

## On Some Perturbation of the Navier-Stokes Equations in $L^p$ Spaces

By Hitoshi ISHII

(Waseda University)

### 1. Introduction.

In [5], Temam proposed a perturbation of the Navier-Stokes equations to avoid the difficulty in numerical analysis caused by the condition of divergence free. Indeed, he showed there that as  $\varepsilon \rightarrow 0$ , the solution  $u_\varepsilon$  of the Cauchy problem for the equation with a parameter  $\varepsilon > 0$ ,

$$(1.1) \quad \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + \left( u_\varepsilon \cdot \text{grad} + \frac{1}{2} \text{div} u_\varepsilon \right) u_\varepsilon - \frac{1}{\varepsilon} \text{grad}(\text{div} u_\varepsilon) = 0,$$

with the initial data  $u_0 \in L^2(\mathbb{R}^n)$ ,  $\text{div} u_0 = 0$ , converges to the solution  $u$  (in the Leray-Hopf class) of the Cauchy problem for the Navier-Stokes equations,

$$(1.2) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \text{grad} u + \text{grad} P = 0 & (x, t) \in \mathcal{Q}_T = \mathbb{R}^n \times (0, T) \\ \text{div} u(x, t) = 0 \\ u(x, 0) = u_0(x), \quad \text{div} u_0 = 0, \end{cases}$$

where  $P$  is the pressure function.

On the other hand, Fabes, Jones and Riviere [1] proved the existence and uniqueness theorem in some  $L^q(0, T; L^p(\mathbb{R}^n))$  spaces,  $p, q \geq 2$ , for the Cauchy problem (1.2) with the initial data in  $L^r(\mathbb{R}^n)$ ,  $r > n$ . One asks naturally whether the solution  $u_\varepsilon$  of the equation (1.1) converges to the solution  $u$  of the problem (1.2) in the function spaces  $L^q(0, T; L^p(\mathbb{R}^n))$ . It, however, seems (perhaps technically) difficult to discuss the problem. The artificial term  $(1/2)(\text{div} u_\varepsilon)u_\varepsilon$  in the equation (1.1) is not suited for the discussion in the  $L^q(0, T; L^p)$  framework and should rather be replaced by  $(\text{div} u_\varepsilon)u_\varepsilon$ .

In this note, we shall consider the Cauchy problem for the equation with a parameter  $\varepsilon > 0$ ,

$$(1.3) \quad \begin{cases} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + (u_\varepsilon \cdot \text{grad} + \text{div} u_\varepsilon)u_\varepsilon - \frac{1}{\varepsilon} \text{grad}(\text{div} u_\varepsilon) = 0, & (x, t) \in \mathcal{Q}_T \\ u_\varepsilon(x, 0) = u_0(x), \end{cases}$$

and show that the solution  $u_\varepsilon$  of the problem (1.3) with the initial value  $u_0$  in  $L^r(\mathbb{R}^n)$ ,  $r > n$ , satisfying  $\text{div} u_0 = 0$  converges in  $L^q(0, T; L^p(\mathbb{R}^n))$  to the solution  $u$  of the problem (1.2) if  $1 \geq \frac{n}{p} + \frac{2}{q} > \frac{n}{r}$ ,  $n < r \leq p < \infty$  and  $T (> 0)$  is sufficiently

small. Here and later  $L^r(R^n)$  denotes the space of all measurable functions  $u = (u_1, \dots, u_n)$  for which

$$\|u\|_r = \sum_{i=1}^n \left( \int_{R^n} |u_i(x)|^r dx \right)^{1/r} < \infty,$$

and  $L^q(a, b; L^p(R^n))$  denotes the space of all measurable functions  $u = (u_1, \dots, u_n)$  satisfying

$$\|u\|_{L^q(a, b; L^p)} = \sum_{i=1}^n \left\{ \int_a^b \left( \int_{R^n} |u_i(x, t)|^p dx \right)^{q/p} dt \right\}^{1/q} < \infty$$

(when  $a=0$ , we simply write this space as  $L^{q,p}(Q_b)$ ).

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## 2. Weak solutions of the problem (1.3).

To study the problem (1.3) in  $L^{q,p}(Q_T)$  spaces,  $p, q \geq 2$ , we make use of the concept of weak solutions.

**Definition.** A function  $u_\varepsilon$  is called a weak solution of the problem (1.3) if  $u_\varepsilon \in L^{q,p}(Q_T)$  and if for any  $\varphi \in C_0^\infty(R^n \times (-\infty, T))$ , the equation

$$(2.1) \quad \int_0^T \int_{R^n} \langle u_\varepsilon, \frac{\partial \varphi}{\partial t} + \Delta \varphi + (J\varphi)u_\varepsilon + \frac{1}{\varepsilon} \text{grad}(\text{div} \varphi) \rangle dx dt \\ + \int_{R^n} \langle u_0(x), \varphi(x, 0) \rangle dx = 0$$

holds. Here  $J\varphi$  denotes the Jacobian matrix, i. e.,  $J\varphi = \left( \frac{\partial \varphi_i}{\partial x_j} \right)_{1 \leq i, j \leq n}$ .

To obtain an equivalent integral equation to the functional equation (2.1), we first seek for the solutions  $e_j^\varepsilon = (e_{1j}^\varepsilon, \dots, e_{nj}^\varepsilon)$ ,  $1 \leq j \leq n$ , of

$$(2.2) \quad \frac{\partial e_j^\varepsilon}{\partial t} - \Delta e_j^\varepsilon - \frac{1}{\varepsilon} \text{grad}(\text{div} e_j^\varepsilon) = \alpha_j \delta(x, t), \quad (x, t) \in R^{n+1}, \quad \varepsilon > 0,$$

where  $\delta$  and  $\alpha_j$ ,  $1 \leq j \leq n$ , stand for the Dirac measure at the origin and the unit vectors whose  $j$ -th components equal to 1, respectively. With the aid of the Fourier transform with respect to the variables  $x$ , we have

$$\frac{d\hat{e}_j^\varepsilon}{dt} + |y|^2 \hat{e}_j^\varepsilon + \frac{1}{\varepsilon} A(y) \hat{e}_j^\varepsilon = \alpha_j \delta(t)$$

where  $A(y)$  denotes the matrix  $(y_i y_j)_{1 \leq i, j \leq n}$ , and where we write  $\hat{e}_j^\varepsilon$  for the Fourier transform of  $e_j^\varepsilon$ . Putting  $E^\varepsilon = (e_{ij}^\varepsilon)_{1 \leq i, j \leq n}$  and solving the system of ordinary differential equations, we get

$$\hat{E}^\varepsilon(y, t) = \exp\left(-|y|^2 I - \frac{1}{\varepsilon} A(y)\right) t \quad \text{for } t > 0.$$

By an elementary calculation,

$$\hat{E}^\varepsilon(y, t) = e^{-|y|^2 I} + \frac{1}{|y|^2} \{e^{-(1+1/\varepsilon)|y|^2 t} - e^{-|y|^2 t}\} A(y) \quad \text{for } t > 0.$$

Therefore,

$$\hat{e}_{ij}^\varepsilon(y, t) = \delta_{ij} e^{-|y|^2 t} - t y_i y_j \int_0^{1/\varepsilon} e^{-(1+\lambda)|y|^2 t} d\lambda, \quad t > 0,$$

and hence

$$(2.3) \quad e_{ij}^\varepsilon(x, t) = \delta_{ij} \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}} + t \int_0^{1/\varepsilon} D_i D_j \frac{e^{-|x|^2/4(1+\lambda)t}}{[4\pi(1+\lambda)t]^{n/2}} d\lambda, \quad t > 0,$$

where  $\delta_{ij}$  and  $D_i$ ,  $1 \leq i, j \leq n$ , stand for the Kronecker's delta and  $\partial/\partial x_i$ , respectively. It follows immediately from this representation that

$$(2.4) \quad e_{ij}^\varepsilon \in C^\infty(R^n \times (0, \infty)),$$

$$(2.5) \quad e_{ij}^\varepsilon(\cdot, t) \in \mathcal{S} \text{ for each } t > 0 \text{ and}$$

$$(2.6) \quad e_{ij}^\varepsilon, D_k e_{ij}^\varepsilon \in L^1(R^n \times (0, T)) \text{ for } T, 0 < T < \infty \text{ and } k = 1, \dots, n.$$

We also have

$$(2.7) \quad E^{\varepsilon*} g(x) = \int_{R^n} E^\varepsilon(x-y, t) g(y) dy \rightarrow g(x) \text{ in } L^p(R^n), \\ 1 \leq p < \infty, \text{ as } t \rightarrow +0$$

by the well-known property of the Weierstrass kernel and the representation (2.3) of  $e_{ij}^\varepsilon$ .

Let us define the bilinear operator  $B_\varepsilon$  by

$$B_\varepsilon(u, v)(x, t) = \int_0^t \int_{R^n} [(u(y, s) \cdot \text{grad}) E^\varepsilon(x-y, t-s)] v(y, s) dy ds$$

for  $(x, t) \in Q_T$  and for  $u, v \in L^{q, p}(Q_T)$ ,  $2 \leq p, q \leq \infty$ , where  $(u \cdot \text{grad}) E^\varepsilon$  denotes the matrix

$$\left( \sum_{k=1}^n u_k D_k e_{ij}^\varepsilon \right)_{1 \leq i, j \leq n}.$$

By (2.6), it is easily seen that the operator  $B_\varepsilon$  is continuous from  $L^{q, p}(Q_T) \times L^{q, p}(Q_T)$  to  $L^{q/2, p/2}(Q_T)$  for  $p, q \geq 2$ . Using this property and the properties (2.4), (2.5), (2.6) and (2.7), we obtain

**Theorem 2.1.** *Let  $u_0 \in L^r(R^n)$ ,  $1 \leq r < \infty$ . A function  $u_\varepsilon \in L^{q, p}(Q_T)$ ,  $0 < T < \infty$ ,  $p, q \geq 2$ ,  $p < \infty$ , is a weak solution of the problem (1.3) if and only if  $u_\varepsilon$*

satisfies the integral equation

$$(2.10) \quad u_\varepsilon + B_\varepsilon(u_\varepsilon, u_\varepsilon) = E^\varepsilon * u_0 \quad \text{for } (x, t) \in Q_T.$$

We can prove this theorem in the same way as the proof of a similar theorem for the Navier-Stokes equations (Theorem (2.1) in [1]) and omit the proof.

### 3. Existence and uniqueness theorem for the problem (3.1).

We begin with a result concerning the continuity of  $B_\varepsilon$ .

**Lemma 3.1.** *If  $\frac{n}{p} + \frac{2}{q} \leq 1$  and  $n < p \leq \infty$ , then the inequality*

$$(3.1) \quad \|B_\varepsilon(u, v)\|_{L^{q,p}(Q_T)} \leq C(n, p, q) T^{\frac{1}{2}\left(1 - \frac{n}{p} - \frac{2}{q}\right)} \|u\|_{L^{q,p}(Q_T)} \|v\|_{L^{q,p}(Q_T)}$$

holds for  $u, v \in L^{q,p}(Q_T)$ . The constant  $C(n, p, q)$  can be taken to be independent of  $\varepsilon$ .

To prove this lemma, we need the next.

**Lemma 3.2.** (Hardy-Littlewood-Sobolev theorem). *Let  $g \in L^r(0, T)$ ,  $1 < r < \infty$ ,  $0 < T \leq \infty$ , and for  $0 < \alpha < \frac{1}{r}$ , define*

$$H_\alpha g(t) = \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad 0 < t < T.$$

If  $\frac{1}{r} - \frac{1}{q} = \alpha$ , then

$$\|H_\alpha g\|_q \leq C(q, r) \|g\|_r.$$

In the above,  $\|\cdot\|_q$  ( $\|\cdot\|_r$ ) stands for the norm of the space  $L^q(0, T)$  ( $L^r(0, T)$ ). For the proof, we refer to [4].

**Proof of Lemma 3.1.** Differentiating the equation (2.3), we get

$$(3.2) \quad |D_k e_{ij}^\varepsilon(x, t)| \leq Ct^{-\frac{n+1}{2}} \left[ h_1\left(\frac{x}{\sqrt{t}}\right) + \int_0^{1/\varepsilon} (1+\lambda)^{-\frac{n+3}{2}} h_2\left(\frac{x}{\sqrt{(1+\lambda)t}}\right) d\lambda \right],$$

where  $h_1(x) = |x|e^{-\frac{1}{4}|x|^2}$  and  $h_2(x) = |x|e^{-\frac{1}{4}|x|^2} + |x|^3 e^{-\frac{1}{4}|x|^2}$  for  $x \in R^n$ . Hence, for  $1 \leq r \leq \infty$ ,

$$\begin{aligned} \|D_k e_{ij}^\varepsilon(\cdot, t)\|_r &\leq Ct^{-\frac{n+1}{2} + \frac{n}{2r}} \left[ \|h_1\|_r + \|h_2\|_r \int_0^{1/\varepsilon} (1+\lambda)^{-\frac{n+3}{2} + \frac{n}{2r}} d\lambda \right] \\ &\leq C't^{-\frac{n+1}{2} + \frac{n}{2r}}. \end{aligned}$$

Since  $-\frac{n+3}{2} + \frac{n}{2r} < -1$ , the above constant can be taken independently of  $\varepsilon$ .

Putting  $r = \frac{p}{p-1}$ , i. e.,  $\frac{1}{p} = \frac{2}{p} + \frac{1}{r} - 1$ , by the Young's inequality, we get

$$\begin{aligned} \|B_\varepsilon(u, v)(\cdot, t)\|_p &\leq C \int_0^t (t-s)^{-\frac{n+1}{2} + \frac{n}{2r}} \| |u(\cdot, s)| |v(\cdot, s)| \|_{p/2} ds \\ &\leq C \int_0^t (t-s)^{-\frac{n+1}{2} + \frac{n}{2r}} \|u(\cdot, s)\|_p \|v(\cdot, s)\|_p ds \end{aligned}$$

In case  $\frac{n}{p} + \frac{2}{q} = 1$ , setting  $\alpha = \frac{2}{q} - \frac{1}{q}$ , we apply Lemma 3.2 to obtain

$$\|B_\varepsilon(u, v)\|_{L^q, p(Q_T)} \leq C \|u\|_{L^q, p(Q_T)} \|v\|_{L^q, p(Q_T)}.$$

In case  $\frac{n}{p} + \frac{2}{q} < 1$ , using the Young's inequality, we have

$$\|B_\varepsilon(u, v)\|_{L^q, p(Q_T)} \leq C T^{\frac{1}{2} \left(1 - \frac{n}{p} - \frac{2}{q}\right)} \|u\|_{L^q, p(Q_T)} \|v\|_{L^q, p(Q_T)}.$$

This completes the proof.

Next we turn to the continuity of  $E^\varepsilon$ .

**Lemma 3.3.** *Let  $g \in L^r(R^n)$ ,  $1 \leq r < \infty$ . If  $q \geq 1$ ,  $r < p$  and  $\frac{n}{p} + \frac{2}{q} > \frac{n}{r}$ , then the inequality*

$$(3.3) \quad \|E^\varepsilon * g\|_{L^q, p(Q_T)} \leq C(n, p, q, r) T^{\frac{1}{2} \left(\frac{n}{p} + \frac{2}{q} - \frac{n}{r}\right)} \|g\|_r$$

holds for some constant  $C(n, p, q, r)$  independent of  $\varepsilon$ .

**Proof.** By the representation (2.3), it is easy to see that

$$|e_{ij}^\varepsilon(x, t)| \leq C t^{-\frac{n}{2}} \left[ k_1 \left( \frac{x}{\sqrt{t}} \right) + \int_0^{1/\varepsilon} (1+\lambda)^{-\frac{n}{2}-1} k_2 \left( \frac{x}{\sqrt{(1+\lambda)t}} \right) d\lambda \right],$$

where  $k_1(x) = e^{-\frac{1}{4}|x|^2}$  and  $k_2(x) = e^{-\frac{1}{4}|x|^2} + |x|^2 e^{-\frac{1}{4}|x|^2}$ . Therefore  $1 < s \leq \infty$ , we get

$$\begin{aligned} \|e_{ij}^\varepsilon(\cdot, t)\|_s &\leq C t^{-\frac{n}{2} + \frac{n}{2s}} \left[ \|k_1\|_s + \|k_2\|_s \int_0^{1/\varepsilon} (1+\lambda)^{-\frac{n}{2}-1 + \frac{n}{2s}} d\lambda \right] \\ &\leq C' t^{-\frac{n}{2} + \frac{n}{2s}}, \end{aligned}$$

where  $C'$  is independent of  $\varepsilon$ . Setting  $\frac{1}{p} = \frac{1}{s} + \frac{1}{r} - 1$ , we have  $1 < s \leq \infty$  by the assumption on  $p$  and  $r$ , and

$$\|E^\varepsilon * g(\cdot, t)\|_p \leq C t^{-\frac{n}{2} + \frac{n}{2s}} \|g\|_r.$$

Since  $q\left(1-\frac{1}{s}\right) < \frac{2}{n}$ , we get

$$\|E^\varepsilon * g\|_{L^{q,p}(Q_T)} \leq CT^{\frac{1}{2}\left(\frac{n}{p} + \frac{2}{q} - \frac{n}{r}\right)} \|g\|_r,$$

which completes the proof.

We are now in a position to state the existence and uniqueness theorem.

**Theorem 3.4.** *If  $1 \geq \frac{n}{p} + \frac{2}{q} > \frac{n}{r}$  and  $r < p < \infty$ , then there exists a positive constant  $C_0 = C_0(n, p, q, r)$  such that the problem (1.3) has only one weak solution  $u_\varepsilon$  in  $L^{q,p}(Q_T)$  for any  $u_0$  and  $T > 0$  which satisfy*

$$(3.4) \quad T^{\frac{1}{2}\left(1-\frac{n}{r}\right)} \|u_0\|_r \leq C_0.$$

Since the method employed to prove the existence and uniqueness theorem for the Navier-Stokes equations by Fabes, Jones and Riviere [1] applies also to the proof of this theorem, we only sketch it.

**Proof.** In view of Theorem 2.1 and Lemma 3.3, it is sufficient to prove that, for some constant  $C_1 = C_1(n, p, q) > 0$ , the integral equation

$$u_\varepsilon + B_\varepsilon(u_\varepsilon, u_\varepsilon) = f$$

has a unique solution  $u_\varepsilon$  in  $L^{q,p}(Q_T)$  for any  $f \in L^{q,p}(Q_T)$  and  $T > 0$  satisfying

$$T^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|f\|_{L^{q,p}(Q_T)} < C_1.$$

Setting  $v_0^\varepsilon = f$ , we define  $v_m^\varepsilon$  by

$$v_m^\varepsilon = -B_\varepsilon(v_{m-1}^\varepsilon, v_{m-1}^\varepsilon) + f, \quad m=1, 2, 3, \dots$$

By Lemma 3.1, we have

$$\|v_m^\varepsilon\|_{L^{q,p}(Q_T)} \leq CT^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|v_{m-1}^\varepsilon\|_{L^{q,p}(Q_T)}^2 + \|f\|_{L^{q,p}(Q_T)}.$$

Let  $C_1 = (4C)^{-1}$ ,

$$(3.5) \quad \rho = T^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|f\|_{L^{q,p}(Q_T)} \left/ \left( 2C_1 - T^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|f\|_{L^{q,p}(Q_T)} \right) \right.$$

and

$$S_T = \{v \in L^{q,p}(Q_T); 2CT^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|v\|_{L^{q,p}(Q_T)} < \rho\}.$$

Then it is not hard to see that every  $v_m^\varepsilon (m \geq 0)$  belongs to  $S_T$  and the mapping  $v \rightarrow -B_\varepsilon(v, v) + f$  is a contraction mapping on  $S_T$  since  $\rho < 1$ . Therefore,  $u_\varepsilon = \lim_{m \rightarrow \infty} v_m^\varepsilon$

exists in  $S_T$  and  $u_\varepsilon + B_\varepsilon(u_\varepsilon, u_\varepsilon) = f$ .

We now turn to the uniqueness of the solution. We may assume  $q < \infty$ . Let  $u_\varepsilon$  and  $v_\varepsilon$  be two arbitrary solutions of (1.3) in  $L^{q,p}(Q_T)$  with the same data  $u_0$ , that is,  $u_\varepsilon + B_\varepsilon(u_\varepsilon, u_\varepsilon) = E^\varepsilon * u_0$  and  $v_\varepsilon + B_\varepsilon(v_\varepsilon, v_\varepsilon) = E^\varepsilon * u_0$ .

Since  $u_\varepsilon - v_\varepsilon = -[B_\varepsilon(u_\varepsilon, u_\varepsilon - v_\varepsilon) + B_\varepsilon(u_\varepsilon - v_\varepsilon, v_\varepsilon)]$ , we easily get

$$\|u_\varepsilon - v_\varepsilon\|_{L^{q,p}(Q_\eta)} \leq C(\|u_\varepsilon\|_{L^{q,p}(Q_\eta)} + \|v_\varepsilon\|_{L^{q,p}(Q_\eta)})\|u_\varepsilon - v_\varepsilon\|_{L^{q,p}(Q_\eta)}$$

for any  $\eta, 0 < \eta < T$ . Thus we obtain  $u_\varepsilon = v_\varepsilon$  in  $L^{q,p}(Q_\eta)$  for sufficiently small  $\eta$ . Repeating this argument, we finally have  $u_\varepsilon = v_\varepsilon$  in  $L^{q,p}(Q_T)$ . The proof is complete.

**Remark 3.5.** (i) If we take the data  $u_0$  as  $\operatorname{div} u_0 = 0$ , then the assertion of Theorem 3.4 holds also for the case  $r = p$ . In fact, the condition  $\operatorname{div} u_0 = 0$  means that  $\sum_{j=1}^n y_j \hat{u}_{0j}(y) = 0$  and, therefore,  $A(y) \hat{u}_0(y) = 0$ . Defining

$$E(x, t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}},$$

we see that  $E^\varepsilon * u_0 = E * u_0$  and the estimate (3.3) is valid for  $r = p$ , which implies the above assertion.

(ii) We shall need in section 4 the fact that for the initial data  $u_0$ ,  $T^{\frac{1}{2}} \left(1 - \frac{n}{r}\right) \|u_0\|_r < C_0$ , the weak solution  $u_\varepsilon$  of (1.3) satisfies the inequality

$$(3.6) \quad 2CT^{\frac{1}{2}} \left(1 - \frac{n}{p} - \frac{2}{q}\right) \|u_\varepsilon\|_{L^{q,p}(Q_T)} < \rho \quad (< 1)$$

for all  $\varepsilon > 0$  and with  $\rho$  defined by (3.5).

(iii) The restriction (3.4) on  $T$  and  $u_0$  is not necessary for the uniqueness part of the above theorem. We have actually proved that there can exist at most one weak solution in  $L^{q,p}(Q_T)$  of (1.3) for any  $u_0 \in L^r(R^n)$  and any  $T > 0$ .

We need the following lemma to obtain a result on the global existence of the weak solution of problem (1.3).

**Lemma 3.6.** Let  $g \in L^r(R^n)$ ,  $1 \leq r < \infty$ . If  $\frac{n}{p} + \frac{2}{q} < \frac{n}{r}$ , then

$$(3.7) \quad \|E^\varepsilon * g\|_{L^q(1, \infty; L^p)} \leq C(n, p, q, r) \|g\|_r.$$

**Proof.** If we put  $\frac{1}{p} = \frac{1}{s} + \frac{1}{r} - 1$ , then  $1 < s \leq \infty$  and by the same reasoning as the proof of Lemma 3.3, we get

$$\|e_{ij}^\varepsilon(\cdot, t)\|_s \leq Ct^{-\frac{n}{2} + \frac{n}{2s}}.$$

Since  $q \left(1 - \frac{1}{s}\right) > \frac{2}{n}$ , we get

$$\|E^\varepsilon * g\|_{L^q(1, \infty; L^p)} \leq C \|g\|_r.$$

We now obtain easily

**Theorem 3.7.** *Suppose  $\frac{n}{p} + \frac{2}{q} = 1$  and  $r_1 < n < r_2 < \infty$ . If both  $\|u_0\|_{r_1}$  and  $\|u_0\|_{r_2}$  is sufficiently small, then there exists a function  $u_\varepsilon$  in  $L^{q,p}(Q_\infty)$  such that for any  $T > 0$ ,  $u_\varepsilon$  is a unique weak solution of the problem (1.3) and for the constant  $C$  in Lemma 3.1,*

$$(3.8) \quad 2C\|u_\varepsilon\|_{L^{q,p}(Q_\infty)} < \rho, \quad 0 < \rho < 1.$$

**Remark 3.8.** If  $\operatorname{div} u_0 = 0$ , then the above statement is also true for  $p = r_2$ .

#### 4. Convergence of $u_\varepsilon$ to the solution of the Navier-Stokes equations.

We shall show in this section that the solution  $u_\varepsilon$  of the problem (1.3) converges to the solution  $u$  of the problem (1.2) in  $L^{q,p}(Q_T)$  as  $\varepsilon \rightarrow 0$ .

According to Fabes, Jones and Riviere [1], we say that a function  $u = (u_1, \dots, u_n)$  in  $L^{q,p}(Q_T)$ ,  $p, q \geq 2$ , is a weak solution of the problem (1.2) if and only if for any  $\varphi \in C_0^\infty(R^n \times (-\infty, T))$ ,

$$(4.1) \quad \int_0^T \int_{R^n} \langle u, \frac{\partial \varphi}{\partial t} + \Delta \varphi + (J\varphi)u \rangle dx dt + \int_{R^n} \langle u_0(x), \varphi(x, 0) \rangle dx = 0,$$

and for almost all  $t \in (0, T)$ ,  $\operatorname{div} u(x, t) = 0$  (in the sense of distributions).

Now we define  $e_{ij}^0$  by

$$\hat{e}_{ij}^0(y, t) = \left( \delta_{ij} - \frac{y_i y_j}{|y|^2} \right) e^{-|y|^2 t}.$$

Setting  $E^0(x, t) = (e_{ij}^0(x, t))_{1 \leq i, j \leq n}$ , we define

$$B_0(u, v)(x, t) = \int_0^t \int_{R^n} [(u(y, s) \cdot \operatorname{grad}) E^0(x - y, t - s)] v(y, s) dy ds$$

for  $u, v \in L^{q,p}(Q_T)$ ,  $p, q \geq 2$ . Note that the function  $e_{ij}^0$  is written explicitly as

$$(4.2) \quad e_{ij}^0(x, t) = \delta_{ij} \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}} + t \int_0^\infty D_i D_j \frac{e^{-|x|^2/4(1+\lambda)t}}{[4\pi(1+\lambda)t]^{n/2}} d\lambda.$$

We list some results obtained by Fabes, Jones and Riviere in [1], which we need in the following discussion.

**Theorem 4.1.** *Let  $u_0 \in L^r(R^n)$ ,  $1 \leq r < \infty$ , and  $\operatorname{div} u_0 = 0$ . Then, a function  $u \in L^{q,p}(Q_T)$ ,  $p, q \geq 2$ ,  $p < \infty$ , is a weak solution of the problem (1.2) if and only if  $u$  is a solution of the integral equation*

$$u + B_0(u, u) = E * u_0.$$

**Theorem 4.2.** *Suppose that  $1 \geq \frac{n}{p} + \frac{2}{q} > \frac{n}{r}$ ,  $r \leq p < \infty$ , and  $u_0 \in L^r(R^n)$  satis-*

fies  $\operatorname{div} u_0 = 0$ . There exists a constant  $C_2 = C_2(n, p, q, r) > 0$  such that  $T^{\frac{1}{2}(1-\frac{n}{r})} \|u_0\|_r < C_2$ , the problem (1.2) has a unique weak solution  $u$  in  $L^{q,p}(Q_T)$  which satisfies

$$(4.3) \quad 2CT^{\frac{1}{2}(1-\frac{n}{p}-\frac{2}{q})} \|u\|_{L^{q,p}(Q_T)} < 1,$$

where the  $C$  is same as in the estimate (3.6).

**Theorem 4.3.** Assume  $\frac{n}{p} + \frac{2}{q} = 1$  and  $r_1 < n < r_2 \leq p < \infty$ . If  $u_0 \in L^{r_1}(R^n) \cap L^{r_2}(R^n)$  and  $\operatorname{div} u_0 = 0$ , and if both  $\|u_0\|_{r_1}$  and  $\|u_0\|_{r_2}$  are sufficiently small, then there exists a function  $u(x, t)$  on  $R^n \times (0, \infty)$  which belongs to  $L^{q,p}(Q_\infty)$  and is a unique weak solution of the problem (1.2) for each  $T > 0$ . Moreover, the estimate

$$(4.4) \quad 2C\|u\|_{L^{q,p}(Q_\infty)} < 1$$

holds with the constant  $C$  in Lemma 3.1.

Now we state our theorem in the next.

**Theorem 4.4** Assume that  $1 \geq \frac{n}{p} + \frac{2}{q} > \frac{n}{r}$  and  $n < r \leq p < \infty$ . There exists a constant  $C_3 = C_3(n, p, q, r) > 0$  such that if

$$T^{\frac{1}{2}(1-\frac{n}{r})} \|u_0\|_r < C_3 \quad \text{and} \quad \operatorname{div} u_0 = 0,$$

the weak solution  $u_\varepsilon$  of the problem (1.3) converges in  $L^{q,p}(Q_T)$  to the weak solution  $u$  of the problem (1.2) as  $\varepsilon \rightarrow 0$  and the convergence rate is as follows:

$$\|u - u_\varepsilon\|_{L^{q,p}(Q_T)} = O\left(\varepsilon^{\frac{1}{2}\left(1+\frac{n}{p}\right)}\right) \quad \text{as } \varepsilon \rightarrow 0.$$

Before going into the proof, we set  $\tilde{e}_{ij}^\varepsilon = e_{ij}^0 - e_{ij}^\varepsilon$ ,  $\tilde{E}^\varepsilon = (\tilde{e}_{ij}^\varepsilon)_{1 \leq i, j \leq n}$ , and we define

$$\tilde{B}_\varepsilon(u, v)(x, t) = \int_0^t \int_{R^n} [(u(y, s) \cdot \operatorname{grad}) \tilde{E}^\varepsilon(x - y, t - s)] v(y, s) dy ds$$

for  $u, v \in L^{q,p}(Q_T)$  and  $0 < T < \infty$ . And we state

**Lemma 4.5.** If  $\frac{n}{p} + \frac{2}{q} \leq 1$  and  $n < p \leq \infty$ , then

$$(4.5) \quad \|\tilde{B}_\varepsilon(u, v)\|_{L^{q,p}(Q_T)} \leq C(T) \varepsilon^{\frac{1}{2}\left(1+\frac{n}{p}\right)} \|u\|_{L^{q,p}(Q_T)} \|v\|_{L^{q,p}(Q_T)}$$

for  $u, v \in L^{q,p}(Q_T)$  and for some constant  $C(T)$  depending on  $n, p, q$  and  $T$  (independent of  $\varepsilon$ ).

**Proof.** It follows from (2.3) and (4.2) that

$$\tilde{e}_{ij}^\varepsilon(x, t) = t \int_{1/\varepsilon}^{\infty} D_i D_j \frac{e^{-|x|^2/4(1+\lambda)t}}{[4\pi(1+\lambda)t]^{n/2}} d\lambda.$$

By the same way as the proof of Lemma 3.1, for a given  $r$ ,  $1 \leq r \leq \infty$ , we get

$$\begin{aligned} \|D_k \tilde{e}_{ij}^\varepsilon(\cdot, t)\|_r &\leq C t^{-\frac{n+1}{2} + \frac{n}{2r}} \|h_2\|_r \int_{1/\varepsilon}^{\infty} (1+\lambda)^{-\frac{n+3}{2} + \frac{n}{2r}} d\lambda \\ &\leq C'(t/\varepsilon)^{-\frac{n+1}{2} + \frac{n}{2r}}. \end{aligned}$$

Putting  $r = \frac{p}{p-1}$  and repeating the same argument as in the proof of Lemma 3.1, we obtain the estimate (4.5). The proof is complete.

**Proof of Theorem 4.4.** Let us set  $C_3 = \min(C_0, C_2)$  and assume that  $u_0$  and  $T(>0)$  satisfy the inequality

$$T^{\frac{1}{2}} \left(1 - \frac{n}{r}\right) \|u_0\|_r < C_3$$

and  $\operatorname{div} u_0 = 0$ . Then the problem (1.3) and (1.2) have the unique weak solutions  $u_\varepsilon$  and  $u$ , respectively, which satisfy

$$u_\varepsilon + B_\varepsilon(u_\varepsilon, u_\varepsilon) = E^\varepsilon * u_0 \quad \text{and} \quad u + B_0(u, u) = E * u_0.$$

It is clear from the definition of  $\tilde{B}_\varepsilon$  that  $B_0(u, u) = B_\varepsilon(u, u) + \tilde{B}_\varepsilon(u, u)$ . Since  $\operatorname{div} u_0 = 0$ , we get  $E^\varepsilon * u_0 = E * u_0$ . Therefore, we have

$$u_\varepsilon - u = B_\varepsilon(u - u_\varepsilon, u_\varepsilon) + B_\varepsilon(u, u - u_\varepsilon) + \tilde{B}_\varepsilon(u, u).$$

By Lemmas 3.1 and 4.5,

$$\begin{aligned} \|u_\varepsilon - u\|_{L^q, p(Q_T)} &\leq C T^{\frac{1}{2}} \left(1 - \frac{n}{p} - \frac{2}{q}\right) (\|u_\varepsilon\|_{L^q, p(Q_T)} + \|u\|_{L^q, p(Q_T)}) \|u_\varepsilon - u\|_{L^q, p(Q_T)} \\ &\quad + C(T) \varepsilon^{\frac{1}{2}} \left(1 + \frac{n}{p}\right) \|u\|_{L^q, p(Q_T)}^2 \end{aligned}$$

and by (3.6) and (4.3),

$$\|u_\varepsilon - u\|_{L^q, p(Q_T)} \leq \rho' \|u_\varepsilon - u\|_{L^q, p(Q_T)} + C(T) \varepsilon^{\frac{1}{2}} \left(1 + \frac{n}{p}\right) \|u\|_{L^q, p(Q_T)}^2$$

for some  $\rho', 0 < \rho' < 1$ . Thus we get

$$\|u_\varepsilon - u\|_{L^q, p(Q_T)} \leq \frac{1}{1 - \rho'} C(T) \varepsilon^{\frac{1}{2}} \left(1 + \frac{n}{p}\right) \|u\|_{L^q, p(Q_T)}^2.$$

This proves the theorem.

Note that Lemmas 3.1 and 4.5 hold also for  $T = \infty$  if  $\frac{n}{p} + \frac{2}{q} = 1$ . In view of (3.8) and (4.4), we obtain in the same way as above

**Theorem 4.6.** Suppose  $\frac{n}{p} + \frac{2}{q} = 1$  and  $r_1 < n < r_2 \leq p < \infty$ . If  $u_0 \in L^{r_1}(R^n)$

$\cap L^{r_2}(R^n)$ ,  $\operatorname{div} u_0 = 0$ , and the norms  $\|u_0\|_{r_1}$  and  $\|u_0\|_{r_2}$  are so small that the functions  $u_\varepsilon$  and  $u$  defined in Theorems 3.7 and 4.3 satisfy (3.8) and (4.4), respectively. Then,  $u_\varepsilon$  converges to  $u$  in  $L^{q,p}(Q_\infty)$  and

$$\|u_\varepsilon - u\|_{L^{q,p}(Q_\infty)} = O\left(\varepsilon^{\frac{1}{2}\left(1 + \frac{n}{p}\right)}\right) \quad \text{as } \varepsilon \rightarrow 0.$$

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nuna adreso:  
 Department of Applied Physics  
 Waseda University  
 Tokyo, Japan

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