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Asymptotic Property of Nonoscillatory Solutions of Second Order Differential Equations

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In [2], Hammett has pointed out that the oscillation of the equation

$$(r(t)x')'+a(t)f(x)=0$$

does not necessarily imply the oscillation of the equation

$$(r(t)x')'+a(t)f(x)=g(t), \tag{1}$$

even if $g(t)$ is sufficiently small, and proved the following;

Theorem 1. *Assume that*

(i) $r(t)\in C[0,\infty)$, $\int_0^\infty 1/r(s)ds=\infty$ *and* $r(t)>k$ *for some positive constant* k,

(ii) $a(t)\in C[0,\infty)$ *and* $a(t)>\tilde{k}$ *for some positive constant* $\tilde{k}$,

(iii) $f(x)\in C(-\infty,\infty)$, $xf(x)>0$ *for* $x\neq 0$ *and* $f'(x)\geqq 0$,

(iv) $g(t)\in C[0,\infty)$ *and* $\int_0^\infty |g(s)|ds<\infty$.

Then a nonoscillatory solution $x(t)$ *of* (1) *satisfies* $\lim_{t\to\infty} x(t)=0$.

Recently, Grimmer has extended Hammett's result to the equation

$$(r(t)x')'+h(x)x'+a(t)f(x)=g(t)$$

and replaced the conditions on $a(t)$ and on $g(t)$ in Theorem 1 by following conditions;

(*) $a(t)\geqq 0$ and if $\{t_n\}$ and $\{s_n\}$ are sequences with $0<\cdots<t_n<s_n<t_{n+1}<\cdots$ and $s_n-t_n>\varepsilon$, where ε is a positive constant, then $\sum_{n=0}^{\infty}\int_{t_n}^{s_n} a(s)ds=\infty$,

and

$$\int_0^t g(s)ds \text{ is bounded,} \tag{**}$$

respectively (see, Theorem 1 in [1]).

In this paper, we shall give some extensions of Hammett's and Grimmer's results mentioned above.

Consider the second order differential equation

$$(r(t)x')'+h(x)x'+f(t,x,x')=g(t), \tag{2}$$

where $r(t)>0$ and $g(t)$ are continuous on $I, I=[0,\infty), h(x)$ is continuous on R, $R=(-\infty,\infty), h(x)$ satisfies $x\cdot\int_0^x h(u)du \geqq 0$ and $f(t,x,y)$ is continuous on $I\times R\times R$. For any continuous function $\varphi(t)$ and $\sigma\in[0,\infty)$, let $\{\varphi(t)\}^+=\max[\varphi(t),0]$, $\{\varphi(t)\}^-=\min[\varphi(t),0]$ and let

$$\operatorname{sgn}\varphi(\sigma)=\begin{cases}1 & \text{for } \varphi(\sigma)>0\\ -1 & \text{for } \varphi(\sigma)<0.\end{cases}$$

Lemma. *Assume that*

$$\int_0^\infty 1/r(s)ds=\infty \tag{3}$$

and

$$\int_0^t g(s)ds \text{ is bounded.} \tag{4}$$

Let $x(t)$ be a solution of (2) which is nonoscillatory on $[\sigma,\infty)$ and satisfies

$$\int_\sigma^\infty \{\operatorname{sgn} x(\sigma)\cdot f(s,x(s),x'(s))\}^- ds>-\infty. \tag{5}$$

Then we have

$$\int_\sigma^\infty \{\operatorname{sgn} x(\sigma)\cdot f(s,x(s),x'(s))\}^+ ds<\infty. \tag{6}$$

In addition to the above conditions, if

$$r(t)\geqq k \text{ for some positive constant } k, \tag{7}$$

then $\operatorname{sgn} x(\sigma)\cdot x'(t)$ *is bounded above for all $t\geqq\sigma$, and futhermore, if*

$$\int_0^\infty |g(s)|ds<\infty \tag{8}$$

and there exists a sequence $\{s_n\}, s_n\to\infty$ as $n\to\infty$ such that $x'(s_n)=0$ and $x(s_n)\to 0$ as $n\to\infty$, then $\limsup_{t\to\infty} \operatorname{sgn} x(\sigma)\cdot x'(t)\leqq 0$. *In particular, if*

$$h(u)\geqq 0 \text{ on } R, \tag{9}$$

then the solution $x(t)$ satisfies $\lim_{t\to\infty} x'(t)=0$.

Proof. Suppose that a solution $x(t)$ of (2) is positive on $[\sigma,\infty)$. An analogous argument will hold if $x(t)<0$. It follows from (2) that

$$r(t)x'(t)-r(\sigma)x'(\sigma)=-\int_{x(\sigma)}^{x(t)} h(u)du-\int_\sigma^t f(s,x(s),x'(s))ds+\int_\sigma^t g(s)ds \tag{10}$$

$$=-\int_{x(\sigma)}^{x(t)} h(u)du-\int_{\sigma}^{t}\{f(s,x(s),x'(s))\}^{+}ds$$
$$-\int_{\sigma}^{t}\{f(s,x(s),x'(s))\}^{-}ds+\int_{\sigma}^{t}g(s)ds.$$

Now, suppose that

$$\int_{\sigma}^{\infty}\{f(s,x(s),x'(s))\}^{+}ds=\infty.$$

Since

$$(11)\qquad -\int_{x(\sigma)}^{x(t)} h(u)du=-\int_{0}^{x(t)} h(u)du+\int_{0}^{x(\sigma)} h(u)du\leqq\int_{0}^{x(\sigma)} h(u)du$$

for all $t\geqq\sigma$, there exist a $\tau>\sigma$ and an $M>0$ such that $r(t)x'(t)<-M$ for all $t\geqq\tau$ by (4), (5), (10) and (11), and hence, by (3),

$$x(t)-x(\tau)<-M\int_{\tau}^{t}1/r(s)ds\longrightarrow-\infty\quad\text{as}\quad t\to\infty$$

This contradicts $x(t)>0$ for $t\geqq\sigma$. Thus (6) is proved.

It follows from (7), (10) and (11) that

$$(12)\qquad x'(t)\leqq\left\{\left|\int_{\sigma}^{t}g(s)ds\right|+\int_{0}^{x(\sigma)}h(u)du-\int_{\sigma}^{t}\{f(s,x(s),x'(s))\}^{-}ds\right.$$
$$\left.+r(\sigma)|x'(\sigma)|\right\}\Big/k,$$

and hence $x'(t)$ is bounded above for all $t\geqq\sigma$ by (4) and (5).

Replacing σ by s_n and noting that by (6) and (8), $\left|\int_{s_n}^{t}g(s)ds\right|\leqq\int_{s_n}^{t}|g(s)|ds$ and $\int_{s_n}^{\infty}|g(s)|ds\to0$, $\int_{0}^{x(s_n)}h(u)du\to0$ and $\int_{s_n}^{\infty}\{f(s,x(s),x'(s))\}^{+}ds\to0$ as $n\to\infty$, the third part follows from the inequality (12).

Now, we shall show the last part. Since

$$r(t)x'(t)+\int_{x(s_n)}^{x(t)}h(u)du\geqq-\int_{s_n}^{t}\{f(s,x(s),x'(s))\}^{+}ds-\int_{s_n}^{t}|g(s)|ds$$

by (10), it follows from (6) and (8) that for any $\varepsilon>0$ there is an $N>0$ such that

$$(13)\qquad r(t)x'(t)+\int_{x(s_n)}^{x(t)}h(u)du\geqq-\varepsilon$$

for all $n\geqq N$ and $t\geqq s_n$. Assume that there exists a $t^*\geqq s_N$ such that $x'(t^*)<-\varepsilon/k$. Then there exists an $N^*\geqq N$ such that $x'(t)\leqq0$ on $t^*\geqq t\geqq s_{N^*}$, which implies that

$$\int_{x(s_{N^*})}^{x(t^*)} h(u)du<0$$

by (9). Hence we have $r(t^*)x'(t^*)>-\varepsilon$ by (13). Since $x'(t^*)>-\varepsilon/k$ by (7), we have a contradiction. Thus $\liminf_{t\to\infty} x'(t)\geqq 0$, and therefore $\lim_{t\to\infty} x'(t)=0$.

The following theorem is the case where $a(t)$ changes the sign.

Theorem 2. *Assume that the conditions* (3), (4) *and* (7) *in Lemma hold and the following conditions are satisfied*;

(a) *For* $t\geqq 0$ *and* $x\geqq 0$, *there exist a continuous function* $a(t)$ *and an* $\alpha(x)$ *such that* $x\alpha(x)>0$ $(x>0)$ *and*

$$a(t)\alpha(x)\leqq f(t,x,y) \tag{14}$$

for all large $t, x\geqq 0$ *and* $|y|<\infty$.

(b) *For* $t\geqq 0$ *and* $x\leqq 0$, *there exist a continuous function* $b(t)$ *and a* $\beta(x)$ *such that* $x\beta(x)>0$ $(x<0)$ *and*

$$f(t,x,y)\leqq b(t)\beta(x) \tag{15}$$

for all large $t, x\leqq 0$ *and* $|y|<\infty$.

Moreover, suppose the following conditions are satisfied;

(c) *If* $\{t_n\}$ *is a sequence with* $0<\cdots<t_n<t_{n+1}<\cdots\to\infty$, *then*

$$\sum_{n=0}^{\infty}\int_{t_n}^{t_n+\varepsilon}\{a(s)\}^{+}ds=\infty \tag{16}$$

and

$$\sum_{n=0}^{\infty}\int_{t_n}^{t_n+\varepsilon}\{b(s)\}^{+}ds=\infty \tag{17}$$

for any $\varepsilon>0$.

(d)

$$\int_0^{\infty}\{a(s)\}^{-}ds>-\infty \tag{18}$$

and

$$\int_0^{\infty}\{b(s)\}^{-}ds>-\infty. \tag{19}$$

Then a bounded nonoscillatory solution $x(t)$ *of* (2) *satisfies* $\lim_{t\to\infty} x(t)=0$. *In particular, if the condition* (8) *holds, then the condition* (c) *is replaced by the condition that*

(c)′ *there exists an* $\varepsilon>0$ *such that if* $\{t_n\}$ *is a sequence with* $0<\cdots<t_n<t_{n+1}<\cdots\to\infty$, *then* (16) *and* (17) *hold.*

Proof. Lex $x(t)$ be a solution of (2) satisfying $0<x(t)\leqq L$ for some

positive constant L and for all $t\in[\sigma,\infty)$, where σ is sufficiently large. An analogous argument will hold if $x(t)$ is a negative bounded solution.

First, we shall show that $\liminf_{t\to\infty} x(t)=0$. Suppose not, then there exists an $\tilde{m}>0$ such that $x(t)\geqq\tilde{m}$ for all $t\geqq\sigma$, and hence there exists an $m>0$ such that $\alpha(x(t))\geqq m$ for all $t\geqq\sigma$. It follows from (14) and (16) that

$$\int_\sigma^\infty\{f(s,x(s),x'(s))\}^+ds\geqq\int_\sigma^\infty\{a(s)\alpha(x(s))\}^+ds\geqq m\int_\sigma^\infty\{a(s)\}^+ds=\infty,$$

which contradicts (6) in Lemma, because

$$\int_\sigma^\infty\{f(s,x(s),x'(s))\}^-ds\geqq M\int_\sigma^\infty\{a(s)\}^-ds>-\infty$$

by (14) and (18), where $M=\sup_{0\leqq x\leqq L}\alpha(x)$.

Next, we shall show that $\limsup_{t\to\infty} x(t)=0$.

(i) The case where conditions (4) and (c) hold. Suppose that $\limsup_{t\to\infty} x(t)\neq0$. Then there are sequences $\{t_n\}$ and $\{r_n\}$ and a constant $K>0$ such that $\cdots<t_n<r_n<t_{n+1}<\cdots\to\infty$ as $n\to\infty$, $x(t_n)=K/2$, $x(r_n)=K$ and $K/2\leqq x(t)\leqq K$ for $t_n\leqq t\leqq r_n$. By Lemma, $x'(t)$ is bounded above, and hence there exists an $\varepsilon>0$ such that $r_n-t_n>\varepsilon$. Thus we have

$$x(t)\geqq K/2 \quad\text{on}\quad t_n+\varepsilon\geqq t\geqq t_n \tag{20}$$

for $n=N,N+1,N+2,\cdots$, where $t_N\geqq\sigma$. It follows from (14), (16) and (20) that

$$\begin{aligned}\int_{t_N}^\infty\{f(s,x(s),x'(s))\}^+ds&\geqq\int_{t_N}^\infty\{a(s)\alpha(x(s))\}^+ds\\&\geqq M'\cdot\sum_{n=N}^\infty\int_{t_n}^{t_n+\varepsilon}\{a(s)\}^+ds\\&\geqq\infty,\end{aligned}$$

where $M'=\inf_{K/2\leqq x\leqq L}\alpha(x)$, which contradicts (6) in Lemma.

(ii) The case where conditions (8) and (c)$'$ hold. Suppose that $\limsup_{t\to\infty} x(t)\neq0$. Then, there are sequences $\{s_n\}$, $\{t_n\}$ and $\{r_n\}$ and a constant $K>0$ such that $s_n\to\infty$ and $x(s_n)\to0$ as $n\to\infty$, $x'(s_n)=0$, $x(t_n)=K/2$, $\cdots<r_n<t_{n+1}<r_{n+1}<\cdots$, $x(r_n)=K$ and $K/2\leqq x(t)\leqq K$ for $t_n\leqq t\leqq r_n$. By Lemma, there exists an $N>0$ such that $x'(t)\leqq K/2\varepsilon$ for $t\geqq t_N$, where ε is the one given in (c)$'$, which implies that $r_n\geqq t_n+\varepsilon$. Thus we have (20). Therefore, by the same argument as in the proof of the case (i), we have a contradiction.

Thus Theorem is proved.

The following theorem is the extension of Grimmer's theorem.

Theorem 3. *Suppose that the conditions* (3), (4) *and* (7) *in Lemma and the conditions* (a), (b) *and* (c) *in Theorem* 2 *hold. Assume that* $a(t)\geqq 0$ *and* $b(t)\geqq 0$ *on* I,

$$\liminf_{x\to\infty}\alpha(x)>0 \tag{21}$$

and

$$\limsup_{x\to-\infty}\beta(x)<0. \tag{22}$$

Then a nonoscillatory solution $x(t)$ *of* (2) *satisfies* $\lim_{t\to\infty}x(t)=0$. *In particular, if the condition* (8) *holds, then the condition* (c) *is replaced by the condition* (c)$'$.

In the proof of Theorem 2, the boundedness of a solution $x(t)$ is used only in showing the existence of constants m and M' and the inequality

$$\int_\sigma^\infty \{f(s,x(s),x'(s))\}^- ds>-\infty.$$

We can easily show the existence of constants m and M' by (21), and it is clear that

$$\int_\sigma^\infty \{f(s,x(s),x'(s))\}^- ds\geqq 0,$$

because $a(t)\geqq 0$, and hence Theorem 3 is proved by the same argument as in the proof of Theorem 2.

We shall consider the homogeneous system.

Theorem 4. *Assume that*

$$f(t,x,y)=a(t)F(x,y), \tag{23}$$

where $a(t)$ *is a nonnegative continuous function on* I *and* $F(x,y)$ *is a continuous function defined on* $R\times R$ *and satisfies*

$$x\cdot F(x,y)>0 \quad (x\neq 0) \tag{24}$$

and

$$F(\lambda x,\lambda y)=\lambda^{2q+1}F(x,y) \tag{25}$$

for every $(x,y)\in R^2$, $\lambda\in R$ *and some nonnegative integer* q. *Suppose that the conditions* (3), (7), (8) *and* (9) *in Lemma and the condition* (c)$'$ *in Theorem* 2 *hold.*

Then a nonoscillatory solution $x(t)$ *of* (2) *satisfies* $\lim_{t\to\infty}x(t)=0$.

Proof. Let a solution $x(t)$ of (2) be positive on $[\sigma,\infty)$, where σ is sufficiently large. An analogous argument will hold if $x(t)<0$ for all $t\geqq\sigma$.

First, we shall show that $\liminf_{t\to\infty} x(t)=0$. Suppose not, then there exists an $m>0$ such that $x(t)>m$ for all $t\geqq\sigma$. By Lemma, $x'(t)$ is bounded for all $t\geqq\sigma$. Hence there exists an $M>0$ by (24) such that

$$\min_{t\geqq\sigma} F(1, x'(t)/x(t))=M. \tag{26}$$

It follows from (16), (23), (25) and (26) that

$$\begin{aligned}\int_\sigma^\infty \{f(s, x(s), x'(s))\}^+ ds &= \int_\sigma^\infty a(s)F(x(s), x'(s))ds\\ &= \int_\sigma^\infty a(s)x(s)^{2q+1}F(1, x'(s)/x(s))ds\\ &\geqq m^{2q+1}\cdot M\cdot\int_\sigma^\infty a(s)ds=\infty,\end{aligned}$$

which contradicts (6) in Lemma.

Next, we shall show that $\limsup_{t\to\infty} x(t)=0$. Suppose not. Then, by the same argument as in the proof of Theorem 2, there exist a sequence $\{t_n\}$ and numbers K and N which satisfy the condition (20) for an $\varepsilon>0$ given in (c)$'$. Hence there exists an $\tilde{M}>0$ such that

$$\min_{t_n\leqq t\leqq t_n+\varepsilon} F(1, x'(t)/x(t))=\tilde{M} \tag{27}$$

for $n=N, N+1, N+2, \cdots$. It follows from (16), (23), (25) and (27) that

$$\begin{aligned}\int_{t_N}^\infty \{f(s, x(s), x'(s))\}^+ ds &= \int_{t_N}^\infty a(s)F(x(s), x'(s))ds\\ &= \int_{t_N}^\infty a(s)x(s)^{2q+1}F(1, x'(s)/x(s))ds\\ &\geqq \sum_{n=N}^\infty \int_{t_n}^{t_n+\varepsilon} a(s)x(s)^{2q+1}F(1, x'(s)/x(s))ds\\ &\geqq (K/2)^{2q+1}\cdot\tilde{M}\cdot\sum_{n=N}^\infty\int_{t_n}^{t_n+\varepsilon} a(s)ds\\ &\geqq\infty,\end{aligned}$$

which contradicts (6) in Lemma. This proves Theorem.

The following theorem contains the case where $r(t)\to 0$ and $r(t)a(t)\to 0$ as $t\to\infty$.

Theorem 5. *Assume that* $h(x)=0$ *on* R *and that the conditions* (3) *and* (4) *in Lemma and the conditions* (a) *and* (b) *in Theorem* 2 *hold. If* $a(t)\geqq 0$ *and* $b(t)\geqq 0$ *on* I, $\alpha(x)$ *and* $\beta(x)$ *are nondecreasing and there exists a function* $\varphi(t)\in C^2([0,\infty), R)$ *such that for a* $\sigma\geqq 0$

$$\varphi(t)>0 \quad on \quad [\sigma,\infty), \tag{28}$$

$$(r(t)\varphi'(t))' > |g(t)| \quad \text{on} \quad [\sigma, \infty) \tag{29}$$

and

$$\int_{\sigma}^{\infty} a(s)\alpha(\varphi(s))ds = \infty, \quad \int_{\sigma}^{\infty} b(s)\beta(\varphi(s))ds = \infty, \tag{30}$$

then a nonoscillatory solution $x(t)$ *of* (2) *satisfies* $|x(t)| \leqq \varphi(t)$ *for all large* t. *Moreover, if* $\varphi(t) \to 0$ *as* $t \to \infty$, *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a solution of (2) satisfying $x(t) > 0$ for $t \geqq T$, where we may assume that $T \geqq \sigma$. An analogous argument will hold if $x(t)$ is negative. By (2), (14) and (29), we have

$$(r(t)x'(t))' \leqq -a(t)\alpha(x(t)) + |g(t)| < (r(t)\varphi'(t))' \tag{31}$$

for all $t \geqq T$. Since $r(t) > 0$ on I, it follows from (31) that either $x(t) \geqq \varphi(t)$ for all large t or $x(t) \leqq \varphi(t)$ for all large t. However, if $x(t) \geqq \varphi(t)$ for all $t \geqq T_1$, $T_1 \geqq T$, then we have

$$\begin{aligned}\int_{T_1}^{\infty} \{f(s, x(s), x'(s))\}^{+} ds = \int_{T_1}^{\infty} f(s, x(s), x'(s))ds &\geqq \int_{T_1}^{\infty} a(s)\alpha(x(s))ds \\ &\geqq \int_{T_1}^{\infty} a(s)\alpha(\varphi(s))ds \geqq \infty\end{aligned}$$

by (28) and (30), which contradicts (6) in Lemma. Thus Theorem is proved.

Example. Consider the differential equation

$$(t^{-1}x')' + x^5 = 3t^{-4} + t^{-5}. \tag{32}$$

Put $\varphi(t) = (1/t)^{1/5}$, then it is easily seen that $\varphi(t)$ satisfies the required conditions in Theorem 5, and nonoscillatory solutions of (32) tends to zero if t tends to infinity. In fact, $x(t) = 1/t$ is a nonoscillatory solution of (32).

References

[1] R. Grimmer, On nonoscillatory solutions of nonlinear differential equation, Proc. Amer. Math. Soc., 34 (1972), 118-120.

[2] M. E. Hammett, Nonoscillatory property of a nonlinear differential equation, Proc. Amer. Math. Soc., 30 (1971), 92-96.

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