q *is block-Hamiltonan if and only if* e^{Qs} *is block-symplectic for all real* s.

Proof:

$$\frac{d}{ds}(e^{Q^T s}\hat{J}e^{Qs}) = e^{Q^T s}Q^T\hat{J}e^{Qs} + e^{Q^T s}\hat{J}e^{Qs} = e^{Q^T s}(Q^T\hat{J} + \hat{J}Q)e^{Qs}.$$

The last expression is zero if Q is Hamiltonian.

Therefore, $e^{Q^T s}\hat{J}e^{Qs}$ is then independent of s. Setting $s=0$ one finds $e^{Q^T s}\hat{J}e^{Qs} = \hat{J}$, *i.e.*, e^{Qs} is block-symplectic. Conversely, if e^{Qs} is block-symplectic, the derivative calculated above must be zero and, hence, $\hat{J}Q + Q^T\hat{J} = 0$.

From here on, the formal reduction procedure is patterned on the one in

[7]. We first observe that the matrices $\tilde{A}_r(t)$ in (2.14) are gentle. For $r=1$ this follows from the relation

$$\tilde{A}_1^{(k)}(t)=\lim_{\varepsilon\to 0+}\varepsilon^{-1}\tilde{A}_1^{(k)}(t,\varepsilon),\qquad k=0,1,\cdots,$$

which is uniformly valid in $[-\infty,\infty]$. The superscript is to indicate differentiation. For $r\geq 2$ one proceeds by induction in an obvious way.

Let us now assume that in the differential equation (2.12) the terms $\tilde{A}_j(t)$, $j=1,2,\cdots,m-1$, with $m\geq 1$, are already block-diagonal in the same partition as $B_0(t)$. If $m=1$ this is no additional hypothesis. We shall construct, somewhat as in [7], a transformation which leaves $B_0(t), \tilde{A}_j(t)$, $j\leq m-1$ unchanged, but block-diagonalizes the next term in the asymptotic expansion without destroying the block-Hamiltonian character or the regularity properties of the differential equation.

Let

$$\tilde{y}=\exp[\varepsilon^m Q(t)]\tilde{z}. \tag{3.1}$$

Expanding the exponential function and inserting into (2.11) we get a differential equation

$$\varepsilon\dot{\tilde{z}}=\tilde{B}(t,\varepsilon)\tilde{z}, \tag{3.2}$$

where $\tilde{B}(t,\varepsilon)$ has the form

$$\tilde{B}=B_0+\sum_{r=1}^{m-1}\tilde{A}_r\varepsilon^r+(\tilde{A}_m+B_0Q-QB_0)\varepsilon^m+O(\varepsilon^{m+1}). \tag{3.3}$$

If Q is bounded in $[-\infty,\infty]$, then the last term in (3.3) is of the indicated order uniformly for all t. We choose for Q a matrix of the special form

$$Q=\begin{bmatrix}0 & Q_{12}\\ Q_{21} & 0\end{bmatrix} \tag{3.4}$$

and impose the further condition that Q be block-Hamiltonian, so as to be able to apply Lemma 3.1. Q will be block-Hamiltonian if

$$J_{2p}Q_{12}+Q_{21}^T J_{2(n-p)}=0. \tag{3.5}$$

To this condition we add the requirement that the factor of ε^m in (3.3) be block-diagonal *i.e.*, that

$$\tilde{A}_m^{12}+B_0^{11}Q_{12}-Q_{12}B_0^{22}=0, \tag{3.6}$$

$$\tilde{A}_m^{21}+B_0^{22}Q_{21}-Q_{21}B_0^{11}=0. \tag{3.7}$$

Here, the superscripts indicate the location of the blocks in the partition. Thanks to Hypothesis (H iv) the matrices $B_0^{11}(t)$, $B_0^{22}(t)$ have no common eigenvalues for any real t, and therefore Q_{12}, Q_{21} can be uniquely calculated from (3.6) and (3.7). Since $\tilde{A}_m(t)$ is gentle and $B_0(t)$ pre-gentle, Q_{12} and Q_{21} are gentle. It remains to be shown that (3.5) is satisfied. To this end we observe that the known block-Hamiltonian character of $\tilde{A}_m$ is equivalent to the three identities

$$(3.8)\qquad \begin{cases} J_{2p}\tilde{A}_m^{11}+(A_m^{11})^T J_{2p}=0 \\ J_{2(n-p)}\tilde{A}_m^{22}+(\tilde{A}_m^{22})^T J_{2(n-p)}=0 \\ J_{2p}\tilde{A}_m^{12}+(\tilde{A}_m^{21})^T J_{2(n-p)}=0. \end{cases}$$

Now muliply (3.6) and (3.7) on the left by J_{2p}, $J_{2(n-p)}$, respectively and add the transpose of the second equation so obtained to the first equation. Thanks to the third relation (3.8) this elimates $\tilde{A}_m^{12}$ and $\tilde{A}_m^{21}$. Remembering also that B_0^{11} B_0^{22} are Hamiltonian one arrives at the relation

$$(3.9)\qquad (B_0^{11})^T[J_{2p}Q_{12}+Q_{21}^T J_{2(n-p)}]+[J_{2p}Q_{12}+Q_{21}^T J_{2(n-p)}]B_0^{22}=0.$$

Since the characteristic polynomial of B_0^{22} contains no odd powers of λ, the matrix $-B_0^{22}$ has the same eigevalues as B_0^{22}, which, by Hypothesis (H iv) are distinct from those of B_0^{11}. We conclude that (3.9) implies (3.5).

Starting with $m=1$ one can construct, in the way just described, an infinite sequence of transformations of the form (3.1) with different matrices $Q=Q_m$ for each m. By Lemma 3.1, each of these transformations is block-symplectic. Therefore, by Lemma 2.1 (iv) the block-Hamiltonian character of (2.11) is preserved under the transformations.

The infinite product of these transformations, for $m=1,2,\cdots$, is, in general, divergent, but the product of the corresponding series in powers of ε exists as a formal power series with leading term I_{2n} and gentle coefficients thereafter. We summarize our result in a theorem, which is the analog of Theorem 4.1 of [7].

Theorem 3.1. *If Hypothesis* (H) *is satisfied, there exists a formal power series* $\sum_{r=0}^{\infty} P_r(t)\varepsilon^r$ *such that the transformation*

$$(3.10)\qquad y=\left(\sum_{r=0}^{\infty} P_r(t)\varepsilon^r\right)z$$

takes the formal differential equation

$$(3.11)\qquad \varepsilon\frac{dy}{dt}=\left(\sum_{r=0}^{\infty} A_r(t)\varepsilon^r\right)y$$

into

$$(3.12)\qquad \varepsilon\frac{dz}{dt}=\left(\sum_{r=0}^{\infty} B_r(t)\varepsilon^r\right)z,$$

where

$$B_r(t)=\begin{bmatrix} B_r^{11}(t) & 0 \\ 0 & B_r^{22}(t) \end{bmatrix}, \qquad r=0,1,\cdots \tag{3.13}$$

with $B_r^{jj}(t)$, $j=1,2$, *Hamiltonian and of dimensions* $2p\times 2p$, $2(n-p)\times 2(n-p)$, *respectively.* $P_r(t)$ *and* $B_r(t)$ *are pre-gentle for* $r=0$, *gentle for* $r>0$. $P_0(t)$ *is identical with the matrix* $M(t)$ *of Theorem* 2.1.

4. Analytic Block-Diagonalization

By Theorem 3.1 of [8] there exists a matrix function $\check{B}(t,\varepsilon)$ which is block-diagonal and has uniformly in $-\infty\leq t\leq\infty$, the asymptotic expansion

$$\check{B}(t,\varepsilon)\sim\sum_{r=0}^{\infty}B_r(t)\varepsilon^r, \qquad \varepsilon\to 0+, \tag{4.1}$$

where the right member is the series in (3.12). Moreover, $\check{B}(t,\varepsilon)-B_0(t)$ is gentle and $\check{B}(t,\varepsilon)$ is pre-gentle. While all the $B_r(t)$ are block-Hamiltonian, $\check{B}(t,\varepsilon)$ may fail to be so. However, the function

$$B(t,\varepsilon)=\frac{1}{2}(\check{B}+\hat{J}\check{B}^T\hat{J})=\begin{bmatrix} B^{11}(t,\varepsilon) & 0 \\ 0 & B^{22}(t,\varepsilon) \end{bmatrix} \tag{4.2}$$

shares all the described properties of $B(t,\varepsilon)$ and is, in addition, block-Hamiltonian, as can be directly verified.

We wish to prove the existence of a transformation

$$y=P(t,\varepsilon)z \tag{4.3}$$

with the asymptotic expansion

$$P(t,\varepsilon)\sim\sum_{r=0}^{\infty}P_r(t)\varepsilon^r, \qquad \varepsilon\to 0+, \tag{4.4}$$

such that (4.3) takes the differential equation (2.2) into the equation

$$\varepsilon\dot{z}=B(t,\varepsilon)z \tag{4.5}$$

with a block-diagonal, block-Hamiltonian coefficient matrix. A matrix function P will have the last mentioned property if it satisfies the differential equation

$$\varepsilon\dot{P}=AP-PB. \tag{4.6}$$

Theorem 3.1 implies that the series $\sum_{r=0}^{\infty}P_r\varepsilon^r$ satisfies (4.6) formally, but this fact is only the very first step towards the proof of the existence of a function $P(t,\varepsilon)$ satisfying (4.4) as well as (4.6). In fact, the results of this section are, as yet, regrettably incomplete.

Theorem 4.1. *There exists a matrix function $P(t,\varepsilon)$ with the following properties.*

(i) $P(t,\varepsilon)$ *admits the expansion* (4.4) *uniformly in* $-\infty\leq t\leq\infty$;

(ii) $P(t,\varepsilon)$ *is pre-gentle, and* $P(t,\varepsilon)-P_0(t)$ *vanishes at* $t=\pm\infty$;

(iii) $\varepsilon\dot{P}=AP-P(B+\Omega)$

with A, B *as in* (4.6) *and*

$$\Omega(t,\varepsilon)\sim 0, \qquad as \quad \varepsilon\rightarrow 0+, \tag{4.7}$$

uniformly in $-\infty\leq t\leq\infty$;

(iv) $P^T JP=\hat{J}$.

Proof: The existence of functions $\check{P}$ with the properties required of P in (i) and (ii) follows immediately from Theorem 3.1 of [8]. Theorem 3.1 of the present paper then shows that $\check{P}$ also is a solution of the differential equation in (iii) with some matrix Ω satisfying (4.7). However, the relation in (iv) will in general, be only approximately true for a matrix $\check{P}$ so constructed, and therefore, $B+\Omega$ will be only approximately block-Hamiltonian.

A matrix P that satisfies (iv) exactly can be constructed as follows: We return from the series in (4.4) to $\sum_{r=0}^{\infty} P_0^{-1}(t)P_r(t)\varepsilon^r$, which is the series constructed in Section 3 by multiplying the expansions for an infinite sequence of exponential matrices all of which were block-symplectic. This implies, in particular, that the m^{th} partial sum of the series above differs from a block-symplectic matrix by a term of the uniform magnitude $O(\varepsilon^{m+1})$. Accordingly, by Lemma 3.1, the m^{th} partial sum of the formal series

$$\sum_{r=1}^{\infty} C_r(t)\varepsilon^r=\log\left(\sum_{r=0}^{\infty} P_0^{-1}(t)P_r(t)\varepsilon^r\right)=P_0^{-1}(t)P_1(t)\varepsilon+\cdots \tag{4.8}$$

differs from a block-Hamiltonian matrix of terms of magnitude of $O(\varepsilon^{m+1})$. Appealing to Theorem 3.1 of [8] we now construct a matrix $\check{C}(t,\varepsilon)$ so that

$$\check{C}(t,\varepsilon)\sim\sum_{r=1}^{\infty} C_r(t)\varepsilon^r, \qquad \text{as} \quad \varepsilon\rightarrow 0+ \tag{4.9}$$

uniformly in $-\infty\leq t\leq\infty$, and that $\check{C}(t,\varepsilon)$ is gentle. The block-Hamiltonian matrix

$$C(t,\varepsilon)=\frac{1}{2}[\check{C}(t,\varepsilon)+\hat{J}\check{C}^T(t,\varepsilon)\hat{J}]$$

has then also the expansion (4.9) and $\exp[C(t,\varepsilon)]$ is therefore block-symplectic, pre-gentle and equal to 1 at $t=\pm\infty$. Finally, we set $P(t,\varepsilon)=P_0(t)\ \exp[C(t,\varepsilon)]$, and the proof is complete.

The preceding theorem reduces the problem of solving the differential equation (2.2) to the same problem for

$$\varepsilon\dot{z}=(B(t,\varepsilon)+\Omega(t,\varepsilon))z. \tag{4.10}$$

Even if the two uncoupled differential equations with coefficient matrices $B^{11}(t,\varepsilon)$, $B^{22}(t,\varepsilon)$ have been solved already, it is not at all certain that the asymptotically negligible perturbation $\Omega(t,\varepsilon)$ causes only asymptotically small changes in those solutions.

Braaksma [1] has made a very general analysis of this question for analytic, not necessarily Hamiltonian matrices $A(t,\varepsilon)$ and described a set of conditions under which $P(t,\varepsilon)$ can be chosen so that $\Omega(t,\varepsilon)$ is strictly zero. However, his conditions are rather artificial in the context of applications to Mechanics.

There is, of course, the older result of Sibuya [3], (see also [6], § 26), which is a special case of Braaksma's and guarantees, in the analytic case, the existence of a transformation (4.5) with the expansion (4.4) for which $\Omega(t,\varepsilon)\equiv 0$, in a sufficiently small neighborhood of some chosen point.

Finally, Leung and K. Meyer [2] have proved that $\Omega(t,\varepsilon)\equiv 0$ can be achieved when in addition to Hypothesis (H) and analyticity the matrix $A(t,\varepsilon)$ satisfies the condition that all its eigenvalues are distinct and purely imaginary throughout.

All these results fail to use the full strength of the Hamiltonian character of the differential equation. The matter deserves further consideration.

As one simple application we mention that a complete block-diagonalization is useful in the study of adiabatic invariants. For our present context it suffices to define an adiabatic invariant of the differential equation

$$\frac{dv}{d\tau}=A(\tau)v \tag{4.11}$$

as a scalar function $I(v,\tau)$ such that for all solution $y=y(t,\varepsilon)$ of

$$\varepsilon\frac{dy}{dt}=A(t)y \tag{4.12}$$

belonging to a certain family $\mathcal{G}$ of solutions and for all $t\in\mathcal{T}$ where $\mathcal{T}=\{t|t_1\leq t\leq t_2\}$, the limit

$$\lim_{\varepsilon\to 0+} I(y(t,\varepsilon),t) \tag{4.13}$$

exists and is independent of t. Note that this property of I is defined relative to $\mathcal{G}$ and $\mathcal{T}$ only. In [9] a more detailed discussion of these matters can be found.

Now assume that a transformation of the form (4.3) takes (4.11) *strictly* into the block-diagonal equation

$$\varepsilon\dot{z}=B(t,\varepsilon)z=\begin{bmatrix}B^{11}(t;\varepsilon) & 0\\ 0 & B^{22}(t,\varepsilon)\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix},$$

at least in some interval $\mathcal{J}$. It may be possible to find, for one or both of the lower order systems

$$\varepsilon\dot{z}_j=B^{jj}(t,\varepsilon)z_j,\qquad j=1,2 \tag{4.14}$$

adiabatic invariants $I_j(v,\tau)$ in the sense that

$$\lim_{\varepsilon\to 0+} I_j(z_j(t,\varepsilon),t)$$

exists and is independent of t with respect to $\mathcal{J}$ and some family $\mathcal{G}_j$ of solutions of (4.14). If it is further known that the $z_j(t,\varepsilon)$ in $\mathcal{G}_j$ are bounded in $\mathcal{J}$ for $0\leq\varepsilon\leq\varepsilon_0$ and that $I_j(v,\tau)$ is continuous with respect to v, then

$$I_j((P_0^{-1}(\tau)y)_j,\tau) \tag{4.15}$$

is an adiabatic invariant for (4.11) relative to the family of solutions of (4.12) originating from $\mathcal{G}_j$ and to the interval $\mathcal{J}$. The symbols $(\cdot)_1,(\cdot)_2$ indicate the vectors of dimensions $2p$ and $2(n-p)$, respectively formed by the first $2p$ and the last $2(n-p)$ components of the vector inside the parentheses.

5. Proof of Theorem 2.1

Lemma 5.1.

Let $M(t)$ be a real pre-gentle m-by-m matrix function of constant rank r for all t, including $t=\pm\infty$. Then there exists a real pre-gentle matrix function $Q(t)$ of m rows and $m-r$ columns having constant rank $m-r$, such that

$$M(t)Q(t)=0,\qquad \textit{for}\quad -\infty\leq t\leq\infty. \tag{5.1}$$

Proof: The lemma and its proof have some resemblence to Theorem 6 of [4]. Locally *i.e.*, in some neighborhood of any point $t=t_0$ the existence of a solution $Q(t)$ of (5.1) of rank $m-r$ and with derivatives of all orders is easily proved, for instance by describing $Q(t)$ explicitly as was done in [5]: If the minor $M_{11}(t)$ formed by the first r rows and columns of $M(t)$ is nonsingular at $t=t_0$, one writes $M(t)$ in the partitioned form

$$M(t)=\begin{bmatrix}M_{11}(t) & M_{12}(t)\\ M_{21}(t) & M_{22}(t)\end{bmatrix}$$

and finds that

$$Q(t)=\begin{bmatrix}M_{11}^{-1}(t)\ \ M_{12}(t)\\ I_{m-r}\end{bmatrix}$$

is such a solution. If $M_{11}(t_0)$ is singular the formula has to be modified appropriately by means of a permutation matrix. Because of the assumed properties of $M(t)$ at infinity there are also two such matrix solutions $Q_{-\infty}(t)$, $Q_{\infty}(t)$ which have rank $m-r$ in certain neighborhoods $I_{-\infty}$, I_{∞}, of $t=-\infty$, $t=\infty$, respectively. Moreover, one sees from the representation above that $\lim_{t\to-\infty} Q_{-\infty}(t)$ and $\lim_{t\to\infty} Q_{\infty}(t)$ exist and that $\dot{Q}_{-\infty}\in L^1(I_{-\infty})$, $\dot{Q}_{\infty}\in L^1(I_{\infty})$. Outside of a certain neighborhood of the chosen point t_0 the matrix $Q(t)$ above, may however become discontinuous, because $\det M_{11}(t)$ may become zero. To construct *one* smooth *global* solution, we cover the t-axis by a finite number of closed intervals $I_{-\infty}, I_1, \cdots, I_N, I_{\infty}$ in each of which the equation (5.1) has a solution $Q_j(t)$ with the indicated properties. By a standard theorem of elementary algebra Q_j and Q_{j+1} must be related by an identity of the form

$$Q_j(t)=Q_{j+1}(t)C_j(t), \qquad t\in I_j\cap I_{j+1}, \tag{5.2}$$

where $C_j(t)$ is some nonsingular $(m-r)\times(m-r)$ matrix. From the fact that $Q_{j+1}(t)$ has one minor of constant rank $m-r$ in I_{j+1} one concludes that $C_j(t)\in C^{\infty}(I_j\cap I_{j+1})$. Now we continue $C_j(t)$ as an indefinitely differentiable nonsingular matrix into all of I_{j+1}. (See, *e.q.*, [4] on details how to construct such a continuation.) The equation (5.2) then defines an extension of $Q_j(t)$ into all of I_{j+1} as an indefinitely differentiable solution of (5.1) of maximal rank. If $j=N$, the continuation of $C_N(t)$ to $t=\infty$ should be chosen so that $C_N(t)$ becomes the identity near ∞. Beginning with $Q_{-\infty}(t)$, one constructs in this manner a solution of (5.1) with all the required properties.

Lemma 5.2. *If $A_0(t)$ has the properties described in Hypothesis* (H), *there exists a nonsingular, pre-gentle matrix $\tilde{M}(t)$ such that*

$$\tilde{M}^{-1}(t)A_0(t)\tilde{M}(t)=\begin{bmatrix}\tilde{B}_{11}(t) & 0\\ 0 & \tilde{B}_{22}(t)\end{bmatrix}, \tag{5.3}$$

where the $\tilde{B}_{jj}(t)$ have all the properties required of the $B_{jj}(t)$ in Theorem 2.1, *except that they may fail to be Hamiltonian.*

The proof of this lemma is based on Lemma 5.1 by a reasoning that differs only trivially from that for Theorem 3 in [4]. It is therefore omitted.

Lemma 5.3. *The matrix $\tilde{M}(t)$ of Lemma* 5.2 *satisfies the relation*

$$\tilde{M}^T(t)J_{2n}\tilde{M}(t)=\begin{bmatrix}R_{11}(t) & 0\\ 0 & R_{22}(t)\end{bmatrix},$$

where $R_{11}(t)$, $R_{22}(t)$ have dimension $2p\times 2p$, $2(n-p)\times 2(n-p)$, respectively.

Proof: Let $\tilde{M}_j(t)$, $j=1,2$ be the first $2p$ and the last $2(n-p)$ columns of $\tilde{M}(t)$, respectively. Then formula (5.3) implies that

(5.4) $$A_0\tilde{M}_j=\tilde{M}_j\tilde{B}_{jj}, \qquad j=1,2.$$

Multiply this equation with $j=1$ by $M_2^T J$ to the left, and multiply the transpose of the equation with $j=2$ by JM_1 to the right. We find that

$$\tilde{M}_2^T J A_0\tilde{M}_1=\tilde{M}_2^T J\tilde{M}_1\tilde{B}_{11}$$

and——since A_0 is Hamiltonian——that

$$-M_2^T J A_0\tilde{M}_1=\tilde{B}_{22}\tilde{M}_2^T J\tilde{M}_1,$$

i.e.,

(5.5) $$\tilde{M}_2^T J\tilde{M}_1\tilde{B}_{11}+\tilde{B}_{22}^T\tilde{M}_2 J\tilde{M}=0.$$

By Lemma 5.2 and Hypothesis (H iv) the matrices $\tilde{B}_{11}(t)$ and $-\tilde{B}_{22}^T$ have no eigenvalues in common for any t. Hence, (5.5) can be true only if $\tilde{M}_2^T J\tilde{M}_1\equiv 0$. As

$$\tilde{M}^T J\tilde{M}=\begin{bmatrix}\tilde{M}_1^T J\tilde{M}_1 & \tilde{M}_1^T J\tilde{M}_2\\ \tilde{M}_2^T J\tilde{M}_1 & \tilde{M}_2^T J\tilde{M}_2\end{bmatrix},$$

the statement of the lemma follows.

Corollary:

(5.6) $$R_{jj}(t)=\tilde{M}_j^T(t)J\tilde{M}_j(t), \qquad j=1,2.$$

Lemma 5.4. *Let let $0<k\leq n$, and let K be a real matrix of $2n$ rows and $2k$ columns such that the $2k\times 2k$ matrix*

$$R=K^T J_{2n}K$$

is nonsingular. Then there exists a real nonsingular $2k\times 2k$ matrix L such that

(5.7) $$(KL)^T J_{2n}KL=J_{2k}.$$

Furthermore, if $K=K(t)$ and $R=R(t)$ are pre-gentle matrix function, then the matrix $L=L(t)$ satisfying (5.7) *can be chosen as pre-gentle.*

Proof: We observe first that the matrix L, if it exists, in (5.7) is not unique. The relation is equally true for any second matrix $\hat{L}$ if, $\hat{L}=LS$, where S is an arbitrary symplectic matrix of order $2k\times 2k$. Next, R is a real nonsingular antisymmetric matrix of even order $2k$. Now, it is well known that the corresponding bilinear form can be reduced to the standard form with matrix J_{2k} by a change of coordinates analogous to the Gram-Schmidt orthonormalization procedure. (See, *e.g.* [10], Ch. VI). In other words, there is a nonsingular matrix $\tilde{L}$ such that $\tilde{L}^T R\tilde{L}=J_{2k}$. This proves formula (5.7) if we set $\tilde{L}=L^{-1}$.

An inspection of the construction of the matrix $\tilde{L}$, as described, *e.g.*, in [10], shows that it can be carried out simultaneously for all t in a sufficiently small neighborhood of any given point t_0 in such a way that the resulting $\tilde{L}$ is there indefinitely differentiable. One also checks directly that near $t=\pm\infty$ these matrices will have finite limits and integrable derivatives if $K(t)$ is given as pre-gentle. To pass from these local solutions to a global one requires an argument that resembles so much that given in the proof of Lemma 5.1 that a repetition is unnecessary.

Proof of Theorem 2.1: We apply Lemma 5.4 with $K=\tilde{M}_1(t)$. Then there is a matrix $L_1(t)$, nonsingular and pre-gentle such that

$$(\tilde{M}_1L)^TJ_{2n}\tilde{M}_1L_1=J_{2p}. \tag{5.8}$$

Analogously, there is a matrix $L_2(t)$ with

$$(\tilde{M}_2L_2)^TJ_{2n}(\tilde{M}_2L_2)=J_{2(n-p)}. \tag{5.9}$$

The matrix

$$M=\tilde{M}\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \tag{5.10}$$

has then all the properties required in Theorem 2.1. In fact, (5.8) and (5.9) combine into

$$\left[\tilde{M}\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}\right]^T J_{2n}\tilde{M}\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}=\begin{pmatrix} J_{2p} & 0 \\ 0 & J_{2(n-p)} \end{pmatrix}$$

which is formula (2.8). To prove (2.9) from (5.3) we note that

$$M^{-1}A_0M=\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}^{-1}\tilde{M}^{-1}A_0\tilde{M}\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}=\begin{pmatrix} B_0^{11} & 0 \\ 0 & B_0^{22} \end{pmatrix}$$

if the B_0^{jj} are defined by

$$B_0^{jj}=L_j^{-1}\tilde{B}_{jj}L_j, \qquad j=1,2.$$

It remains to show that these B_{jj} are Hamiltonian. This is an easy consequence of (2.8), (2.9) and the fact that A_0 is Hamiltonian:

$$\begin{aligned}\hat{J}B+B^T\hat{J}&=\hat{J}M^{-1}A_0M+M^TA_0^T(M^{-1})^T\hat{J}\\ &=M^TJA_0M+M^TA_0^TJM=M^TJA_0M-M^TJA_0M=0.\end{aligned}$$

Bibliography

[1] Braaksma, B. L. J., Global reduction of linear differential equations involving a small singular parameter, *SIAM J. Math. Am.* **2** (1971), 149–165.

[2] Leung, A. and K. Meyer, Adiabatic Invariants for Linear Hamiltonian Systems. To be published.

[3] Sibuya, Y., Sur réduction analytique d'un système d'équation différentielles

ordinaires linéaires contenant un paramètre, *J. Fac. Sci. Univ. Tokyo* (1) **7** (1958), 527–540.
[4] Sibuya, Y., Some global properties of matrices of functions of one variable, *Math. Ann.* **16** (1965), 67–77.
[5] Wasow, W., On holomorphically similar matrices, *J. Math. Anal. Appl.* **4** (1962), 202–206.
[6] Wasow, W., *Asymptotic Expansions for Ordinary Differential Equations* 362 pp., John Wiley and Sons, New York, 1965.
[7] Wasow, W., Asymptotic simplification of self-adjoint differential equations with a parameter, *J. of Diff. Equ.* **2** (1966), 378–390.
[8] Wasow, W., Adiabatic invariance of a simple oscillator, *SIAM J. Math. Anal.* **4** (1973), 78–88.
[9] Wasow, W., Some recent results in the theory of adiabatic invariants. To be published in the proceedings of the International Conference on Differential Equations, U. of S. Calf., Los Angeles, Sept. 3–7, 1974.
[10] Weyl, H., *The Classical Group*, Princeton University Press, 1939.

nuna adreso:
Department of Mathematics
University of Wisconsin
Van Vleck Hall
Madison, Wisconsin 53706, U. S. A.

(Ricevita la 9-an de oktobro, 1974)