Funkcialaj Ekvacioj, **17** (1974), 181-191

Functional Measures in Information Theory

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Abstract.

Daróczy has obtained non-additive entropy of type β after generalizing Kendall's functional equation. Differently from Kendall, Chaundy and McLeod have characterized Shannon's entropy through another functional equation. It is natural to expect Daróczy's entropy in terms of solutions of some generalized form of Chaundy and McLeod's functional equation. This problem has been tackled in this paper which gives rise to generalized functional entropies of which Daróczy's entropy is a very special case.

The study has been extended to the bivariate generalized functional equation. Here we get generalizations of Kerridge's inaccuracy and Kullback's information.

1. Introdution.

Generalizing Kendall's [6] functional equation Daróczy [3] formed the functional equation

$$(1.1)\qquad f(x)+(1-x)^{\beta}f\left(\frac{y}{1-x}\right)=f(y)+(1-y)^{\beta}f\left(\frac{x}{1-y}\right),\quad (\beta>0,\ \beta\neq 1).$$

Under the conditions $f(0)=f(1)$ and $f(1/2)=1$, its only continuous solution is

$$(1.2)\qquad f(x)=(2^{1-\beta}-1)^{-1}(x^{\beta}+(1-x)^{\beta}-1)=f_{\beta}(x)$$

Now given a probability distribution $P=(p_1,\cdots,p_n)$, $\sum_{i=1}^{n}p_i=1$, entropy of type β defined in term of continuous solution $f_{\beta}(x)$ of (1.1) is

$$(1.3)\qquad H_{\beta}(P)=\sum_{i=2}^{n}s_i^{\beta}f_{\beta}\left(\frac{p_i}{s_i}\right),\quad s_i=p_1+\cdots+p_i$$

$$(1.4)\qquad =(2^{1-\beta}-1)^{-1}\left(\sum_{i=1}^{n}p_i^{\beta}-1\right),\quad (\beta>0,\ \beta\neq 1)$$

An axiomatic characterization of quantity (1.4) has been given earlier by Havrda and Charvat [4].

Generalization of (1.1) in more than one variable has also been studied (refer [9], [10], [11]).

Chaundy and McLeod [2] considered the functional equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\sum_{j=1}^{n} f(y_j), \tag{1.5}$$

where $\sum_{i=1}^{m} x_i=\sum_{j=1}^{n} y_j=1, x_i\geq 0, y_j\geq 0$.

The only continuous solution of this functional equation turns out to be

$$f(x)=\begin{cases} Ax\log x, & \text{when } x\in(0,1) \\ 0 & \text{, when } x=0 \text{ or } 1, \end{cases} \tag{1.6}$$

where A is an arbitrary constant.

Generalization of (1.5) in two variables has also been studied by Kannappan [5].

In this paper we characterize functional entropies which include Daróczy's entropy $H_\beta(P)$ given in (1.4) as a special case. This characterization is done through a functional equation which is a generalization of (1.5).

Investigations have also been made by extending the functional equation in two variables. The eight solutions of the functional equation in two variables characterize quantities which in particular cases include Shannon's entropy, Kerridge's inaccuracy, Kullback's information and a quantity which is sum of entropy and inaccuracy.

In what follows we shall take $0\log 0=0\log(0/0)=0$ and $0^\alpha=(0/0)^\alpha=0$ for $\alpha>0$, also all logarithms shall be considered to the base 2.

2. A Functional Equation in One Variable.

Let f be a real valued continuous function defined in $[0,1]$, satisfying the functional equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\sum_{j=1}^{n} f(y_j)+C(\beta)\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i)f(y_j), \tag{2.1}$$

where $\sum_{i=1}^{m} x_i=\sum_{j=1}^{n} y_j=1, x_i\geq 0, y_j\geq 0$ and $C(\beta)$ is a constant which is a continuous function of the parameter β such that $C(1)=0$.

It would be seen that if $C(\beta)\equiv 0$, then (2.1) reduces to (1.5).

In the following theorem we obtain all the continuous solution of (2.1).

Theorem 2.1. *All the continuous solutions of the functional equation* (2.1) *may be put in the following form*

$$_kf^1_{h(\beta)}(x)=C(\beta)^{-1}(x^{1-k+h(\beta)}-x), \quad C(\beta)\neq 0 \tag{2.2}$$

and

$$_kf^2_{h(\beta)}(x)=C(\beta)^{-1}(x^{1+k-h(\beta)}-x), \quad C(\beta)\neq 0 \tag{2.3}$$

where $h(\beta)$ is a continuous function of the parameter β such that $h(1)=k$, k

being an arbitrary constant.

Note. Continuity at $x=0$ demands that for $\beta, 1-k+h(\beta)>0$ in (2.2) and $1+k-h(\beta)>0$ in (2.3).

Proof. Let p, q, r and s be positive integers such that $1\le r\le p$, $1\le s\le q$. Setting $m=p-r+1$, $n=q-s+1$,

$$x_i=\frac{1}{p},\ (i=1,2,\cdots,p-r),\ x_{p-r+1}=\frac{r}{p},$$
$$y_j=\frac{1}{q},\ (j=1,2,\cdots,q-s),\ y_{q-s+1}=\frac{s}{q},$$

in (2.1), we obtain

$$(2.4)\quad (p-r)(q-s)f\left(\frac{1}{pq}\right)+(p-r)f\left(\frac{s}{pq}\right)+(q-s)f\left(\frac{r}{pq}\right)+f\left(\frac{rs}{pq}\right)$$
$$=(p-r)f\left(\frac{1}{p}\right)+f\left(\frac{r}{p}\right)+(q-s)f\left(\frac{1}{q}\right)+f\left(\frac{s}{q}\right)$$
$$+C(\beta)\left[(p-r)f\left(\frac{1}{p}\right)+f\left(\frac{r}{p}\right)\right]\left[(q-s)f\left(\frac{1}{q}\right)+f\left(\frac{s}{q}\right)\right].$$

Multiplying by pq and putting $f(x)=x\,\phi(x)$, in (2.4), we obtain

$$(2.5)\quad (p-r)(q-s)\phi\left(\frac{1}{pq}\right)+s(p-r)\phi\left(\frac{s}{pq}\right)+r(q-s)\phi\left(\frac{r}{pq}\right)+rs\phi\left(\frac{rs}{pq}\right)$$
$$=q(p-r)\phi\left(\frac{1}{p}\right)+rq\,\phi\left(\frac{r}{p}\right)+p(q-s)\,\phi\left(\frac{1}{q}\right)+sp\,\phi\left(\frac{s}{q}\right)$$
$$+C(\beta)\left[(p-r)\phi\left(\frac{1}{p}\right)+r\phi\left(\frac{r}{p}\right)\right]\left[(q-s)\phi\left(\frac{1}{q}\right)+s\phi\left(\frac{s}{q}\right)\right].$$

Equation (2.5), when $r=s=1$ reduces to

$$(2.6)\qquad \phi\left(\frac{1}{pq}\right)=\phi\left(\frac{1}{p}\right)+\phi\left(\frac{1}{q}\right)+C(\beta)\,\phi\left(\frac{1}{p}\right)\phi\left(\frac{1}{q}\right).$$

However if we take $s=1$, leaving r undetermined (2.5) gives

$$q(p-r)\phi\left(\frac{1}{pq}\right)+rq\,\phi\left(\frac{r}{pq}\right)=q(p-r)\phi\left(\frac{1}{p}\right)+rq\,\phi\left(\frac{r}{p}\right)+pq\,\phi\left(\frac{1}{q}\right)$$
$$+C(\beta)\,q\phi\left(\frac{1}{q}\right)\left[(p-r)\phi\left(\frac{1}{p}\right)+r\phi\left(\frac{r}{p}\right)\right],$$

so that substituting value of $\phi(1/pq)$ from (2.6) this yields

$$(2.7)\qquad \phi\left(\frac{r}{pq}\right)=\phi\left(\frac{r}{p}\right)+\phi\left(\frac{1}{q}\right)+C(\beta)\,\phi\left(\frac{r}{p}\right)\phi\left(\frac{1}{q}\right),$$

Similarly putting $r=1$ and once again using (2.6), we obtain

$$(2.8)\qquad \phi\left(\frac{s}{pq}\right)=\phi\left(\frac{1}{p}\right)+\phi\left(\frac{s}{q}\right)+C(\beta)\,\phi\left(\frac{1}{p}\right)\phi\left(\frac{s}{q}\right).$$

Finally, (2.5) together with (2.6), (2.7) and (2.8) yields

$$\phi\left(\frac{rs}{pq}\right)=\phi\left(\frac{r}{p}\right)+\phi\left(\frac{s}{q}\right)+C(\beta)\,\phi\left(\frac{r}{p}\right)\phi\left(\frac{s}{q}\right), \tag{2.9}$$

$$\text{i.e.}\quad \phi(xy)=\phi(x)+\phi(y)+C(\beta)\,\phi(x)\,\phi(y),$$

for all rational numbers $x, y\in(0,1]$. Invoking now the continuity of f we find that (2.9) is valid for all real numbers $x, y\in(0,1]$.

We wish to transform the functional equation (2.9) to a standard form for that we set

$$1+C(\beta)\phi(x)=g(x),\quad C(\beta)\neq 0.$$

Functional equation (2.9) now gives

$$g(xy)=g(x)\,g(y) \tag{2.10}$$

The most general continuous solution of (2.10) (Aczél [1] page 44) is

$$g(x)=x^{\lambda},$$

giving $C(\beta)\phi(x)=x^{\lambda}-1$, where λ is an arbitrary constant.

Now from condition $C(1)=0$, we have to have $x^{\lambda}\to 1$ as $\beta\to 1$. Leaving the degenerate case $\lambda=0$, we get $\lambda=\pm(k-h(\beta))$, where $h(\beta)$ is some continuous function of β such that $h(1)=k$ and k is an arbitrary constant.

When

$$\lambda=h(\beta)-k,\qquad \phi(x)=C(\beta)^{-1}(x^{h(\beta)-k}-1),$$

so that

$$f(x)=x\phi(x)=C(\beta)^{-1}(x^{1-k+h(\beta)}-x)={}_{k}f^{1}_{h(\beta)}(x),$$

and when

$$\lambda=k-h(\beta),\qquad \phi(x)=C(\beta)^{-1}(x^{k-h(\beta)}-1),$$

giving

$$f(x)=x\phi(x)=C(\beta)^{-1}(x^{1+k-h(\beta)}-x)={}_{k}f^{2}_{h(\beta)}(x).$$

Note. We have taken $h(\beta)$ and $C(\beta)$ as continuous functions of β to avoid mathematical complexities. It would be noted that $f(x)=C(\beta)^{-1}(x^{\lambda+1}-x)$, $(C(\beta)\neq 0)$ for an arbitrary λ satisfies the functional equation (2.1).

3. Functional Entropies.

Consider now a discrete random variate X taking finite number of values $x_1,\cdots,x_n$ with probability distribution $P=(p_1,\cdots,p_n)$, $\sum_{i=1}^{n}p_i=1$. In terms of a solution ${}_kf_{h(\beta)}(x)$ of (2.1) we define the entropy of type $(h(\beta), k)$ of a distribution P as

$$(3.1)\qquad {}_kH_{h(\beta)}(P)=\sum_{i=1}^{n} {}_kf_{h(\beta)}(p_i).$$

In the next theorem we get a characterization of entropies of type $(h(\beta), k)$ under the usual normalizing condition $f(1/2)=1/2$.

Theorem 3.1. *The entropies of type* $(h(\beta), k)$ *of a distribution* P *under the condition* $f(1/2)=1/2$, *corresponding to the continuous solutions of* (2.1) *are given by*

$$(3.2)\qquad {}_kH^1_{h(\beta)}(P)=\sum_{i=1}^{n} {}_kf^1_{h(\beta)}(p_i)=(2^{k-h(\beta)}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{1-k+h(\beta)}-1\right],$$

and

$$(3.3)\qquad {}_kH^2_{h(\beta)}(P)=\sum_{i=1}^{n} {}_kf^2_{h(\beta)}(p_i)=(2^{h(\beta)-k}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{1+k-h(\beta)}-1\right],$$
$$(\beta\neq 1)$$

The proof which requires only the calculation of $C(\beta)$ under the condition $f(1/2)=1/2$ is rather straight.

Expressions in (3.2) and (3.3) are functional generalizations of Shannon's entropy as in the limiting case $\beta\to 1$ both these reduce to $-\sum_{i=1}^{n} p_i \log p_i$.

It would also be seen that

$${}_kH^1_{h(\beta)}(P)={}_kH^2_{2k-h(\beta)}(P).$$

Also when $k=1, h(\beta)=\beta$, (3.2) reduces to (1.4), Daróczy's entropy of type β. Thus above expressions generalize Shannon's as well as Daróczy's entropy.

4. A Functional Equation in Two Variables.

In this section we consider the functional equation

$$(4.1)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} F(x_iy_j, u_iv_j)=\sum_{i=1}^{m} F(x_i, u_i)+\sum_{j=1}^{n} F(y_j, v_j)$$
$$+C(\beta)\sum_{i=1}^{m}\sum_{j=1}^{n} F(x_i, u_i)F(y_j, v_j),$$

satisfied by a real valued continuous function $F(x, u)$ defined on $[0, 1]\times[0, 1]$ such that

$$\sum_{i=1}^{m} x_i=\sum_{j=1}^{n} y_j=1,\quad x_i\geq 0, y_j\geq 0,$$

and

$$\sum_{i=1}^{m} u_i \le 1, \ \sum_{j=1}^{n} v_j \le 1, \quad u_i \ge 0, v_j \ge 0,$$

where $C(\beta)$ is a constant which is a continuous function of the parameter β such that $C(1)=0$.

We now obtain all the continuous solutions of (4.1) in the following:

Theorem 4.1. *All the continuous solutions of the functional equation* (4.1) *are given by*

$$ {}_kF^1_{h(\beta)}(x,u)=C(\beta)^{-1}(x^{1-k+h(\beta)}-x), \tag{4.2}$$

$$ {}_kF^2_{h(\beta)}(x,u)=C(\beta)^{-1}(x^{1+k-h(\beta)}-x), \tag{4.3}$$

$$ {}_kF^3_{h(\beta)}(x,u)=C(\beta)^{-1}(xu^{h(\beta)-k}-x), \tag{4.4}$$

$$ {}_kF^4_{h(\beta)}(x,u)=C(\beta)^{-1}(xu^{k-h(\beta)}-x), \tag{4.5}$$

$$ {}_kF^5_{h(\beta)}(x,u)=C(\beta)^{-1}(x^{1-k+h(\beta)}u^{k-h(\beta)}-x), \tag{4.6}$$

$$ {}_kF^6_{h(\beta)}(x,u)=C(\beta)^{-1}(x^{1+k-h(\beta)}u^{h(\beta)-k}-x), \tag{4.7}$$

$$ {}_kF^7_{h(\beta)}(x,u)=C(\beta)^{-1}(x^{1-k+h(\beta)}u^{h(\beta)-k}-x), \tag{4.8}$$

and

$$ {}_kF^8_{h(\beta)}(x,u)=C(\beta)^{-1}(x^{1+k-h(\beta)}u^{k-h(\beta)}-x), \quad (C(\beta)\neq 0) \tag{4.9}$$

where $h(\beta)$ *is a continuous function of* β *such that* $h(1)=k$, k *being an arbitrary constant.*

Note. As is evident, the above solutions are being studied for characterizing information theoretic measures associated with two probability distributions say $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$. For obvious reasons we shall take $p_i=0$ if a corresponding $q_i=0$ but not necessarily otherwise. This in view of our assumption $0^\alpha=(0/0)^\alpha=0$ for $\alpha>0$ requires that for β, $h(\beta)-k>0$ in (4.4), (4.6) and (4.8) and $k-h(\beta)>0$ in (4.5), (4.7) and (4.9). (Here in these solutions $x=0$ whenever $u=0$).

Proof. Let m, n, r and s be positive integers such that $1 \le m \le r$, $1 \le n \le s$.

Setting $x_i=\frac{1}{m}$, $(i=1,2,\cdots,m)$; $y_j=\frac{1}{n}$, $(j=1,2,\cdots,n)$,

$u_i=\frac{1}{r}$, $(i=1,2,\cdots,m)$; $v_j=\frac{1}{s}$, $(j=1,2,\cdots,n)$,

in (4.1), we get

$$mn\,F\left(\frac{1}{mn},\frac{1}{rs}\right)=mF\left(\frac{1}{m},\frac{1}{r}\right)+nF\left(\frac{1}{n},\frac{1}{s}\right) + C(\beta)\,mnF\left(\frac{1}{m},\frac{1}{r}\right)F\left(\frac{1}{n},\frac{1}{s}\right). \tag{4.10}$$

Next putting

$$F(x,u)=x\phi(x,u), \tag{4.11}$$

functional equation (4.10) gives

$$\phi\left(\frac{1}{mn},\frac{1}{rs}\right)=\phi\left(\frac{1}{m},\frac{1}{r}\right)+\phi\left(\frac{1}{n},\frac{1}{s}\right) \\ +C(\beta)\phi\left(\frac{1}{m},\frac{1}{r}\right)\phi\left(\frac{1}{n},\frac{1}{s}\right). \tag{4.12}$$

Again setting

$$1+C(\beta)\phi(x,u)=G(x,u),\ (C(\beta)\neq 0) \tag{4.13}$$

(4.12) becomes

$$G\left(\frac{1}{mn},\frac{1}{rs}\right)=G\left(\frac{1}{m},\frac{1}{r}\right)G\left(\frac{1}{n},\frac{1}{s}\right), \tag{4.14}$$

$$\text{i.e.}\quad G(ab,cd)=G(a,c)G(b,d),$$

where $a=1/m$, $b=1/n$, $c=1/r$ and $d=1/s$.

The most general continuous solutions of (4.14) (Aczél [1]) are given by

$$G(a,c)=a^{\lambda}c^{\mu}, \tag{4.15}$$

where λ and μ are arbitrary constants. Thus

$$F\left(\frac{1}{m},\frac{1}{r}\right)=C(\beta)^{-1}\left(\frac{1}{m}\right)\left[\left(\frac{1}{m}\right)^{\lambda}\left(\frac{1}{r}\right)^{\mu}-1\right]. \tag{4.16}$$

The solution (4.15) can be extended to the case when a and c are rational numbers. For this let $x=m/n(m<n)$, $u=p/q(p<q)$ be two rational numbers. Choose an integer t sufficiently large such that $tp\geq m$ and $t\geq q(n-m)/n(q-p)$.

Setting $m=n-m+1, n=m$

$$x_1=\frac{m}{n},\ x_2=\cdots=x_{n-m+1}=\frac{1}{n};\ y_1=\cdots=y_m=\frac{1}{m},$$

$$u_1=\frac{p}{q},\ u_2=\cdots=u_{n-m+1}=\frac{1}{tn};\ v_1=\cdots=v_m=\frac{1}{tp},$$

in (4.1), we get

$$mF\left(\frac{1}{n},\frac{1}{qt}\right)+(n-m)F\left(\frac{1}{mn},\frac{1}{pnt^2}\right)=F\left(\frac{m}{n},\frac{p}{q}\right) \\ +(n-m)F\left(\frac{1}{n},\frac{1}{tn}\right)+mF\left(\frac{1}{m},\frac{1}{pt}\right) \\ +C(\beta)mF\left(\frac{1}{m},\frac{1}{pt}\right)\left[F\left(\frac{m}{n},\frac{p}{q}\right)+(n-m)F\left(\frac{1}{n},\frac{1}{tn}\right)\right]. \tag{4.17}$$

Thus (4.17) together with (4.16) yields

$$F\left(\frac{m}{n}, \frac{p}{q}\right) = C(\beta)^{-1}\left(\frac{m}{n}\right)\left[\left(\frac{m}{n}\right)^{\lambda}\left(\frac{p}{q}\right)^{\mu} - 1\right],$$

i. e. $F(x, u) = C(\beta)^{-1}x(x^{\lambda}u^{\mu} - 1)$, for all rational numbers $x, u \in [0, 1]$.

(4.18) i. e. $C(\beta)\phi(x, u) = x^{\lambda}u^{\mu} - 1$, $(C(\beta) \neq 0)$,

where λ and μ are arbitrary constants and x, u are rational numbers such that $0 < x \leq 1$, $0 \leq u \leq 1$.

From the continuity of F, we can say that (4.18) is valid for all real numbers x, u such that $0 < x \leq 1$, $0 \leq u \leq 1$.

Now from the condition $C(1) = 0$, we have to have $x^{\lambda}u^{\mu} \rightarrow 1$ as $\beta \rightarrow 1$. Leaving the degenerate case $\lambda = 0$, $\mu = 0$, the sets of all admissible values of λ and μ are

(a$_1$) $\lambda = \pm(k - h(\beta))$, $\mu = 0$,

(a$_2$) $\lambda = 0$, $\mu = \pm(k - h(\beta))$,

and

(b) $\lambda = \pm(k - h(\beta))$, $\mu = \pm(k - h(\beta))$,

where $h(\beta)$ is a continuous function of β such that $h(1) = k$ and k is an arbitrary constant.

We discuss the above possibilities one by one.

Under the set of values in (a$_1$), we get

$$\phi(x, u) = C(\beta)^{-1}(x^{h(\beta)-k} - 1) \quad \text{and} \quad C(\beta)^{-1}(x^{k-h(\beta)} - 1),$$

giving the solutions (4.2) and (4.3) respectively.

Next under the set of values in (a$_2$), we get

$$\phi(x, u) = C(\beta)^{-1}(u^{h(\beta)-k} - 1) \quad \text{and} \quad C(\beta)^{-1}(u^{k-h(\beta)} - 1),$$

giving the solutions (4.4) and (4.5) respectively.

Finally, under the set of values in (b), we get the other four solutions (4.6), (4.7), (4.8) and (4.9) of the theorem.

Since the solutions (4.2) and (4.3) turn out to be independent of second variable u and also arise as solutions of the functional equation in one variable studied in section 2 the functions obtained from possibilities (a$_2$) and (b) warrant further study.

Note. It would be noted that $F(x, u) = C(\beta)^{-1}(x^{\lambda+1}u^{\mu} - x)$, $(C(\beta) \neq 0)$ for an arbitrary λ and μ satisfies the functional equation (4.1).

5. Functional Information Theoretic Measures Associated with Two Distributions of a Random Variate.

Consider now a discrete random variate x taking finite number of values $x_1, \cdots, x_n$ and let there be two distributions

$$P=(p_1, \cdots, p_n),\ \sum_{i=1}^{n} p_i=1 \quad \text{and} \quad Q=(q_1, \cdots, q_n),\ \sum_{i=1}^{n} q_i=1$$

attached with it (depending upon the method of proof of the theorem 4.1, the condition over the second distribution

$$Q=(q_1, \cdots, q_n) \quad \text{is} \quad \sum_{i=1}^{n} q_i \leq 1.$$

Corresponding to the solution $F(x, u)$ we shall associate an Information Theoretic measure involving distributions P and Q given by

$$I\left(\frac{P}{Q}\right)=\sum_{i=1}^{n} F(p_i, q_i), \tag{5.1}$$

with suitable normalizing condition.

We enunciate these in the following theorems. The proofs, which require only the calculation of $C(\beta)$ under the normalizing condition $F(1, 1/2)=1$, are omitted.

Theorem 5.1. *The measure associated with the distributions P and Q corresponding to the continuous solutions* (4.4) *and* (4.5) *under the condition* $F(1, 1/2)=1$, are

$${}_kI^1_{h(\beta)}\left(\frac{P}{Q}\right)=(2^{k-h(\beta)}-1)^{-1}\left[\sum_{i=1}^{n} p_i q_i^{h(\beta)-k}-1\right], \tag{5.2}$$

and

$${}_kI^2_{h(\beta)}\left(\frac{P}{Q}\right)=(2^{h(\beta)-k}-1)^{-1}\left[\sum_{i=1}^{n} p_i q_i^{k-h(\beta)}-1\right], \tag{5.3}$$

$$(\beta \neq 1)$$

respectively.

When $\beta \to 1$, (5.2) and (5.3) reduce to Kerridge's inaccuracy viz.

$$-\sum_{i=1}^{n} p_i \log q_i.$$

As before

$${}_kI^1_{h(\beta)}\left(\frac{P}{Q}\right)={}_kI^2_{2k-h(\beta)}\left(\frac{P}{Q}\right).$$

We define (5.2) and (5.3) as functional inaccuracies of type $(h(\beta), k)$.

When $k=1, h(\beta)=\beta$, (5.2) reduces to inaccuracy function of type β.

Theorem 5.2. *The measures associated with the distributions P and Q corresponding to the continuous solutions* (4.6) *and* (4.7) *under the condition* $F(1, 1/2)=1$, *are*

$$(5.4)\qquad {}_kI^3_{h(\beta)}\left(\frac{P}{Q}\right)=(2^{h(\beta)-k}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{1-k+h(\beta)}q_i^{k-h(\beta)}-1\right],$$

and

$$(5.5)\qquad {}_kI^4_{h(\beta)}\left(\frac{P}{Q}\right)=(2^{k-h(\beta)}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{1+k-h(\beta)}q_i^{h(\beta)-k}-1\right],\qquad (\beta\neq 1)$$

respectively.

When $\beta\to 1$, (5.4) and (5.5) reduce to Kullback's information viz.

$$\sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

Furthermore

$${}_kI^3_{h(\beta)}\left(\frac{P}{Q}\right)={}_kI^4_{2k-h(\beta)}\left(\frac{P}{Q}\right).$$

We define (5.4) and (5.5) as functional directed-divergence (Kullback's information) of type $(h(\beta), k)$.

When $k=1$ and $h(\beta)=\beta$, (5.4) reduces to directed-divergence (Kullback's information) function of type β studied by Rathie and Kannappan [9].

Theorem 5.3. *The measures associated with the distributions P and Q corresponding to the continuous solutions* (4.8) *and* (4.9) *under the condition* $F(1, 1/2)=1$, *are*

$$(5.6)\qquad {}_kI^5_{h(\beta)}\left(\frac{P}{Q}\right)=(2^{k-h(\beta)}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{1-k+h(\beta)}q_i^{h(\beta)-k}-1\right],$$

and

$$(5.7)\qquad {}_kI^6_{h(\beta)}\left(\frac{P}{Q}\right)=(2^{h(\beta)-k}-1)^{-1}\left[\sum_{i=1}^{n} p_i^{1+k-h(\beta)}q_i^{k-h(\beta)}-1\right],\qquad (\beta\neq 1)$$

respectively.

When $\beta\to 1$, (5.6) and (5.7) reduce to

$$-\sum_{i=1}^{n} p_i \log p_i q_i = -\sum_{i=1}^{n} p_i \log p_i - \sum_{i=1}^{n} p_i \log q_i = \text{Entropy}+\text{Inaccuracy}$$

These in a way generalize the quantity which is the sum of the entropy of P and the inaccuracy of Q with respect to P. This quantity has so far not been studied as such and we wonder if it can arouse interest.

Here also it is noted that

$${}_kI^5_{h(\beta)}\left(\frac{P}{Q}\right)={}_kI^6_{2k-h(\beta)}\left(\frac{P}{Q}\right).$$

Remark. In the theorem 5.3, if we replace condition $F(1, 1/2)=1$ by $F(1/2, 1/2)=1$, then the constant $C(\beta)$ has values $2^{-1}(2^{2(h(\beta)-k)}-1)$ and $2^{-1}(2^{2(k-h(\beta))}-1)$ respectively. However the limiting case $\beta \rightarrow 1$, for all these is

$$-\sum_{i=1}^{n} p_i \log p_i q_i.$$

A generalization in three variables interestingly leads to multiplicity of results and will be discussed elsewhere.

Acknowledgements

The authors are thankful to Professor U. N. Singh, Dean, Faculty of Mathematics, University of Delhi for providing facilities in the department. The second author is thankful to C. S. I. R. (INDIA) for providing research fellowship.

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(Ricevita la 20-an de Aŭgusto, 1973)
(Reviziita la 21-an de Majo, 1974)