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Surjectivity of Operators Involving Linear Noninvertible Part and Nonlinear Compact Perturbation

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Abstract.

Consider an equation $A(u)-S(u)=h$, where A is a linear selfadjoint Fredholm operator in a real Hilbert space H with a nonzero null-space $N(A)$ and S is a nonlinear completely continuous mapping. The operator S is of linear growth at infinity and satisfies a certain coercivity condition with respect to $N(A)$. It is proved that the equation has a solution for an arbitrary $h\in H$. The abstract theorem is applied to show the existence of weak solutions of boundary value problems for nonlinear differential equations.

1. Introduction.

This paper deals with solvability of nonlinear equations of the type $A(u)-S(u)=h$, where A is a linear selfadjoint operator in a real Hilbert space H with a nonzero null-space $N(A)$, $h\in H$, S is a nonlinear completely continuous operator. Under some additional assumptions on the operator S it is proved (see Theorem 1) that the equation considered is solvable for an arbitrary $h\in H$. The same assertion under the assumption $N(A)=\{0\}$ and under suitable conditions on the operator S is proved e.g. in [1] and in other papers which deal with the so-called "Fredholm alternative for nonlinear operators". A result concerning solvability of the homogeneous Dirichlet problem for nonlinear elliptic partial differential equations whose linear part has a trivial null-space is established in [3] and generalized in [4]. The case of nontrivial null-space $N(A)$ is considered in [2] where the following growth condition on the nonlinear operator S is used: There exist $\mu_1\geqq 0$, $\mu_2>0$, $\delta\in(0,1)$ such that

$$\|S(u)\|\leqq\mu_1+\mu_2\|u\|^{\delta}$$

for each $u\in H$.

In the present paper an analogous growth condition with $\delta=1$ is introduced, the constant μ_2 being assumed sufficiently small. As an example of applicability of Theorem 1 we shall give an assertion concerning weak solvability of boundary value problems for nonlinear differential equations (see Theorem 2).

2. Main result.

The main result of this paper is the following.

Theorem 1. *Let H be a real Hilbert space with the inner product $(\cdot, \cdot)$. Let A be a linear bounded selfadjoint operator from H into H with a closed range $R(A)$ and finite-dimensional null-space $N(A)$,* $\dim N(A) \geqq 1$.

Let S be a completely continuous operator from H into H (i.e., S is continuous and maps bounded subsets of H into compact sets). Suppose that there are two constants $\mu_1 \geqq 0$, $\mu_2 > 0$ such that

(S 1) $$\|S(u)\| \leqq \mu_1 + \mu_2 \|u\|$$

for all $u \in H$.

Moreover, suppose that there exists $\delta > 0$ satisfying the following condition: for any $K > 0$ we have $t_K > 0$ such that

(S 2) $$(S(t(w+v)), w) \geqq K$$

for all $t \geqq t_K$, $\|w\| = 1$, $w \in N(A)$, $v \in R(A)$, $\|v\| \leqq \delta$.

Then the equation

(2.1) $$A(u) - S(u) = h$$

is solvable for each $h \in H$ provided the constant μ_2 satisfies the inequality

(2.2) $$\mu_2 < \delta \|M\|^{-1} (1+\delta)^{-1},$$

where $M\colon R(A) \to R(A)$ is the right inverse of the operator A [1].

Proof. A. Let $h \in H$ be arbitrary but fixed. To prove that the equation (2.1) is solvable with the right hand side h it is sufficient to show that the equation

(2.3) $$A(u) - U(u) = 0$$

is solvable, where $U(u) = S(u) + h$.

The operator U is completely continuous and satisfies

(2.4) $$\|U(u)\| \leqq (\mu_1 + \|h\|) + \mu_2 \|u\| = \bar{\mu}_1 + \mu_2 \|u\|.$$

Put

$$K = 1 - \min_{\substack{w \in N(A) \\ \|w\| = 1}} (h, w).$$

Then

(2.5) $$(U(t(w+v)), w) \geqq (S(t(w+v)), w) + \min_{\substack{w \in N(A) \\ \|w\| = 1}} (h, w) \geqq 1$$

1) For the definition and properties of the right inverse of the operator A see part B of the proof of Theorem 1.

for all $t \geqq t_K$, $w \in N(A)$, $\|w\|=1$, $v \in R(A)$, $\|v\| \leqq \delta$.

B. Denote by P the orthogonal projection from H onto $N(A)$ and let $Q = I - P$ (I is the identity operator), i.e., Q is the orthogonal projection from H onto $R(A)$.

Under our assumptions there exists a linear continuous map (the so-called right inverse) $M: R(A) \to R(A)$ satisfying

$$MA(x) = Q(x) \quad (x \in H),$$
$$AM(y) = y \quad (y \in R(A)).$$

C. Define the family $\{V_\varepsilon\}_{\varepsilon>0}$ of mappings

$$V_\varepsilon : N(A) \times R(A) \to N(A) \times R(A)$$

by

$$V_\varepsilon : [w, v] \mapsto [w - \varepsilon PU(w + MQU(w+v)),\ MQU(w+v)].$$

It is easy to see that if for some $\varepsilon > 0$ the operator V_ε has a fixed point $[w_0, v_0]$, i.e.,

$$V_\varepsilon(w_0, v_0) = [w_0, v_0]$$

then the equation (2.3) has a solution $w_0 + v_0$. For any $\varepsilon > 0$ the mapping V_ε is completely continuous from $N(A) \times R(A)$ into $N(A) \times R(A)$.

D. To prove that for some $\varepsilon > 0$ the operator V_ε has a fixed point in $N(A) \times R(A)$ it is sufficient to show (by Schauder Fixed Point Theorem and by part **C** of this proof) the existence of a nonempty convex closed and bounded set $\mathcal{K} \subset N(A) \times R(A)$ and $\varepsilon > 0$ such that

$$V_\varepsilon(\mathcal{K}) \subset \mathcal{K}.$$

E. Put

$$a_0 = \|M\| \bar{\mu}_1 (1 - \mu_2 \|M\|)^{-1}.$$

Then for each $b \in R_1$ satisfying

$$b \geqq \mu_2 \|M\| (1 - \mu_2 \|M\|)^{-1}, \tag{2.6}$$

each $\rho > 0$, $w \in N(A)$, $\|w\| \leqq \rho$ and any $v \in R(A)$, $\|v\| \leqq a_0 + b\rho$, we have

$$\|MQU(w+v)\| \leqq a_0 + b\rho. \tag{2.7}$$

Indeed, if $a > 0$, $b \geqq \mu_2 \|M\| (1 - \mu_2 \|M\|)^{-1}$ then for $w \in N(A)$, $\|w\| \leqq \rho$ and $v \in R(A)$, $\|v\| \leqq a + b\rho$, it is

$$\|MQU(w+v)\| \leqq \|M\| \bar{\mu}_1 + \|M\| \mu_2 (\rho + a + b\rho).$$

Set

$$z(b, \rho) = \|M\| \bar{\mu}_1 + (\mu_2 \|M\| + \mu_2 \|M\| b - b)\rho.$$

The function $z(b,\rho)$ is bounded from above for $\rho\geqq 0$, $b\geqq\mu_2\|M\|(1-\mu_2\|M\|)^{-1}$. Put

$$r(a)=(1-\mu_2\|M\|)a.$$

Then condition (2.2) implies $\mu_2<\|M\|^{-1}$ and hence

$$z(b,\rho)\leqq r(a_0)$$

for all b and ρ considered. Thus the inequality (2.7) holds.

F. For $w\in N(A)$, $\|w\|=\rho$ we have

$$\begin{aligned}(2.8)\qquad &\|w-\varepsilon PU(w+MQU(w+v))\|^2\\ &=\rho^2-2\varepsilon(U(w+MQU(w+v)),w)+\varepsilon^2\|PU(w+MQU(w+v))\|^2\end{aligned}$$

and

$$\begin{aligned}(2.9)\qquad &(U(w+MQU(w+v)),w)\\ &=\rho(U(\rho(\rho^{-1}w+\rho^{-1}MQU(w+v))),\rho^{-1}w).\end{aligned}$$

Fix $b\in\langle\mu_2\|M\|(1-\mu_2\|M\|)^{-1},\delta)$[2]. Then for each $\rho\geqq(\delta-b)^{-1}a_0$, $v\in R(A)$, $\|v\|\leqq a_0+b\rho$ and $w\in N(A)$, $\|w\|=\rho$, we have $\|\rho^{-1}MQU(w+v)\|\leqq\delta$ by virtue of (2.7).

Thus by the relation (2.5) there exists $\rho_0>2(\delta-b)^{-1}a_0$ such that for each $\rho\geqq\rho_0/2$,

$$(2.10)\qquad (U(\rho(\rho^{-1}w+\rho^{-1}MQU(w+v))),\rho^{-1}w)\geqq 1$$

holds provided $\|w\|=\rho$ and $\|v\|\leqq a_0+b\rho$.

From (2.8)-(2.10) we have:

$$\|w-\varepsilon PU(w+MQU(w+v))\|^2\leqq\rho^2-2\varepsilon\rho+\varepsilon^2[\bar{\mu}_1+\mu_2(\rho+a+b\rho)]^2$$

for each $\varepsilon>0$, $\rho\geqq\rho_0/2$, $\|w\|=\rho$, $\|v\|\leqq a_0+b\rho$. Thus for $\rho\in\langle\rho_0/2,\rho_0\rangle$ and

$$\varepsilon\in(0,\rho_0[\bar{\mu}_1+\mu_2(\rho_0+a_0+b\rho_0)]^{-2}\rangle$$

we obtain

$$(2.11)\qquad \|w-\varepsilon PU(w+MQU(w+v))\|\leqq\rho_0$$

whenever $\|w\|=\rho$ and $\|v\|\leqq a_0+b\rho_0$.

For $\|w\|\leqq\rho_0/2$, $\|v\|\leqq a_0+b\rho_0$ and arbitrary

$$\varepsilon\in\left(0,\ \frac{1}{4}\rho_0\left(\bar{\mu}_1+\mu_2\left[\frac{\rho_0}{2}+a_0+b\rho_0\right]\right)^{-1}\right\rangle$$

we have

$$(2.12)\qquad \|w-\varepsilon PU(w+MQU(w+v))\|\leqq\frac{\rho_0}{2}+\frac{\rho_0}{4}<\rho_0.$$

2) For example, $\langle\alpha,\beta)$ denotes the interval $\alpha\leqq x<\beta$.

G. Set

$$\varepsilon_0=\min\left\{\rho_0(\bar{\mu}_1+\mu_2[\rho_0+a_0+b\rho_0])^{-2},\ \frac{1}{4}\rho_0\left(\bar{\mu}_1+\mu_2\left[\frac{\rho_0}{2}+a_0+b\rho_0\right]\right)^{-1}\right\}$$

and

$$\mathcal{K}=\{[w,v]\in N(A)\times R(A):\|w\|\leqq\rho_0,\ \|v\|\leqq a_0+b\rho_0\}.$$

The set $\mathcal{K}$ is convex closed and nonempty and relations (2.7), (2.11) and (2.12) imply $V_{\varepsilon_0}(\mathcal{K})\subset\mathcal{K}$. Hence the mapping V_{ε_0} has a fixed point and thus equation (2.1) is solvable as follows from part D.

3. Application to boundary value problems.

Let Ω be a bounded domain in N-dimensional Euclidean space $R_N(N\geqq1)$ with boundary $\partial\Omega$ which is lipschitzian in case of $N>1$. If α_i are non-negative integers $(i=1,\cdots,N)$ we denote $\alpha=(\alpha_1,\cdots,\alpha_N)$, and

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}},$$

where $|\alpha|=\sum_{i=1}^N\alpha_i$. For $k>0$ (integer), let $W_2^k(\Omega)$ be the Sobolev space of real functions $u\in L_2(\Omega)$ for which (in the sense of distributions) $D^\alpha u\in L_2(\Omega)$ if $|\alpha|\leqq k$, with the inner product

$$(u,v)_k=\sum_{|\alpha|\leqq k}\int_\Omega D^\alpha u(x)D^\alpha v(x)dx$$

and with the norm $\|u\|_k=(u,u)_k^{1/2}$.

Denoting by $\mathcal{D}(\Omega)$ the set of all infinitely differentiable functions on Ω with compact supports in Ω, we define $\mathring{W}_2^k(\Omega)$ to be the closure of $\mathcal{D}(\Omega)$ in $W_2^k(\Omega)$. Let V be a closed subspace of $W_2^k(\Omega)$ such that

$$\mathring{W}_2^k(\Omega)\subset V\subset W_2^k(\Omega).$$

Let

(3.1) $$a_{ij}(x)\in L_\infty(\Omega),\quad a_{ij}=a_{ji}\quad(|i|,|j|\leqq k).$$

Suppose that there exists $c>0$ such that

(3.2) $$\sum_{|i|=|j|=k}a_{ij}(x)\xi_i\xi_j\geqq c\sum_{|i|=k}\xi_i^2$$

for all $\xi_i\in R_1(|i|=k)$ and almost all $x\in\Omega$.

Let

(3.3) $$A_{ij}\in L_\infty(\partial\Omega),\quad A_{ij}=A_{ji}\quad(|i|,|j|<k).$$

Since

$$A(v, u)=\sum_{|i|,|j|\leqq k}\int_{\Omega}a_{ij}(x)D^{i}v(x)D^{j}u(x)dx+\sum_{|i|,|j|<k}\int_{\partial\Omega}A_{ij}D^{i}vD^{j}u\,dS$$ [3)]

is a symmetric bounded and bilinear form on $W_2^k(\Omega)\times W_2^k(\Omega)$, we can define the mapping $A: V\rightarrow V$ by the relation

$$(3.4)\qquad (A(v), u)_k=A(v, u)$$

for all $u, v\in V$.

Under the assumptions (3.1) and (3.3) the operator A is selfadjoint. Moreover, the operator $A_1: V\rightarrow V$ defined by

$$(A_1(v), u)_k=\sum_{|i|=|j|=k}\int_{\Omega}a_{ij}(x)D^{i}v(x)D^{j}u(x)dx$$

for all $u, v\in V$ is an isomorphism from V onto V (by condition (3.2)). The mapping $A_2=A-A_1$ is completely continuous by virtue of the complete continuity of the imbeddings from $W_2^k(\Omega)$ into $W_2^{k-1}(\Omega)$ and from $W_2^1(\Omega)$ into $L_2(\partial\Omega)$ (see e.g. [5, Chapter II]). Thus the range $R(A)$ is a closed subspace of V and the null-space $N(A)$ is finite-dimensional.

Let l be a positive integer and suppose

$$(3.5)\qquad 2(k-l+1)>N.$$

Inequality (3.5) implies that the Sobolev space $W_2^k(\Omega)$ is continuously imbedded into the Schauder space $C^{l-1}(\bar{\Omega})$ and, moreover, the imbedding is completely continuous (see e.g. [5]). For our convenience, denote by c^* the norm of the identity mapping from $W_2^k(\Omega)$ into $C^{l-1}(\bar{\Omega})$.

Let $\mathcal{M}$ be a nonempty set of multiindices $\beta=(\beta_1, \cdots, \beta_N)$ such that $|\beta|\leqq l-1$.

Lemma. *Assume* (3.1)-(3.5) *and let* $\dim N(A)\geqq 1$. *Moreover, suppose that*

(3.6) *if $w\in N(A)$ and the derivative $D^{\alpha}w$ for some $\alpha\in\mathcal{M}$ vanishes on the set of positive measure in Ω, then $w=0$.*

For $w\in N(A)$, $\|w\|_k=1$, $\alpha\in\mathcal{M}$ and $\varepsilon>0$ put

$$\Phi_{\varepsilon}^{\alpha}(w)=\{x\in\Omega : |D^{\alpha}w(x)|\geqq\varepsilon\},$$
$$\Psi_{\varepsilon}^{\alpha}(w)=\{x\in\Omega : 0<|D^{\alpha}w(x)|<\varepsilon\}.$$

Then

$$\min_{\alpha\in\mathcal{M}}\inf\{\varepsilon>0 : \inf_{\substack{w\in N(A)\\ \|w\|_k=1}}(\operatorname{meas}\Phi_{\varepsilon}^{\alpha}(w)-\operatorname{meas}\Psi_{\varepsilon}^{\alpha}(w))\leqq 0\}=\varepsilon^*>0,$$

3) In the surface integral the derivatives $D^{i}v, D^{j}u$ are considered in the sense of traces. Since we suppose that Ω is a domain with lipschitzian boundary $\partial\Omega$ and, moreover, $D^{i}v, D^{j}u\in W_2^1(\Omega)$ for $|i|, |j|<k$, the traces are well-defined (see e.g. [5, p. 15]).

where meas *denotes the N-dimensional Lebesgue measure.*

Proof. Suppose that $\varepsilon^*=0$. Then there exist $\alpha_0\in\mathcal{M}$, a monotone sequence of positive numbers ε_n with $\lim_{n\to\infty}\varepsilon_n=0$, and a sequence $w_n\in N(A)$, $\|w_n\|_k=1$ such that w_n converges to w_0 in the space $W_2^k(\Omega)$ (since the subspace $N(A)$ is finite-dimensional), $D^\alpha w_n(x)$ converges uniformly to $D^\alpha w_0(x)$ on $\bar{\Omega}$ for $|\alpha|\leqq l-1$ (since the imbedding from $W_2^k(\Omega)$ into $C^{l-1}(\bar{\Omega})$ is completely continuous) so that

$$\text{meas}\,\Phi_{\varepsilon_n}^{\alpha_0}(w_n)\leqq\frac{1}{n}+\text{meas}\,\Psi_{\varepsilon_n}^{\alpha_0}(w_n),\quad n=1,2,\cdots.$$

The last inequality together with the assumption (3.6) implies

$$\text{(3.7)}\qquad \frac{1}{2}\,\text{meas}\,\Omega\leqq\frac{1}{2n}+\text{meas}\,\{x\in\Omega:|D^{\alpha_0}w_n(x)|<\varepsilon_n\}$$

for $n=1,2,\cdots$.

Let $\eta>0$ be arbitrary but fixed. Then there exists a positive integer n_0 such that for each $n\geqq n_0$ and $x\in\bar{\Omega}$ we have

$$|D^{\alpha_0}w_n(x)-D^{\alpha_0}w_0(x)|<\eta$$

and so

$$\text{(3.8)}\qquad \{x\in\Omega:|D^{\alpha_0}w_n(x)|<\varepsilon_n\}\subset\{x\in\Omega:|D^{\alpha_0}w_0(x)|<\eta+\varepsilon_n\}$$

for $n\geqq n_0$.

From (3.7) and (3.8) we obtain

$$\frac{1}{2}\,\text{meas}\,\Omega\leqq\frac{1}{2n}+\text{meas}\,\{x\in\Omega:|D^{\alpha_0}w_0(x)|<\eta+\varepsilon_n\}$$

for $n\geqq n_0$ and thus

$$\text{(3.9)}\qquad \frac{1}{2}\,\text{meas}\,\Omega\leqq\text{meas}\,\{x\in\Omega:|D^{\alpha_0}w_0(x)|\leqq\eta\}.$$

Since the inequality (3.9) holds for arbitrary $\eta>0$, we have

$$\frac{1}{2}\,\text{meas}\,\Omega\leqq\text{meas}\,\{x\in\Omega:D^{\alpha_0}w_0(x)=0\}.$$

Consequently, $w_0=0$ by the assumption (3.6). This is a contradiction with $\|w_0\|_k=1$.

For every $\alpha\in\mathcal{M}$ let g_α be a continuous monotone odd function on R_1. Suppose that there exist constants $c_1,c_2>0$ such that

$$\text{(3.10)}\qquad |g_\alpha(\xi)|\leqq c_1+c_2|\xi|$$

for all $\xi\in R_1$ and suppose that

$$\text{(3.11)}\qquad \lim_{\xi\to\infty}g_\alpha(\xi)=\infty.$$

Define the operator $S: V \to V$ by

$$(S(u), v)_k = \sum_{\alpha\in\mathcal{M}} \int_\Omega g_\alpha(D^\alpha u(x)) D^\alpha v(x)\, dx$$

for all $u, v \in V$.

Using the results about the continuity of the Nemyckij operator from $L_2(\Omega)$ into $L_2(\Omega)$ (see e.g. [6]), we obtain by virtue of the complete continuity of the imbedding from $W_2^k(\Omega)$ into $W_2^{k-1}(\Omega)$ that the mapping S is completely continuous.

Further,

$$\|S(u)\|_k = \sup_{\|v\|_k=1} \left| \sum_{\alpha\in\mathcal{M}} \int_\Omega g_\alpha(D^\alpha u(x)) D^\alpha v(x)\, dx \right|$$
$$\leqq c^* \sum_{\alpha\in\mathcal{M}} \int_\Omega |g_\alpha(D^\alpha u(x))|\, dx \leqq c^*(\operatorname{card}\mathcal{M}) c_1 (\operatorname{meas}\Omega)$$
$$+ c^*(\operatorname{meas}\Omega)^{1/2}(\operatorname{card}\mathcal{M})^{1/2} c_2 \|u\|_k$$

($\operatorname{card}\mathcal{M}$ is the number of points in the set $\mathcal{M}$) and

$$\|S(u)\|_k = \sup_{\|v\|_k=1} \left| \sum_{\alpha\in\mathcal{M}} \int_\Omega g_\alpha(D^\alpha u(x)) D^\alpha v(x)\, dx \right|$$
$$\leqq \sum_{\alpha\in\mathcal{M}} \left(\int_\Omega |g_\alpha(D^\alpha u(x))|^2 dx \right)^{1/2} \leqq c_1(\operatorname{card}\mathcal{M})(2 \operatorname{meas}\Omega)^{1/2}$$
$$+ c_2(2 \operatorname{card}\mathcal{M})^{1/2}\|u\|_k.$$

Thus there exists $\mu_1 > 0$ such that for each $u \in V$ we have

$$\|S(u)\|_k \leqq \mu_1 + \mu_2 \|u\|_k,$$

where

$$\mu_2 = \min\{c^* c_2(\operatorname{meas}\Omega \operatorname{card}\mathcal{M})^{1/2},\ c_2(2\operatorname{card}\mathcal{M})^{1/2}\}.$$

Theorem 2. *Let the notation introduced in this Section be observed. Suppose* (3.1)-(3.6), (3.10), (3.11). *Let*

(3.12) $\quad c_2 < \varepsilon^*[\|M\|(\operatorname{card}\mathcal{M})^{1/2}(\varepsilon^* + 2c^*) \min\{\sqrt{2},\ c^*(\operatorname{meas}\Omega)^{1/2}\}]^{-1}.$

Then for each $f \in L_2(\Omega)$ there exists a weak solution $u_0 \in V$ of the equation $A(u) - S(u) = f$, i.e.,

$$A(u_0, v) - (S(u_0), v)_k = \int_\Omega f(x) v(x)\, dx$$

for all $v \in V$.

Proof. Let $\varepsilon \in (0, \varepsilon^*)$ and $\delta \in (0, \varepsilon/2c^*)$ be fixed. To obtain Theorem 2 we shall apply Theorem 1. It is sufficient to verify condition (S2) with δ. Suppose that this condition is not fulfilled. Then there exist $K > 0$ and sequences,

$$t_n \in R_1,\ \lim_{n\to\infty} t_n = \infty,\ v_n \in R(A),\ \|v_n\|_k \leqq \delta,\ w_n \in N(A),\ \|w_n\|_k = 1,$$

such that

$$(S(t_n(w_n+v_n)), w_n)_k \leqq K. \tag{3.13}$$

By virtue of condition (3.5) we have $|D^\alpha v_n(x)| \leqq c^*\delta$ for all $x \in \bar{\Omega}$, every n and every $\alpha \in \mathcal{M}$. Thus (using the assertion of Lemma) we have

$$\begin{aligned}
&(S(t_n(w_n+v_n)), w_n)_k \\
&= \sum_{\alpha\in\mathcal{M}} \int_\Omega g_\alpha(t_n(D^\alpha w_n(x) + D^\alpha v_n(x))) D^\alpha w_n(x)\,dx \\
&= \sum_{\alpha\in\mathcal{M}} \left(\int_{\Phi_\varepsilon^\alpha(w_n)} g_\alpha(t_n(D^\alpha w_n(x) + D^\alpha v_n(x))) D^\alpha w_n(x)\,dx \right. \\
&\quad \left. + \int_{\Psi_\varepsilon^\alpha(w_n)} g_\alpha(t_n(D^\alpha w_n(x) + D^\alpha v_n(x))) D^\alpha w_n(x)\,dx \right) \\
&\geqq \sum_{\alpha\in\mathcal{M}} \varepsilon \{ g_\alpha(t_n(\varepsilon - c^*\delta)) \operatorname{meas} \Phi_\varepsilon^\alpha(w_n) - g_\alpha(t_n c^*\delta) \operatorname{meas} \Psi_\varepsilon^\alpha(w_n) \} \\
&= \sum_{\alpha\in\mathcal{M}} \varepsilon g_\alpha(t_n c^*\delta) \left[\frac{g_\alpha(t_n(\varepsilon - c^*\delta))}{g_\alpha(t_n c^*\delta)} \operatorname{meas} \Phi_\varepsilon^\alpha(w) - \operatorname{meas} \Psi_\varepsilon^\alpha(w) \right] \\
&\geqq \varepsilon \sum_{\alpha\in\mathcal{M}} g_\alpha(t_n c^*\delta) \inf_{\substack{w\in N(A) \\ \|w\|_k=1}} [\operatorname{meas} \Phi_\varepsilon^\alpha(w) - \operatorname{meas} \Psi_\varepsilon^\alpha(w)].
\end{aligned}$$

The last inequality is a contradiction with assumption (3.13).
(Note that the integral

$$\int_{\Psi_\varepsilon^\alpha(w_n)} g_\alpha(t_n(D^\alpha w_n(x) + D^\alpha v_n(x))) D^\alpha w_n(x)\,dx$$

is estimated by means of the following inequalities: if $x \in \Psi_\varepsilon^\alpha(w_n)$, $D^\alpha w_n(x) > 0$ then $g_\alpha(t_n(D^\alpha w_n(x) + D^\alpha v_n(x))) D^\alpha w_n(x) \geqq \varepsilon g_\alpha(-t_n c^*\delta) = -\varepsilon g_\alpha(t_n c^*\delta)$, and if $x \in \Psi_\varepsilon^\alpha(w_n)$, $D^\alpha w_n(x) < 0$ then $g_\alpha(t_n(D^\alpha w_n(x) + D^\alpha v_n(x))) D^\alpha w_n(x) \geqq -\varepsilon g_\alpha(t_n c^*\delta)$.)

4. Remarks.

Remark 1. From part G of the proof of Theorem 1 (Section 2) we obtain for a solution u_0 of equation (2.1) the following estimate:

$$\|u_0\|^2 \leqq (a_0 + 2bT_K)^2 + (2T_K)^2,$$

where

$$\begin{aligned}
&0 < b < \delta, \\
&a_0 = \|M\|(\mu_1 + \|h\|)(1 - \mu_2\|M\|)^{-1}, \\
&K = 1 - \min_{\substack{w\in N(A) \\ \|w\|=1}} (h, w), \\
&T_K = \max\{(\delta - b)^{-1} a_0, t_K\}.
\end{aligned}$$

Remark 2. If condition (3.10) is replaced by

$$|g_\alpha(\xi)|\leqq c(1+|\xi|^\delta)$$

($\delta\in(0,1)$), then the assertion of Theorem 2 is true with no restriction on $c>0$. This case is solved in [2].

Remark 3. We consider the boundary value problem

$$\begin{aligned} &\lambda u''+u+g(u)=f \\ &u(0)=u(\pi)=0, \end{aligned} \tag{4.1}$$

where g is a continuous odd and monotone function on R_1,

$$\lim_{\xi\to\infty} g(\xi)=\infty, \tag{4.2}$$

$$|g(\xi)|\leqq c_1+c_2|\xi|. \tag{4.3}$$

For the sake of simplicity we consider $\lambda=1$. Then $\|M\|=4/3$, $\varepsilon^*=\pi/4$, $c^*=\pi^{1/2}$. From (3.12) we obtain that the sufficient condition for the weak solvability of (4.1) for any right hand side $f\in L_2(0,\pi)$ is

$$c_2<\frac{3}{4}\pi^{1/2}[\sqrt{2}(\pi^{1/2}+8)]^{-1}=c_2^*.$$

Cleary $c_2^*<3/4$=distance to the next eigenvalue. An open problem is whether equation (4.1) is weakly solvable for each $f\in L_2(0,\pi)$ provided $c_2\in\langle c_2^*, 3/4)$.

Remark 4. Theorem 1 can be applied to the boundary value problem for partial differential equations, where the nonlinear perturbation is of the type

$$\sum_{|i|,|j|<k}(-1)^j D^j(g_i(x,u,\cdots)).$$

However, the conditions on the functions g_i are very complicated and it is better to verify assumption (S2) in every particular case. An example may be obtained by modifying slightly the example from paper [2].

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