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A New Generalized Functional Equation for Inaccuracy and Entropy of Kind β

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Abstract.

Arimoto has defined entropy of a complete probability distribution $(p_1, p_2, \cdots, p_n)$ as inf. $\sum_{i=1}^{n} p_i f(q_i)$ where infimum is taken over all complete probability distributions $(q_1, q_2, \cdots, q_n)$. The expression $\sum_{i=1}^{n} p_i f(q_i)$ whose particular case is $-\sum_{i=1}^{n} p_i \log q_i$ may be called inaccuracy. In this paper a characterization of the function which leads to inaccuracy of kind β through a functional equation is taken up. This functional equation arises from the recursivity and symmetry of the inaccuracy of kind β. Interestingly the infimum of the inaccuracy of kind β gives rise to entropy studied by Arimoto.

1. Introduction.

A number of researchers have taken up the problem of characterizing Shannon's entropy under various set of postulates [1]. The change in postulates in different ways has led to generalizations of Shannon's entropy. A method apparently much at variance from axiomatic characterization has been given by Arimoto [2], where he has been guided by the well known inequality

$$(1.1) \qquad -\sum_{i=1}^{n} p_i \log p_i \leqq -\sum_{i=1}^{n} p_i \log q_i$$

in proposing the entropy of a distribution $P=(p_1, p_2, \cdots, p_n)$ as

$$(1.2) \qquad H_n^f(p_1, p_2, \cdots, p_n) = \text{Inf.} \sum_{i=1}^{n} p_i f(q_i)$$

where the operation of infimum is taken over all probablity distributions such as $Q=(q_1, q_2, \cdots, q_n)$, $\sum_{i=1}^{n} q_i = 1$ and $q_i > 0$.

Actually the quantity $\sum_{i=1}^{n} p_i f(q_i)$ is a general expression embodying the Kerridge's inaccuracy [5] $-\sum_{i=1}^{n} p_i \log q_i$, as a particular case. As such it is important to explore the meeting ground of axiomatic characterizations with the inequalities (1.2) (with (1.1) as particular case) in search of new and useful measures.

In this paper we shall form a generalized functional equation in two variables, involving a parameter β, which under suitable boundary conditions would give a new measure i. e., the inaccuracy of kind β, as we shall call it.

A characterization of this new measure is given along with some of its properties. In the last section, we deal with a characterization and some properties of entropy of kind β arising from inaccuracy of kind β and (1.2).

2. Functional Equation and Inaccuracy of Kind β.

Kendall [4] has formed a functional equation for information function, which gives Shannon's entropy. This has been generalized by Darcozy [3] with a parameter. Some generalizations of this in two or more variables involving parameters have been studied by Sharma and Ram Autar [6,7].

Under mainly the recursivity and symmetry of inaccuracy of kind β, we obtain the functional equation,

$$\begin{aligned} (2.1) \qquad & f(x, y)+(1-x)(1-y)^{1-\beta} f\left(\frac{u}{1-x}, \frac{v}{1-y}\right) \\ & =f(u, v)+(1-u)(1-v)^{1-\beta} f\left(\frac{x}{1-u}, \frac{y}{1-v}\right) \\ & x, y, u, v \in[0,1) \text{ with } x+u, y+v \in[0,1], \end{aligned}$$

which an inaccuracy function $f(x, y)$ defined below satisfies. If we put $y=x$ and $v=u$, and take $f(x, x)=f(x)$, (2.1) reduces to Kendall's functional equation for information function when $\beta=1$.

We adopt the following:

Difinition. A real valued function $f(x, y)$ satisfying the functional equation (2.1), defined in $[0,1]\times[0,1]$ would be called an *inaccuracy function of kind* $\beta(\neq 1)$ if it satisfies the boundary conditions

$$(2.2) \qquad f(1,1)=f(0,0) \text{ and}$$

$$(2.3) \qquad f(\beta, 1)=f(1-\beta, 0)=1 \quad (0<\beta<1).$$

Again if $f(x, y)$ is an inaccuracy function of kind $\beta(\neq 1)$ then the *inaccuracy of kind* β for a distribution P with respect to Q is defined as

$$(2.4) \qquad H_n^\beta\begin{pmatrix} p_1, p_2, \cdots, p_n \\ q_1, q_2, \cdots, q_n \end{pmatrix}=\sum_{i=2}^n r_i s_i^{1-\beta} f\left(\frac{p_i}{r_i}, \frac{q_i}{s_i}\right)$$

where $r_i=p_1+p_2+\cdots+p_i$, $s_i=q_1+q_2+\cdots+q_i$; $i=1,2, \cdots, n$, with $r_n=s_n=1$.

3. Characterization of Inaccuracy of Kind β.

Let $P=(p_1, p_2, \cdots, p_n)$ and $Q=(q_1, q_2, \cdots, q_n)$ be two complete probability distributions of a random variable $X=(x_1, x_2, \cdots, x_n)$, then L_n the inaccuracy of kind β of P with respect to Q is taken to satisfy the following postulates:

Postulate I.

$$L_n\begin{pmatrix} p_1, p_2, \cdots, p_n \\ q_1, q_2, \cdots, q_n \end{pmatrix} = L_{n-1}\begin{pmatrix} p_1+p_2, p_3, \cdots, p_n \\ q_1+q_2, q_3, \cdots, q_n \end{pmatrix}$$
$$+(p_1+p_2)(q_1+q_2)^{1-\beta}L_2\begin{pmatrix} \dfrac{p_1}{p_1+p_2}, \dfrac{p_2}{p_1+p_2} \\ \dfrac{q_1}{q_1+q_2}, \dfrac{q_2}{q_1+q_2} \end{pmatrix}; \quad (n \geq 3)$$

with $p_1+p_2, q+q_2>0$.

Postulate II.

$L_3\begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix}$ is a symmetric function such that for any permutation of p's there is the same permutation of q's.

Postulate III.

$$L_2\begin{pmatrix} \beta, 1-\beta \\ 1, \quad 0 \end{pmatrix} = 1 \qquad (0<\beta<1).$$

Theorem I. A real valued function $f(x, y)$ given by

$$f(x, y) = L_2\begin{pmatrix} x, 1-x \\ y, 1-y \end{pmatrix}$$

Where L_n, $n=2, 3, \cdots$, that satisfies the postulates I-III, satisfies the functional equation

$$(3.2) \qquad f(x, y)+(1-x)(1-y)^{1-\beta}f\left(\frac{u}{1-x}, \frac{v}{1-y}\right)$$
$$= f(u, v)+(1-u)(1-v)^{1-\beta}f\left(\frac{x}{1-u}, \frac{y}{1-v}\right),$$

for all $x, y, u, v \in [0, 1)$, $x+u$, $y+v \in [0, 1]$ and the boundary conditions,

$$(3.3) \qquad f(1, 1) = f(0, 0) = 0,$$

$$(3.4) \qquad f(\beta, 1) = f(1-\beta, 0) = 1 \quad (0<\beta<1).$$

Proof. Postulate I for $n=3$ gives

$$(3.5) \qquad L_3\begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix} = L_2\begin{pmatrix} p_1+p_2, p_3 \\ q_1+q_2, q_3 \end{pmatrix}$$
$$+(p_1+p_2)(q_1+q_2)^{1-\beta}L_2\begin{pmatrix} \dfrac{p_1}{p_1+p_2}, \dfrac{p_2}{p_1+p_2} \\ \dfrac{q_1}{q_1+q_2}, \dfrac{q_2}{q_1+q_2} \end{pmatrix}$$

for $p_1+p_2, q_1+q_2>0$.

Interchanging p_1 and p_2 and q_1 and q_2 in (3.5) and then equating the right hand sides, we get

$$(3.6)\qquad L_2\begin{pmatrix} \frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2} \\ \frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2} \end{pmatrix} = L_2\begin{pmatrix} \frac{p_2}{p_1+p_2}, \frac{p_1}{p_1+p_2} \\ \frac{q_2}{q_1+q_2}, \frac{q_1}{q_1+q_2} \end{pmatrix},$$

for $p_1, p_2, q_1, q_2 \in [0, 1)$ with $p_1+p_2, q_1+q_2 \in (0, 1]$.

Hence (3.1) and (3.6) yield

$$f\left(\frac{p_1}{p_1+p_2}, \frac{q_1}{q_1+q_2}\right) = f\left(\frac{p_2}{p_1+p_2}, \frac{q_2}{q_1+q_2}\right),$$

which is equivalent to

$$(3.7)\qquad f(x, y) = f(1-x, 1-y) \quad \text{for all} \quad x, y \in [0, 1].$$

Rewriting postulate II, we have

$$(3.8)\qquad L_3\begin{pmatrix} x, 1-x-u, u \\ y, 1-y-v, v \end{pmatrix} = L_3\begin{pmatrix} u, 1-x-u, x \\ v, 1-y-v, y \end{pmatrix},$$

where $x, y, u, v, 1-x-u, 1-y-v \in [0, 1]$.

Now (3.8) with the help of postulate I for $n=3$ and (3.1) gives

$$(3.9)\qquad f(1-u, 1-v) + (1-u)(1-v)^{1-\beta} f\left(\frac{x}{1-u}, \frac{y}{1-v}\right)$$
$$= f(1-x, 1-y) + (1-x)(1-y)^{1-\beta} f\left(\frac{u}{1-x}, \frac{v}{1-y}\right),$$

for $x, y, u, v \in [0, 1)$ with $x+u, y+u \in [0, 1]$.

In view of (3.7), (3.9) takes the form

$$f(u, v) + (1-u)(1-v)^{1-\beta} f\left(\frac{x}{1-u}, \frac{y}{1-v}\right)$$
$$= f(x, y) + (1-x)(1-y)^{1-\beta} f\left(\frac{u}{1-x}, \frac{v}{1-y}\right),$$

which is precisely (3.2).

Putting $x=\beta$, $y=1-\beta$, $u=v=0$ in (3.2), we get

$$(3.10)\qquad f(0, 0) = 0, \quad \text{for} \quad 0<\beta<1.$$

Next postulate II on using postulate I and (3.1) gives $f(1, 1) = f(0, 0) = 0$, by using (3.10), which is precisely (3.3).

Furthermore (3.4) results from postulate III, (3.1) and (3.7) for $x=\beta$ and $y=1$.

Theorem 2. *The only solution* $f(x, y)$ *of* (3.2) *satisfying the additional conditions* (3.3) *and* (3.4) *is given by*

$$(3.11)\qquad f(x, y) = [1 - x y^{1-\beta} - (1-x)(1-y)^{1-\beta}](1-\beta)^{-1}, \quad \text{for} \quad x, y \in [0, 1].$$

(*We shall take* $0^{\alpha}=0$ $(\alpha\neq 0)$)

Proof. Substituting

$$p=\frac{u}{1-x},\quad q=\frac{v}{1-y},\quad r=1-x \text{ and } s=1-y$$

in (3.2), it reduces to

$$f(1-r,1-s)+rs^{1-\beta}f(p,q)=f(pr,qs)+(1-pr)(1-qs)^{1-\beta}f\left(\frac{1-r}{1-pr},\frac{1-s}{1-qs}\right), \tag{3.12}$$

for all $p,q\in[0,1]$, $r,s\in(0,1]$ such that $pr\neq 1$ and $qs\neq 1$.

Let $r=\beta$, $s=1$ in (3.12), then we have

$$f(1-\beta,0)+\beta f(p,q)=f(p\beta,q)+(1-p\beta)(1-q)^{1-\beta}f\left(\frac{1-\beta}{1-p\beta},0\right), \tag{3.13}$$

for $p\in[0,1]$, $q\in[0,1)$.

Taking $p=0$ in (3.13) and using (3.4), we get

$$f(0,q)=[1-(1-q)^{1-\beta}](1-\beta)^{-1},\quad \text{for}\quad q\in[0,1). \tag{3.14}$$

Setting $p=0$, $q=1$ in (3.12), we get

$$f(1-r,1-s)+rs^{1-\beta}f(0,1)=f(0,s)+(1-s)^{1-\beta}f(1-r,1), \tag{3.15}$$

for $s\in(0,1)$, $r\in(0,1]$.

Now (3.14) and (3.15) for $r=1$ give

$$f(0,1)=(1-\beta)^{-1}. \tag{3.16}$$

Again taking $s=1$, $p=0$ in (3.12), we get

$$f(1-r,0)+rf(0,q)=f(0,q)+(1-q)^{1-\beta}f(1-r,0), \tag{3.17}$$

which on using (3.14) gives

$$f(1-r,0)=\frac{1-r}{1-\beta},\quad \text{for}\quad r\in(0,1].$$

or

$$f(r,0)=\frac{r}{1-\beta},\quad \text{for}\quad r\in[0,1). \tag{3.18}$$

Also for $p=1$, $q=0$, (3.12) becomes

$$f(1-r,1-s)+rs^{1-\beta}f(1,0)=f(r,0)+(1-r)f(1,1-s), \tag{3.19}$$

for $r\in(0,1), s\in(0,1]$.

Now (3.19) with $s=1$ and (3.18) will give

(3.20) $$f(1,0)=(1-\beta)^{-1}.$$

Further (3.15) and (3.19) on using (3.17) and (3.20) give

(3.21) $$f(0,s)+(1-s)^{1-\beta}f(1-r,1)=f(r,0)+(1-r)f(1,1-s),\quad \text{for } r,s\in(0,1).$$

From (3.21) for $r=1-\beta$, (3.14) and (3.4), we have

(3.22) $$f(1,1-s)=[1-(1-s)^{1-\beta}](1-\beta)^{-1},\quad \text{for } s\in(0,1).$$

Hence (3.21), (3.22), (3.14) and (3.18) give

(3.23) $$f(1-r,1)=r(1-\beta)^{-1},\quad \text{for } r\in(0,1).$$

Thus (3.14), (3.16), (3.18), (3.20), (3.22) and (3.23) give $f(0,x), f(x,0)$, $f(1,x), f(x,1)$ for $x\in[0,1]$.

Also (3.19), (3.22), (3.18) and (3.20) give

(3.24) $$f(1-r,1-s)=[1-rs^{1-\beta}-(1-r)(1-s)^{1-\beta}](1-\beta)^{-1},\quad \text{for } r,s\in(0,1).$$

It is easy to see from (3.14), (3.16), (3.18), (3.20), (3.22), (3.23) and (3.24) that f is given by (3.11) with the notation $0^{\alpha}=0$ $(\alpha\neq 0)$ and that (3.7) is true for all $x,y\in[0,1]$.

Theorem 3. *Given two distributions $P=(p_1,p_2,\cdots,p_n)$, $Q=(q_1,q_2,\cdots,q_n)$ the inaccuracy of kind β derived from the inaccuracy function $f(x,y)$ is in general given by*

(3.25) $$H_n^{\beta}(P;\,Q)=H_n^{\beta}\begin{pmatrix}p_1, p_2, \cdots, p_n\\ q_1, q_2, \cdots, q_n\end{pmatrix}=\sum_{i=2}^{n} r_i s_i^{1-\beta} f\left(\frac{p_i}{r_i}, \frac{q_i}{s_i}\right),$$

where $r_i=p_1+p_2+\cdots+p_i$, $s_i=q_1+q_2+\cdots+q_i$; $i=1,2,\cdots,n$, with $r_n=s_n=1$, and if the inaccuracy function is $f(x,y)$ obtained in Theorem 2, then the corresponding inaccuracy of kind β is

(3.26) $$H_n^{\beta}(P;\,Q)=\frac{1-\sum_{i=1}^{n} p_i q_i^{1-\beta}}{1-\beta},\qquad (0<\beta<1).$$

Proof. By successive applications of postulate I and the use of (3.1), we get

$$L_n=H_n^{\beta}\begin{pmatrix}p_1, p_2, \cdots, p_n\\ q_1, q_2, \cdots, q_n\end{pmatrix}=\sum_{i=2}^{n} r_i s_i^{1-\beta} f\left(\frac{p_i}{r_i}, \frac{q_i}{s_i}\right),$$

where $r_i=p_1+p_2+\cdots+p_i$, $s_i=q_1+q_2+\cdots+q_i$; $i=1,2,\cdots,n$, with $r_n=s_n=1$. Substituting the expression for $f(x,y)$ from (3.11) in (3.25), (3.26) is obtained for inaccuracy of kind β.

For the purposes of characterizing inaccuracy of kind β, we have taken $0<\beta<1$. But the expression would be taken for other values of β also by definition.

The inaccuracy of kind β studied above is non-negative, a monotonic increasing function of β and expansible. Further it has the normalized property

$$H_2^{\beta}\begin{pmatrix}\beta, 1-\beta\\ 1, \ \ 0\end{pmatrix}=1 \qquad (0<\beta<1).$$

We now list two other properties of this measure.

1. *Recursive property*:

$$H_n^{\beta}\begin{pmatrix}p_1, p_2, \cdots, p_n\\ q_1, q_2, \cdots, q_n\end{pmatrix}-H_{n-1}^{\beta}\begin{pmatrix}p_1+p_2, p_3, \cdots, p_n\\ q_1+q_2, q_3, \cdots, q_n\end{pmatrix}$$
$$=(p_1+p_2)(q_1+q_2)^{1-\beta}H_2^{\beta}\begin{pmatrix}\dfrac{p_1}{p_1+p_2}, \dfrac{p_2}{p_1+p_2}\\ \dfrac{q_1}{q_1+q_2}, \dfrac{q_2}{q_1+q_2}\end{pmatrix}; \quad (n\geqq 3),$$

with $p_1+p_2, q_1+q_2>0$.

2. *Strongly-additive property*:

$$H_{mn}^{\beta}\begin{pmatrix}p_1p_{11}, p_1p_{21}, \cdots, p_1p_{m1}, \cdots, p_np_{1n}, \cdots, p_np_{mn}\\ q_1q_{11}, q_1q_{21}, \cdots, q_1q_{m1}, \cdots, q_nq_{1n}, \cdots, q_nq_{mn}\end{pmatrix}$$
$$=H_n^{\beta}\begin{pmatrix}p_1, p_2, \cdots, p_n\\ q_1, q_2, \cdots, q_n\end{pmatrix}+\sum_{i=1}^{n} p_i q_i^{1-\beta} H_m^{\beta}\begin{pmatrix}p_{1i}, p_{2i}, \cdots, p_{mi}\\ q_{1i}, q_{2i}, \cdots, q_{mi}\end{pmatrix},$$

where $\sum_{j=1}^{m} p_{ji}=1$ and $\sum_{j=1}^{m} q_{ji}=1$ for all $i=1, 2, \cdots, n$.

An interesting special case of 2 may be put as

$$H_{mn}^{\beta}\begin{pmatrix}P*P'\\ Q*Q'\end{pmatrix}=H_n^{\beta}\begin{pmatrix}P\\ Q\end{pmatrix}+H_m^{\beta}\begin{pmatrix}P'\\ Q'\end{pmatrix}+(\beta-1)H_n^{\beta}\begin{pmatrix}P\\ Q\end{pmatrix}H_m^{\beta}\begin{pmatrix}P'\\ Q'\end{pmatrix}$$

where $P=(p, p_2, \cdots, p_n)$, $Q=(q_1, q_2, \cdots, q_n)$, $P'=(P_1, P_2, \cdots, P_m)$, $Q'=(Q_1, Q_2, \cdots, Q_m)$, $P*P'=(p_1P_1, q_1P_2, \cdots, p_1P_m, \cdots, p_nP_1, p_nP_2, \cdots, p_nP_m)$ and $\sum_{j=1}^{m} P_j=1$, $\sum_{j=1}^{m} Q_j=1$.

4. Entropy of Kind β.

Following theorem is a characterization of entropy of Kind β.

Theorem 4. *If the inaccuracy*

$$L_n\begin{pmatrix}p_1, p_2, \cdots, p_n\\ q_1, q_2, \cdots, q_n\end{pmatrix}; \quad n=2, 3, \cdots,$$

of kind β of $P=(p_1, p_2, \cdots, p_n)$, $Q=(q_1, q_2, \cdots, q_n)$ *satisfies the postulates* I-III, *then the entropy of kind* β, $H_n^\beta(p_1, p_2, \cdots, p_n)$ *under the additional postulate*:

$$H_n^\beta(p_1, p_2, \cdots, p_n)=\operatorname*{Inf.}_Q L_n\begin{pmatrix} p_1, p_2, \cdots, p_n \\ q_1, q_2, \cdots, q_n \end{pmatrix} \tag{4.1}$$

is given by

$$H_n^\beta(p_1, p_2, \cdots, p_n)=\begin{cases} 1-\max. p_k, & \text{for } \beta\neq 0 \\ \dfrac{1-\left(\sum_{k=1}^n p_k^{\frac{1}{\beta}}\right)^\beta}{1-\beta}, & \textit{for } \beta\neq 1 \textit{ and } \beta>0 \\ -\sum_{k=1}^n p_k \log p_k, & \text{for } \beta=1. \end{cases} \tag{4.2}$$

Proof. From Theorem 3,

$$H_n^\beta(p_1, p_2, \cdots, p_n)=\operatorname*{Inf.}_Q \left[\frac{1-\sum_{i=1}^n p_i q_i^{1-\beta}}{1-\beta}\right]. \tag{4.3}$$

Now the right hand side of (4.2) is the infimum of an expression in the bracket (refer [2]).

The entropy $H_n^\beta(p_1, p_2, \cdots, p_n)$ is non-negative, expansible, symmetric in p's, monotonic increasing function of β, concave function of the distribution (refer [2]). Some more properties are listed in the next theorem.

Theorem 5. *The entropy* $H_n^\beta(p_1, p_2, \cdots, p_n)$ *has the following three properties*:

1. *Non-Commutative Property*:

$$H_{mn}^\beta(P*Q)=H_m^\beta(P)+\left(\sum_{i=1}^m p_i^{\frac{1}{\beta}}\right)^\beta H_n^\beta(Q)$$
$$=H_n^\beta(Q)+\left(\sum_{j=1}^n q_j^{\frac{1}{\beta}}\right)^\beta H_m^\beta(P)$$

where $H_{mn}^\beta(P*Q)=H_{mn}^\beta(p_1q_1, \cdots, p_1q_n, \cdots, p_mq_1, \cdots, p_mq_n)$.

2. *Maximal Property*:

$$H_n^\beta(p_1, p_2, \cdots, p_n)\leqq H_n^\beta\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right), \quad \textit{since}$$

$$\operatorname*{Max.}_P H_n^\beta(P)=\operatorname*{Min.}_P H_n^\beta(p_1, p_2, \cdots, p_n)=\left[\frac{1-\operatorname*{Min.}_P\left(\sum_{i=1}^n p_i^{\frac{1}{\beta}}\right)^\beta}{1-\beta}\right]$$
$$=H_n^\beta\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right); \quad \textit{for } \beta<1, \quad \textit{and}$$

$$\underset{P}{\text{Max.}}\ H_n^\beta(P)=\underset{P}{\text{Max.}}\ H_n^\beta(p_1, p_2, \cdots, p_n)=\left[\frac{1-\underset{P}{\text{Max.}}\left(\sum_{i=1}^n p_i^{\frac{1}{\beta}}\right)^\beta}{1-\beta}\right]$$

$$=H_n^\beta\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right); \quad \textit{for} \quad \beta>1.$$

3. $H_n^\beta\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$ *is an increasing function of* n, *i.e.*,

$$H_{n+1}^\beta\left(\frac{1}{n+1}, \frac{1}{n+1}, \cdots, \frac{1}{n+1}\right)\geqq H_n^\beta\left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right).$$

Theorem 6. *Let* $A=a_{ij}$ *be a doubly stochastic matrix, that is,* $a_{ij}\geqq 0$ *for all* i, j;

$$\sum_{j=1}^n a_{ij}=1, \quad i=1, 2, \cdots, n;$$

$$\sum_{i=1}^n a_{ij}=1, \quad j=1, 2, \cdots, n.$$

Given a set of probabilities $p_1, p_2, \cdots, p_n$, *define a new set of probabilities* $p_1', p_2', \cdots, p_n'$ *by*

$$p_i'=\sum_{j=1}^n a_{ij}p_j, \quad i=1, 2, \cdots, n.$$

Then $H_n^\beta(p_1', p_2', \cdots, p_n')\geqq H_n^\beta(p_1, p_2, \cdots, p_n)$ *and there is equality if* $(p_1', p_2', \cdots, p_n')$ *is a rearrangement of* $(p_1, p_2, \cdots, p_n)$.

Proof.

$$H_n^\beta(p_1', p_2', \cdots, p_n')=\frac{1-\left(\sum_{i=1}^n p_i'^{\frac{1}{\beta}}\right)^\beta}{1-\beta} \quad \textit{for} \quad \beta\neq 1, \quad \beta>0 \tag{4.4}$$

$$=\frac{1-\left(\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}p_j\right)^{\frac{1}{\beta}}\right)^\beta}{1-\beta}$$

Now from the inequalities

$$\left(\sum_{j=1}^n a_{ij}p_j\right)^{\frac{1}{\beta}}\leqq\sum_{j=1}^n a_{ij}p_j^{\frac{1}{\beta}}; \quad \beta<1,$$

$$\geqq\sum_{j=1}^n a_{ij}p_j^{\frac{1}{\beta}}; \quad \beta>1,$$

we get in either case

$$\frac{1-\left(\sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}p_j\right)^{\frac{1}{\beta}}\right)^\beta}{1-\beta}\geqq\frac{1-\left(\sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j^{\frac{1}{\beta}}\right)^\beta}{1-\beta} \tag{4.5}$$

Thus

$$H_n^\beta(p_1', p_2', \cdots, p_n') \geqq \frac{1-\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} p_j^{\frac{1}{\beta}}\right)^\beta}{1-\beta} = \frac{1-\left[\left(\sum_{j=1}^{n} p_j^{\frac{1}{\beta}}\right)\left(\sum_{i=1}^{n} a_{ij}\right)\right]^\beta}{1-\beta}$$

$$= \frac{1-\left(\sum_{j=1}^{n} p_j^{\frac{1}{\beta}}\right)^\beta}{1-\beta} = H_n^\beta(p_1, p_2, \cdots, p_n),$$

$$\left(\text{Note } \sum_{i=1}^{n} a_{ij}=1\right)$$

so that $H_n^\beta(p_1', p_2', \cdots, p_n') \geqq H_n^\beta(p_1, p_2, \cdots, p_n)$ for $\beta \neq 1$.

It would be clearly noted that if p_i' is equal to some p_j i. e., if $(p_1', p_2', \cdots, p_n')$ is a rearrangement of $(p_1, p_2, \cdots, p_n)$, then $H_n^\beta(p_1', p_2', \cdots, p_n') = H_n^\beta(p_1, p_2, \cdots, p_n)$.

The new measure of entropy proposed would be found useful for finite parameter estimation problem in a generalized way. However, this needs further investigations. The kind β inaccuracy can be considered as some sort of difference between two distributions and may be useful in comparing them.

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