Funkcialaj Ekvacioj, **16** (1973) 169-180

An Analytical Method for Certain Highly Nonlinear Periodic Differential Equations

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Abstract.

An analytical method is introduced for certain highly nonlinear periodic differential equations by inserting artificially a small parameter ε in the coefficients of certain harmonics of the trigonometric polynomial for an approximate periodic solution. This method is an appropriate combination of the harmonic balance method and the perturbation method. The higher improved approximations can be systematically obtained and the error in the equation in the nth improved approximation is proved to be of the order of ε^n. As an illustration of this method the equation $\ddot{x}+x^3=\sin t$ is examined here. The same example had been studied independently and by different view points by L. Cesari and by M. Urabe.

1. Introduction.

In [1] M. Urabe and A. Reiter calculated Galerkin approximations up to the order 15 for periodic solutions of certain ordinary differential equations with a very high precision and using an electronic computer, which was programmed to perform the entire design. They applied their numerical method to a weakly nonlinear van der Pol equation, as well as to certain highly nonlinear equations like the equation

$$\ddot{x}+x^3=\sin t \tag{1.1}$$

which had been first studied by L. Cesari [2] under a different viewpoint.

In order to avoid the numerical character of M. Urabe's method for Galerkin's procedure we introduced in [3] and [4] an analytical method which is a combination of two classical methods, the harmonic balance method and the perturbation method, for certain weakly nonlinear periodic differential systems which can be written as follows

$$\ddot{x}_i+\omega_i^2 x_i=\varepsilon f_i(x, \dot{x}, t), \qquad i=1,2,\cdots,p \tag{1.2}$$

where $x, \dot{x}$ and f are vectors of the same dimension p, and ε is a small parameter. A dot means differentiation with respect to time t. When applied to the van der Pol equation considered by M. Urabe and A. Reiter and also to coupled

Duffing equations with two-degrees-of-freedom, this analytical method yields excellent results compared to the Galerkin approximations of high order.

It is very surprising that a similar procedure as for weakly nonlinear periodic differential systems can be applied to certain highly nonlinear periodic differential equations of the following form

$$\ddot{x}=f(x, \dot{x}, t) \tag{1.3}$$

by introducing artificially a small parameter ε in the coefficients of certain harmonics of the trigonometric polynomial for an approximate periodic solution. In view to apply this analytical method to equation (1.1) and analogous ones, we assume in this paper that $f(x, \dot{x}, t)$ contains the fundamental harmonics $\sin t$ and $\cos t$, whose coefficients are not small, and that it is a polynomial in x and $\dot{x}$. In the first improved approximation of our method, essentially derived by a harmonic balance method, we have to solve a nonlinear system as in any other analytical method. In all higher improved approximations, which are systematically introduced by a perturbation method, the problem reduces to that of solving linear equations.

The error in equation (1.3) in the nth improved approximation is then $O(\varepsilon^n)$. As a numerical application of our method we consider equation (1.1) previously studied by L. Cesari [2] and by M. Urabe and A. Reiter [1]. We show that the higher improved approximations are comparable with the Galerkin approximation of order 15 calculated by M. Urabe and A. Reiter using an electronic computer. Our results, compared with those of M. Urabe and A. Reiter, are summarized in Table 1 at the end of section 6.

2. The first improved approximation.

We consider a real periodic differential equation of the following form

$$\ddot{x}=f(x, \dot{x}, t) \tag{2.1}$$

where a dot means differentiation with respect to time t. We assume that $f(x, \dot{x}, t)$ contains the fundamental harmonics $\sin t$ and $\cos t$ with coefficients which are not small and that it is a polynomial in x and $\dot{x}$ whose coefficients are trigonometric polynomials.

We propose to look for an approximate periodic solution of equation (2.1) with the same period 2π as the period of the forcing terms $\sin t$ and $\cos t$ in that equation, represented by a trigonometric polynomial (a truncated Fourier series) in which the first harmonics are excited due to the occurrence in equation (2.1) of the fundamental harmonics whose coefficients are not small

$$x_m(t, \varepsilon)=a_0+a_1 \sin t+a_2 \cos t+\varepsilon \sum_{n=2}^{m} (a_{2n-1} \sin nt+a_{2n} \cos nt) \tag{2.2}$$

Thus we introduce artificially a small parameter ε in the coefficients of all harmonics of order higher than one to take into account that the higher harmonics are negligible with respect to the first harmonics. The small parameter ε is not inserted in the constant term a_0 of the trigonometric polynomial (2.2), for otherwise this should lead to certain contradictions in the resulting system (2.9) determining the unknown coefficients a_ν. These coefficients a_ν will be determined by a harmonic balance procedure and m will be chosen sufficiently large. From (2.2) we have

$$(2.3)\qquad \dot{x}_m(t,\varepsilon)=-a_2\sin t+a_1\cos t+\varepsilon\sum_{n=2}^{m} n(-a_{2n}\sin nt+a_{2n-1}\cos nt)$$

Hence for $\varepsilon=0$

$$(2.4)\qquad \begin{aligned} x_m(t,0)&=x_1(t)=a_0+a_1\sin t+a_2\cos t\\ \dot{x}_m(t,0)&=\dot{x}_1(t)=\quad -a_2\sin t+a_1\cos t \end{aligned}$$

Since $f(x,\dot{x},t)$ is a polynomial in x and $\dot{x}$, we can expand $f(x_1(t),\dot{x}_1(t),t)$ in a trigonometric polynomial using some trigonometric identities as follows:

$$(2.5)\qquad f(x_1(t),\dot{x}_1(t),t)=f_0+\sum_{n=1}^{n_1}(f_{2n-1}\sin nt+f_{2n}\cos nt)$$

in which f_ν are usually nonlinear functions of a_0, a_1, a_2.

Let us determine the unknown coefficients by the following equation:

$$(2.6)\qquad \ddot{x}_m(t,\varepsilon)=f_m(x_1(t),\dot{x}_1(t),t)$$

with

$$(2.7)\qquad \begin{aligned} f_m(x_1(t),\dot{x}_1(t),t)&=f_0+\sum_{n=1}^{m}(f_{2n-1}\sin nt+f_{2n}\cos nt)\\ f_{2n-1}=f_{2n}&=0 \quad \text{if} \quad n=n_1+1,\cdots,m,\quad m>n_1 \end{aligned}$$

In this procedure we replace x and $\dot{x}$ respectively by $x_1(t)$ and $\dot{x}_1(t)$ in the right member $f(x,\dot{x},t)$ of equation (2.1) which contains the nonlinear terms. Compared to Galerkin's procedure [5] where we should replace x and $\dot{x}$ respectively by x_m and $\dot{x}_m$ everywhere in the equation, we here avoid the tedious computations which result from the occurrence of the nonlinear terms in $f(x,\dot{x},t)$.

The left member of (2.6) becomes

$$(2.8)\qquad \ddot{x}_m(t,\varepsilon)=-a_1\sin t-a_2\cos t-\varepsilon\sum_{n=2}^{m} n^2(a_{2n-1}\sin nt+a_{2n}\cos nt)$$

Applying a harmonic balance procedure by equating the coefficients of 1, $\sin nt$, $\cos nt$ in (2.6), we obtain

$$(2.9\,\text{abc})\qquad \begin{cases} 0=f_0 \\ -a_1=f_1 \\ -a_2=f_2 \end{cases}$$

$$(2.9)\qquad \begin{aligned} &\begin{cases} -n^2\varepsilon a_{2n-1}=f_{2n-1} \\ -n^2\varepsilon a_{2n}\ \ =f_{2n} \end{cases} \qquad n=2,\cdots,n_1 \\ &\begin{cases} -n^2\varepsilon a_{2n-1}=0 \\ -n^2\varepsilon a_{2n}\ \ =0 \end{cases} \qquad n=n_1+1,\cdots,m \end{aligned}$$

Let $\bar{a}_0, \bar{a}_1, \bar{a}_2$ be an approximate solution of the three nonlinear equations (2.9 abc) for which the determinant of the Jacobian matrix with respect to a_0, a_1, a_2 calculated for $\bar{a}_0, \bar{a}_1, \bar{a}_2$ different from zero. Then the other coefficients are given by

$$(2.10)\qquad \begin{aligned} &\begin{cases} \varepsilon\bar{a}_{2n-1}=-n^{-2}\bar{f}_{2n-1} \\ \varepsilon\bar{a}_{2n}\ \ =-n^{-2}\bar{f}_{2n} \end{cases} \qquad n=2,\cdots,n_1 \\ &\varepsilon\bar{a}_{2n-1}=\varepsilon\bar{a}_{2n}=0 \qquad n=n_1+1,\cdots,m. \end{aligned}$$

The symbol $\bar{f}_\nu$ means that the value of the corresponding expression is calculated for $a_0=\bar{a}_0, a_1=\bar{a}_1, a_2=\bar{a}_2$.

Thus an approximate periodic solution of (2.1) is

$$(2.11)\qquad \bar{x}_m(t,\varepsilon)=\bar{a}_0+\bar{a}_1\sin t+\bar{a}_2\cos t+\varepsilon\sum_{n=2}^{m}(\bar{a}_{2n-1}\sin nt+\bar{a}_{2n}\cos nt)$$

We call this approximate solution which is evidently an approximation of the Galerkin approximation, the *first improved approximation* of our method. The terminology "improved" is used in the sense of N. BOGOLIUBOFF and I. MITROPOLSKI [6] to indicate an amelioration of the first approximation $\bar{x}_1(t)$ given by

$$(2.12)\qquad \bar{x}_1(t)=\bar{a}_0+\bar{a}_1\sin t+\bar{a}_2\cos t$$

which is not as good as a second approximation where a modification of the coefficients of the constant term and the first harmonics is also taken into account.

3. The second improved approximation.

Let

$$(3.1)\qquad x(t,\varepsilon)=\bar{x}_m(t,\varepsilon)+\varepsilon y(t,\varepsilon)$$

Substituting this into equation (2.1) and taking into account that $\bar{x}_m(t,\varepsilon)$ is the first improved approximation satisfying (2.6) and that m is sufficiently large $(m>n_1)$, we obtain that $y(t,\varepsilon)$ satisfies the equation

$$\ddot{y}=g(y,\dot{y},t,\varepsilon) \tag{3.2}$$

where

$$g(y,\dot{y},t,\varepsilon)=\varepsilon^{-1}[f(\bar{x}_m+\varepsilon y,\dot{\bar{x}}_m+\varepsilon\dot{y},t)-f(\bar{x}_1,\dot{\bar{x}}_1,t)] \tag{3.3}$$

Following a similar procedure as in the first improved approximation, we look for an approximate periodic solution of equation (3.2) with period 2π represented by the trigonometric polynomial

$$y_m(t,\varepsilon)=b_0+b_1\sin t+b_2\cos t+\varepsilon\sum_{n=2}^{m}(b_{2n-1}\sin nt+b_{2n}\cos nt) \tag{3.4}$$

with undetermined coefficients b_ν and m sufficiently large. Hence for $\varepsilon=0$

$$\begin{aligned} y_m(t,0)&=y_1(t)=b_0+b_1\sin t+b_2\cos t\\ \dot{y}_m(t,0)&=\dot{y}_1(t)=\quad -b_2\sin t+b_1\cos t. \end{aligned} \tag{3.5}$$

Expanding $g(y_1(t),\dot{y}_1(t),t,0)$ in a trigonometric polynomial we have

$$g(y_1(t),\dot{y}_1(t),t,0)=g_0+\sum_{n=1}^{n_2}(g_{2n-1}\sin nt+g_{2n}\cos nt) \tag{3.6}$$

The coefficients b_ν will be determined by the following equation

$$\ddot{y}_m(t,\varepsilon)=g_m(y_1(t),\dot{y}_1(t),t,0) \tag{3.7}$$

where

$$\begin{gathered} g_m(y_1(t),\dot{y}_1(t),t,0)=g_0+\sum_{n=1}^{m}(g_{2n-1}\sin nt+g_{2n}\cos nt)\\ g_{2n-1}=g_{2n}=0 \quad \text{if} \quad n=n_2+1,\cdots,m, \qquad m>n_2. \end{gathered} \tag{3.8}$$

Then we equate the coefficients of 1, $\sin nt$, $\cos nt$ in (3.7). This yields

$$\begin{cases} \quad\;\; =g_0\\ -b_1=g_1\\ -b_2=g_2 \end{cases} \tag{3.9 abc}$$

$$\begin{aligned} &\begin{cases} -n^2\varepsilon b_{2n-1}=g_{2n-1}\\ -n^2\varepsilon b_{2n}\;\;\, =g_{2n} \end{cases} \qquad n=2,\cdots,n_2\\ &\begin{cases} -n^2\varepsilon b_{2n-1}=0\\ -n^2\varepsilon b_{2n}\;\;\, =0 \end{cases} \qquad n=n_2+1,\cdots,m \end{aligned} \tag{3.9}$$

Now $g(y_1(t),\dot{y}_1(t),t,0)$ is linear b_0,b_1,b_2 as can be easily seen by expanding $f(\bar{x}_m+\varepsilon y,\dot{\bar{x}}_m+\varepsilon\dot{y},t)$ in a Taylor series with respect to $(\bar{x}_1(t),\dot{\bar{x}}_1(t),t)$ indicated hereafter by a lower index 0,

$$g(y,\dot{y},t,\varepsilon)=\left(\frac{\partial f}{\partial x}\right)_0(\hat{A}+y)+\left(\frac{\partial f}{\partial\dot{x}}\right)_0(\dot{\hat{A}}+\dot{y})+O(\varepsilon) \tag{3.10}$$

with

$$\hat{A}=\sum_{n=2}^{m}(\bar{a}_{2n-1}\sin nt+\bar{a}_{2n}\cos nt)$$

$$\dot{\hat{A}}=\sum_{n=2}^{m}n(-\bar{a}_{2n}\sin nt+\bar{a}_{2n-1}\cos nt) \tag{3.11}$$

and consequently

$$g(y_1(t),\dot{y}_1(t),t,0)=\left(\frac{\partial f}{\partial x}\right)_0[\hat{A}+y_1(t)]+\left(\frac{\partial f}{\partial \dot{x}}\right)_0[\dot{\hat{A}}+\dot{y}_1(t)]. \tag{3.12}$$

Thus the three equations (3.9 abc) are linear in b_0, b_1, b_2. We remark that the non-vanishing condition of the determinant for this linear system is the same as the non-vanishing condition applied to the determinant of the Jacobian matrix related to the three equations (2.9 abc) with respect to a_0, a_1, a_2 and calculated for $\bar{a}_0, \bar{a}_1, \bar{a}_2$ determined by (2.9 abc). Since we assumed that this determinant is different from zero we have the unique solution $\bar{b}_0, \bar{b}_1, \bar{b}_2$. Then the remaining equations of (3.9) are satisfied by

$$\begin{cases}\varepsilon\bar{b}_{2n-1}=-n^{-2}\bar{g}_{2n-1}\\ \varepsilon\bar{b}_{2n}\ \ =-n^{-2}\bar{g}_{2n}\end{cases}\quad n=2,\cdots,n_2 \tag{3.13}$$

$$\varepsilon\bar{b}_{2n-1}=\varepsilon\bar{b}_{2n}=0,\quad n=n_2+1,\cdots,m.$$

Consequently an approximate periodic solution of (3.2) is given by

$$\bar{y}_m(t,\varepsilon)=\bar{b}_0+\bar{b}_1\sin t+\bar{b}_2\cos t+\varepsilon\sum_{n=2}^{m}(\bar{b}_{2n-1}\sin nt+\bar{b}_{2n}\cos nt) \tag{3.14}$$

and the second improved approximation of a periodic solution of (2.1) is given by

$$\begin{aligned}x(t,\varepsilon)=&(\bar{a}_0+\varepsilon\bar{b}_0)+(\bar{a}_1+\varepsilon\bar{b}_1)\sin t+(\bar{a}_2+\varepsilon\bar{b}_2)\cos t\\&+\sum_{n=2}^{m}[(\varepsilon\bar{a}_{2n-1}+\varepsilon^2\bar{b}_{2n-1})\sin nt+(\varepsilon\bar{a}_{2n}+\varepsilon^2\bar{b}_{2n})\cos nt]\end{aligned} \tag{3.15}$$

4. The higher improved approximations.

The higher improved approximations can be found following the same procedure as the one determining the second improved approximation. In all higher improved approximations the equations determining the first three coefficients of the considered trigonometric polynomial are linear and the determinant of this linear system is the same as the one in the second improved approximation and consequently by assumption different from zero. Further details are omitted here.

Finally we conclude that the improved approximations are obtained systematically by an analytical method which is an appropriate combination of the harmonic balance method and the perturbation method.

5. The error in the equation.

The error in equation (2.1) is defined by

$$\tilde{\eta}_m(t)=\ddot{\tilde{x}}_m-f(\tilde{x}_m, \dot{\tilde{x}}_m, t) \tag{5.1}$$

with $\tilde{x}_m$ an approximate periodic solution of the equation (2.1).

Let us determine the error in equation (2.1) in the second improved approximation. In this approximation the approximate periodic solution of (2.1) is

$$\tilde{x}_m(t, \varepsilon)=\bar{x}_m(t, \varepsilon)+\varepsilon\bar{y}_m(t, \varepsilon) \tag{5.2}$$

in which $\bar{x}_m(t, \varepsilon)$ and $\bar{y}_m(t, \varepsilon)$ satisfy the equations

$$\begin{aligned}\ddot{\bar{x}}_m(t, \varepsilon)&=f_m(\bar{x}_1(t), \dot{\bar{x}}_1(t), t)\\ \ddot{\bar{y}}_m(t, \varepsilon)&=g_m(\bar{y}_1(t), \dot{\bar{y}}_1(t), t, 0)\end{aligned} \tag{5.3}$$

hence

$$\ddot{\tilde{x}}_m(t, \varepsilon)=f_m(\bar{x}_1(t), \dot{\bar{x}}_1(t), t)+\varepsilon g_m(\bar{y}_1(t), \dot{\bar{y}}_1(t), t, 0). \tag{5.4}$$

Consequently

$$\tilde{\eta}_m(t)=-[f(\tilde{x}_m, \dot{\tilde{x}}_m, t)-f_m(\bar{x}_1(t), \dot{\bar{x}}_1(t), t)-\varepsilon g_m(\bar{y}_1(t), \dot{\bar{y}}_1(t), t, 0)]. \tag{5.5}$$

Expanding $f(\tilde{x}_m, \dot{\tilde{x}}_m, t)$ with respect to $(\bar{x}_1(t), \dot{\bar{x}}_1(t), t)$ and taking into account that $g(y_1(t), \dot{y}_1(t), t, 0)$ is given by (3.12), yields

$$\begin{aligned}f(\tilde{x}_m, \dot{\tilde{x}}_m, t)&=f(\bar{x}_1(t), \dot{\bar{x}}_1(t), t)+\varepsilon g(\bar{y}_1(t), \dot{\bar{y}}_1(t), t, 0)+O(\varepsilon^2)\\ &=f_m(\bar{x}_1(t), \dot{\bar{x}}_1(t), t)+\varepsilon g_m(\bar{y}_1(t), \dot{\bar{y}}_1(t), t, 0)+O(\varepsilon^2)\end{aligned} \tag{5.6}$$

if we choose m sufficiently large as follows

$$m>\max(n_1, n_2). \tag{5.7}$$

Substitution of (5.6) into (5.5) proves that the error in equation (2.1) in the second improved approximation is of the order of ε^2

$$\tilde{\eta}_m(t)=O(\varepsilon^2). \tag{5.8}$$

The result can be easily extended to the nth improved approximation where we obtain that the error in equation (2.1) is of the order of ε^n.

6. An application of our method.

Let us apply our analytical method to the equation studied by L. CESARI and M. URABE (see the 1st and 2nd column in table 1 for their results)

$$\ddot{x}+x^3=\sin t. \tag{6.1}$$

The function f is defined by

$$f(x,t)=-x^3+\sin t. \tag{6.2}$$

It contains the fundamental harmonic $\sin t$ and a cubic term in x whose coefficients are not small.

The 1st improved approximation:

The three nonlinear equations corresponding to (2.9 abc) are

$$\begin{aligned} 0&=-\frac{1}{2}a_0(2a_0^2+3a_1^2+3a_2^2) \\ -a_1&=1-\frac{3}{4}a_1(4a_0^2+a_1^2+a_2^2) \\ -a_2&=\ -\frac{3}{4}a_2(4a_0^2+a_1^2+a_2^2) \end{aligned} \tag{6.3}$$

These equations are satisfied by

$$\bar{a}_0=\bar{a}_2=0 \tag{6.4}$$

and $\bar{a}_1$ a solution of the cubic equation

$$3a_1^3-4a_1-4=0. \tag{6.5}$$

This equation has one and only one real root

$$\bar{a}_1=1.4922\cdots \tag{6.6}$$

The determinant of the Jacobian matrix related to the three nonlinear equations (6.3) with respect to a_0, a_1, a_2 and calculated for $\bar{a}_0, \bar{a}_1, \bar{a}_2$ determined by (6.4) and (6.5) is different from zero since $\bar{a}_1$ is not a double zero of the equation (6.5).

For the other coefficients we find

$$\bar{a}_3=\bar{a}_4=\bar{a}_6=0 \tag{6.7}$$

$$\varepsilon\bar{a}_5=-\bar{a}_1^3/36=-0.0923\cdots \tag{6.8}$$

Consequently the first improved approximation of a periodic solution of (6.1) is

$$x(t,\varepsilon)=\bar{a}_1\sin t+\varepsilon\bar{a}_5\sin 3t. \tag{6.9}$$

The 2nd improved approximation:

We obtain

$$x(t,\varepsilon)=(\bar{a}_1+\varepsilon\bar{b}_1)\sin t+(\varepsilon\bar{a}_5+\varepsilon^2\bar{b}_5)\sin 3t+\varepsilon^2\bar{b}_9\sin 5t. \tag{6.10}$$

with

$$\begin{aligned} \bar{b}_1&=3\bar{a}_1^2\bar{a}_5/(9\bar{a}_1^2-4) \\ \bar{b}_5&=\bar{a}_1^2(2\bar{a}_5-\bar{b}_1)/(12\varepsilon) \\ \bar{b}_9&=-3\bar{a}_1^2\bar{a}_5/(100\varepsilon). \end{aligned} \tag{6.11}$$

The 3rd improved approximation:

$$\begin{aligned} x(t,\varepsilon)=&(\bar{a}_1+\varepsilon\bar{b}_1+\varepsilon^2\bar{c}_1)\sin t+(\varepsilon\bar{a}_5+\varepsilon^2\bar{b}_5+\varepsilon^3\bar{c}_5)\sin 3t\\ &+(\varepsilon^2\bar{b}_9+\varepsilon^3\bar{c}_9)\sin 5t+\varepsilon^3\bar{c}_{13}\sin 7t \end{aligned} \tag{6.12}$$

$$\begin{aligned} \bar{c}_1&=3\bar{a}_1(\bar{a}_1\bar{b}_5-2\bar{a}_5^2+2\bar{a}_5\bar{b}_1-3\bar{b}_1^2)/(9\bar{a}_1^2-4)\\ \bar{c}_5&=\bar{a}_1(2\bar{a}_1\bar{b}_5-\bar{a}_1\bar{b}_9-\bar{a}_1\bar{c}_1+4\bar{a}_5\bar{b}_1-\bar{b}_1^2)/(12\varepsilon)\\ \bar{c}_9&=3\bar{a}_1(-\bar{a}_1\bar{b}_5+2\bar{a}_1\bar{b}_9+\bar{a}_5^2-2\bar{a}_5\bar{b}_1)/(100\varepsilon)\\ \bar{c}_{13}&=-3\bar{a}_1(\bar{a}_1\bar{b}_9+\bar{a}_5^2)/(196\varepsilon) \end{aligned} \tag{6.13}$$

The 4th improved approximation:

$$\begin{aligned} x(t,\varepsilon)=&(\bar{a}_1+\varepsilon\bar{b}_1+\varepsilon^2\bar{c}_1+\varepsilon^3\bar{d}_1)\sin t\\ &+(\varepsilon\bar{a}_5+\varepsilon^2\bar{b}_5+\varepsilon^3\bar{c}_5+\varepsilon^4\bar{d}_5)\sin 3t\\ &+(\varepsilon^2\bar{b}_9+\varepsilon^3\bar{c}_9+\varepsilon^4\bar{d}_9)\sin 5t\\ &+(\varepsilon^3\bar{c}_{13}+\varepsilon^4\bar{d}_{13})\sin 7t\\ &+\varepsilon^4\bar{d}_{17}\sin 9t \end{aligned} \tag{6.14}$$

$$\begin{aligned} \bar{d}_1=&3[\bar{a}_1^2\bar{c}_5-4\bar{a}_1\bar{a}_5\bar{b}_5+2\bar{a}_1\bar{a}_5\bar{b}_9+2\bar{a}_1(\bar{a}_5\bar{c}_1+\bar{b}_1\bar{b}_5)-6\bar{a}_1\bar{b}_1\bar{c}_1\\ &-2\bar{a}_5^2\bar{b}_1+\bar{a}_5\bar{b}_1^2-\bar{b}_1^3]/(9\bar{a}_1^2-4)\\ \bar{d}_5=&[6\bar{a}_1^2\bar{c}_5-3\bar{a}_1^2\bar{c}_9-3\bar{a}_1^2\bar{d}_1+6\bar{a}_1\bar{a}_5\bar{b}_9+12\bar{a}_1(\bar{a}_5\bar{c}_1+\bar{b}_1\bar{b}_5)-6\bar{a}_1\bar{b}_1\bar{b}_9\\ &-6\bar{a}_1\bar{b}_1\bar{c}_1+3\bar{a}_5^3+6\bar{a}_5\bar{b}_1^2-\bar{b}_1^3]/(36\varepsilon)\\ \bar{d}_9=&3[-\bar{a}_1^2\bar{c}_5+2\bar{a}_1^2\bar{c}_9-\bar{a}_1^2\bar{c}_{13}+2\bar{a}_1\bar{a}_5\bar{b}_5-2\bar{a}_1(\bar{a}_5\bar{c}_1+\bar{b}_1\bar{b}_5)+4\bar{a}_1\bar{b}_1\bar{b}_9\\ &+\bar{a}_5^2\bar{b}_1-\bar{a}_5\bar{b}_1^2]/(100\varepsilon)\\ \bar{d}_{13}=&3(-\bar{a}_1^2\bar{c}_9-2\bar{a}_1^2\bar{c}_{13}-2\bar{a}_1\bar{a}_5\bar{b}_5+2\bar{a}_1\bar{a}_5\bar{b}_9-2\bar{a}_1\bar{b}_1\bar{b}_9-\bar{a}_5^2\bar{b}_1)/(196\varepsilon)\\ \bar{d}_{17}=&-(3\bar{a}_1^2\bar{c}_{13}+6\bar{a}_1\bar{a}_5\bar{b}_9+\bar{a}_5^3)/(324\varepsilon). \end{aligned} \tag{6.15}$$

The 5th improved approximation:

$$\begin{aligned} x(t,\varepsilon)=&(\bar{a}_1+\varepsilon\bar{b}_1+\varepsilon^2\bar{c}_1+\varepsilon^3\bar{d}_1+\varepsilon^4\bar{e}_1)\sin t\\ &+(\varepsilon\bar{a}_5+\varepsilon^2\bar{b}_5+\varepsilon^3\bar{c}_5+\varepsilon^4\bar{d}_5+\varepsilon^5\bar{e}_5)\sin 3t\\ &+(\varepsilon^2\bar{b}_9+\varepsilon^3\bar{c}_9+\varepsilon^4\bar{d}_9+\varepsilon^5\bar{e}_9)\sin 5t\\ &+(\varepsilon^3\bar{c}_{13}+\varepsilon^4\bar{d}_{13}+\varepsilon^5\bar{e}_{13})\sin 7t\\ &+(\varepsilon^4\bar{d}_{17}+\varepsilon^5\bar{e}_{17})\sin 9t\\ &+\varepsilon^5\bar{e}_{21}\sin 11t \end{aligned} \tag{6.16}$$

$$\begin{aligned} \bar{e}_1=&-3[-\bar{a}_1^2\bar{d}_5+2\bar{a}_1\bar{b}_5^2+2\bar{a}_1\bar{b}_9^2+3\bar{a}_1\bar{c}_1^2-2\bar{a}_1\bar{b}_5\bar{b}_9-2\bar{a}_1\bar{b}_5\bar{c}_1+4\bar{a}_1\bar{a}_5\bar{c}_5\\ &-2\bar{a}_1\bar{a}_5\bar{c}_9-2\bar{a}_1(\bar{a}_5\bar{d}_1+\bar{b}_1\bar{c}_5)+6\bar{a}_1\bar{b}_1\bar{d}_1+\bar{a}_5^2\bar{b}_9+2(2\bar{a}_5\bar{b}_1\bar{b}_5+\bar{a}_5^2\bar{c}_1)\\ &-2\bar{a}_5\bar{b}_1\bar{b}_9-(\bar{b}_1^2\bar{b}_5+2\bar{a}_5\bar{b}_1\bar{c}_1)+3\bar{b}_1^2\bar{c}_1]/(9\bar{a}_1^2-4)\\ \bar{e}_5=&[2\bar{a}_1^2\bar{d}_5-\bar{a}_1^2\bar{d}_9-\bar{a}_1^2\bar{e}_1-\bar{a}_1\bar{c}_1^2+2\bar{a}_1\bar{b}_5\bar{b}_9+4\bar{a}_1\bar{b}_5\bar{c}_1-2\bar{a}_1\bar{b}_9\bar{c}_1\\ &+2\bar{a}_1\bar{a}_5\bar{c}_9-2\bar{a}_1\bar{a}_5\bar{c}_{13}+4\bar{a}_1(\bar{a}_5\bar{d}_1+\bar{b}_1\bar{c}_5)-2\bar{a}_1\bar{b}_1\bar{c}_9-2\bar{a}_1\bar{b}_1\bar{d}_1 \end{aligned}$$

$\bar{x}$	Galerkin Urabe $m=15$	Galerkin $m=3$	Analytical method				
			1st improved approximation	2nd improved approximation	3rd improved approximation	4th improved approximation	5th improved approximation
$\bar{x}_1$	1.431189037	1.434	1.492257983	1.453817114	1.438508632	1.433061463	1.431453028
$\bar{x}_5$	−0.126915530	−0.124	−0.092305851	−0.119430698	−0.126220070	−0.127324148	−0.127219538
$\bar{x}_9$	0.009754734	0.000	0.000000000	0.006166494	0.008866213	0.009663646	0.009808681
$\bar{x}_{13}$	−0.000763601	0.000	0.000000000	0.000000000	−0.000404791	−0.000648939	−0.000742176
$\bar{x}_{17}$	0.000059845	0.000	0.000000000	0.000000000	0.000000000	0.000026503	0.000047002
$\bar{x}_{21}$	−0.000004691	0.000	0.000000000	0.000000000	0.000000000	0.000000000	−0.000001734
$\bar{x}_{25}$	0.000000368	0.000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
$\bar{x}_{29}$	−0.000000029	0.000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000

Table 1: The coefficients $\bar{x}_\nu$ of an approximate periodic solution $\bar{x}$ with period 2π of the equation $\ddot{x}+x^3=\sin t$.

$$
\begin{aligned}
&\quad +3\bar a_5^2\bar b_5+2\bar a_5\bar b_1\bar b_9+2(\bar b_1^2\bar b_5+2\bar a_5\bar b_1\bar c_1)-\bar b_1^2\bar b_9-\bar b_1^2\bar c_1]/(12\varepsilon)\\
\bar e_9 &=3[-\bar a_1^2\bar d_5+2\bar a_1^2\bar d_9-\bar a_1^2\bar d_{13}+\bar a_1\bar b_5^2-2\bar a_1\bar b_5\bar c_1+4\bar a_1\bar b_9\bar c_1\\
&\quad +2\bar a_1\bar a_5\bar c_5+2\bar a_1\bar a_5\bar c_{13}-2\bar a_1(\bar a_5\bar d_1+\bar b_1\bar c_5)+4\bar a_1\bar b_1\bar c_9-2\bar a_1\bar b_1\bar c_{13}-(\bar b_1^2\bar b_5\\
&\quad +2\bar a_5\bar b_1\bar c_1)+2\bar a_5^2\bar b_9+(2\bar a_5\bar b_1\bar b_5+\bar a_5^2\bar c_1)+2\bar b_1^2\bar b_9]/(100\varepsilon)\\
\bar e_{13} &=3[-\bar a_1^2\bar d_9+2\bar a_1^2\bar d_{13}-\bar a_1^2\bar d_{17}-\bar a_1\bar b_5^2+2\bar a_1\bar b_5\bar b_9-2\bar a_1\bar b_9\bar c_1-2\bar a_1\bar a_5\bar c_5\\
&\quad +2\bar a_1\bar a_5\bar c_9-2\bar a_1\bar b_1\bar c_9+4\bar a_1\bar b_1\bar c_{13}-(2\bar a_5\bar b_1\bar b_5+\bar a_5^2\bar c_1)+2\bar a_5\bar b_1\bar b_9\\
&\quad -\bar b_1^2\bar b_9]/(196\varepsilon)\\
\bar e_{17} &=3(-\bar a_1^2\bar d_{13}+2\bar a_1^2\bar d_{17}+\bar a_1\bar b_9^2-2\bar a_1\bar b_5\bar b_9-2\bar a_1\bar a_5\bar c_9+2\bar a_1\bar a_5\bar c_{13}\\
&\quad -2\bar a_1\bar b_1\bar c_{13}-\bar a_5^2\bar b_5-2\bar a_5\bar b_1\bar b_9)/(324\varepsilon)\\
\bar e_{21} &=-3(\bar a_1^2\bar d_{17}+\bar a_1\bar b_9^2+2\bar a_1\bar a_5\bar c_{13}+\bar a_5^2\bar b_9)/(484\varepsilon)
\end{aligned}
\tag{6.17}
$$

In this example m has been chosen so that

$$m>11 \tag{6.18}$$

The numerical results for the first five improved approximations obtained by our method are given in the last five columns of Table 1 in which $\bar x_\nu$ are the coefficients of the trigonometric polynomial of the following form representing an approximate periodic solution $\bar x$ with period 2π of equation (6.1):

$$\bar x(t)=\bar x_0+\sum_{n=1}^{m}(\bar x_{2n-1}\sin nt+\bar x_{2n}\cos nt) \tag{6.19}$$

all other coefficients not mentioned in this table being zero. The 5th improved approximation is a very good approximation of the Galerkin approximation of order 15 calculated numerically with a very high precision by M. URABE and A. REITER [1] using the electronic computer CDC 1604 at the University of Wisconsin.

The largest difference in the coefficients $\bar x_\nu$ is smaller than 3.1×10^{-4}. Hence an appropriate combination of the harmonic balance method and the perturbation method can give very reasonable approximations to periodic solutions of highly nonlinear periodic differential equations such as equation (6.1).

Acknowledgement.

I would like to express my appreciation for the encouragement I have received from Professor P. JANSSENS while doing this work.

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(Ricevita la 5-an de marto, 1973)