

Representation of Dynamical Systems

By Otomar HÁJEK*

(Case Western Reserve University)

Bebutov proved the following theorem: Given a dynamical system π on X , assume that

- a) X is compact metric,
- b) π has at most one critical point.

Then π can be embedded within the shift system on $C(R^1)$. (See [8, p. 33]; the literature gives conflicting references to the original.) (Definitions are given below.) Kakutani [5] weakened (b) to

b') the set of critical points is homeomorphic to a subset of R^1 (also see [6, p. 52 a] for an earlier reference unavailable to this author).

In this paper we present a further modification:

- a'') X is locally compact, metric, separable,
- b'') the set of critical points is homeomorphic to a closed subset of R^N , concluding that π can be embedded within the shift system on $C(R^1)^N$ (with further information, see Theorem 13).

The passage from dimension 1 to N presents no difficulties; weakening of the compactness requirement is the point. An obvious mode of attack would be to reduce the present situation to Kakutani's by taking the one-point compactification of X . However, then the augmented set of critical points may well be S^N , which cannot be embedded in R^N [2, p. 349, 6.2(2)]. For this reason our proof proceeds in a different manner, analogous to Kakutani's.

We also prove a theorem on representation of dynamical systems, as described above, as solutions of differential equations in $C[-1, 1]$ (even for $N > 1$); this parallels Grabar's application [3] of the Bebutov theorem.

Conventions and Notation.

A dynamical system on a topological space X is a continuous mapping $\pi : X \times R^1 \rightarrow X$ (we will write $x\pi t$ in place of $\pi(x, t)$) satisfying the following two axioms:

$$x\pi 0 = x, \quad (x\pi t)\pi s = x\pi(t+s)$$

for all $x \in X$, and t, s in R^1 . The carrier X is then called the phase space of π . With each point $x \in X$ we associate the solution $x\pi$ through x :

* This research was supported, in part, by the National Science Foundation under grants no. NSF-GP-12261 and NSF-GP-22689.

$${}_x\pi : R^1 \rightarrow X, \quad {}_x\pi(t) = x\pi t.$$

A homomorphism between dynamical systems $\pi \rightarrow \pi'$ is a continuous mapping $f: X \rightarrow X'$ between their respective phase spaces such that

$$f(x\pi t) = f(x)\pi' t \quad (\text{all } x \in X, t \in R^1).$$

Iff f is a homeomorphism into X' , we say that it is an embedding of π in π' .

The Bebutov system over a topological space Y has as its phase space the set Y^{R^1} of all continuous mappings $R^1 \rightarrow Y$, endowed with the compact-open topology; the dynamical system (or shift system) σ is then defined thus: for $x: R^1 \rightarrow Y$ and $t \in R^1$ let $x\sigma t: R^1 \rightarrow Y$ be

$$(x\sigma t)(s) = x(t+s) \quad \text{for all } s \in R^1.$$

(Note that the Bebutov system over Y is on Y^{R^1} .)

1. Lemma. *Let $F: X \rightarrow Y^{R^1}$ be continuous, let π be a dynamical system on X . Then F is a homomorphism between π and the Bebutov system over Y if and only if $F(x) = f \circ {}_x\pi$ for some continuous $f: X \rightarrow Y$, whereupon f is defined by $x \mapsto F(x)(0)$.*

Proof. If $f: X \rightarrow Y$, then it is readily verified that $x \mapsto f \circ {}_x\pi$ defines a homomorphism F . Conversely, define f by $x \mapsto F(x)(0)$, and conclude $F(x) = f \circ {}_x\pi$ from

$$\begin{aligned} (f \circ {}_x\pi)(t) &= f(x\pi t) = F(x\pi t)(0) \\ &= (F(x)\sigma t)(0) = F(x)(t) \end{aligned}$$

for all $t \in R^1$.

We may also conclude that if f is a (topological) embedding, then F embeds the dynamical systems.

As an example, let X be separable metric, $Y = C[0, 1]$, $f: X \rightarrow Y$ the classical embedding; then F embeds any π on X within the Bebutov system over $C[0, 1]$. Analogously, $f = \text{identity}$ embeds any system on X within the Bebutov system over X .

We will later need the simple topological result below. A known theorem states that, if X is compact and Y Hausdorff, then every continuous one-to-one map $f: X \rightarrow Y$ is necessarily a homeomorphism (onto a compact subset of Y). An analysis of the proof yields the following results, useful in the case that X is not compact.

2. Lemma. *Let $f: X \rightarrow Y$ be continuous, Y Hausdorff, and assume that $f(x_i)$ has no cluster points in Y whenever the net x_i in X is ultimately outside each compact subset. Then f is a proper mapping.*

3. Corollary. *Under the above assumptions, let f be one-to-one; then f maps X homeomorphically onto a closed subset of Y .*

(Proof: proper maps are closed.)

The condition on f appearing in Lemma 2 will be abbreviated to: $f(x) \rightarrow \infty$

as $x \rightarrow \infty$ in X .

Given a dynamical system π on X and a function $f: X \rightarrow R^N$, we define, wherever possible, the time-derivative ∂f of f (relative to π) as

$$\partial f(x) = \left[\frac{\partial}{\partial t} f(x\pi t) \right]_{t=0} = \lim_{t \rightarrow 0} \frac{1}{t} (f(x\pi t) - f(x)).$$

Constructions.

In the absence of over-riding remarks, throughout this section we make the following assumptions.

4. Convention. π is a dynamical system on a space X which is metrizable, separable, locally compact; the set of all critical points of π is denoted by Q ; there exists a homeomorphism, h , mapping Q onto a closed subset of R^N , $N \geq 1$.

5. Lemma. $h(x) \rightarrow \infty$ as $x \rightarrow \infty$ in Q .

(Proof: h is a homeomorphism onto a closed subset of R^N .)

6. Lemma. h has a continuous extension $f: X \rightarrow R^N$ such that also $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ in X .

Proof. Take a sequence of compact sets X_n with

$$\bigcup_{n=1}^{\infty} X_n = X, \quad X_n \subset \text{Int } X_{n+1}.$$

If α_n is the infimum of $|h(x)|$ for $x \in Q - \text{Int } X_n$, then $\alpha_n \rightarrow +\infty$ from Lemma 5. Thus

$$(1) \quad |h(x)| \geq \alpha_n \quad \text{for } x \in Q - \text{Int } X_n.$$

Using the Tietze theorem, extend h successively over $\partial X_1, X_1, \partial X_2, X_2, \dots$, preserving inequalities such as (1) on ∂X_n and $X_{n+1} - \text{Int } X_n$. There results a continuous extension f of h with $|f(x)| \geq \alpha_n$ for $x \in \text{Int } X_n$. Since $\alpha_n \rightarrow +\infty$, f has the required property.

7. Lemma. We may assume that, in addition,

$$|f(x\pi t) - f(x)| \leq |t|$$

for all $x \in X, t \in R^1$.

Proof. Let f be as described in Lemma 6. Consider first the special case that $N=1$ and $f \geq 0$. Set

$$(2) \quad g(x) = \inf \left\{ \frac{1}{2}|t| + f(x\pi t) : t \in R^1 \right\}.$$

Then, obviously, $g: X \rightarrow R^+$, $g = f = h$ on Q , $0 \leq g \leq f$, $g(x) = 0$ iff $f(x) = 0$. Easily, for all $x \in X$ and $t \in R^1$,

$$g(x\pi t) \leq g(x) + \frac{1}{2}|t|.$$

Since the infimum in (2) is actually attained, at some t with $(1/2)|t| \leq g(x) \leq$

$f(x)$, it follows that g is continuous, and also that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$.

In the second step omit the assumption $f \geq 0$. Let $f = f^+ - f^-$ be the decomposition into positive and negative parts; apply the preceding to each, obtaining g^+ , g^- ; and then set $g = g^+ - g^-$. Evidently $g : X \rightarrow R^1$ is continuous, $g = f = h$ on Q ,

$$|g(x\pi t) - g(x)| \leq \frac{1}{2}|t| + \frac{1}{2}|t| = |t|.$$

If $g(x) \rightarrow \infty$, then both $g^+(x) \rightarrow \infty \leftarrow g^-(x)$, hence both $f^+(x) \rightarrow \infty \leftarrow f^-(x)$, and so $x \rightarrow \infty$.

For general N , apply the preceding construction to each coordinate f_k of f . Finally, re-name the resulting function.

8. Lemma. *We may even assume that $\partial f : X \rightarrow R^N$ is continuous, and $|\partial f| \leq 1$.*

Proof. Let f be as described in Lemma 7, and define g by

$$g(x) = \int_0^1 f(x\pi s) ds.$$

Thus $g : X \rightarrow R^N$ is continuous, and $g = f = h$ on Q . Also

$$\partial g(x) = \left[\frac{\partial}{\partial t} g(x\pi t) \right]_{t=0} = \left[\frac{\partial}{\partial t} \int_t^{t+1} f(x\pi s) ds \right]_{t=0} = f(x\pi 1) - f(x),$$

so that ∂g is continuous, and $|\partial g| \leq 1$ from Lemma 7. If $x \rightarrow \infty$ in X , then, from Lemma 7 again, $f(x\pi s) \rightarrow \infty$ uniformly for $s \in [0, 1]$, and so $g(x) \rightarrow \infty$. Again, re-name the resulting function.

9. Lemma. *(On smooth approximation.) Given, a continuous function $f : X \rightarrow R^N$ such that the collection $f \circ_x \pi$, $x \in X$, is uniformly equicontinuous. Then, for any $\epsilon > 0$, there exists a continuous $g : X \rightarrow R^N$ such that*

$$(3) \quad g \text{ is uniformly } \epsilon\text{-close to } f.$$

$$(4) \quad g|_Q = f|_Q.$$

$$(5) \quad \partial g : X \rightarrow R^N \text{ is continuous and bounded.}$$

Proof. For any $\delta > 0$ (to be fixed subsequently) define $g : X \rightarrow R^N$ by

$$g(x) = \frac{1}{\delta} \int_0^\delta f(x\pi s) ds.$$

Thus g is continuous, and (4) holds. Also

$$(6) \quad \begin{aligned} \partial g(x) &= \left[\frac{\partial}{\partial t} g(x\pi t) \right]_{t=0} = \left[\frac{\partial}{\partial t} \frac{1}{\delta} \int_t^{t+\delta} f(x\pi s) ds \right]_{t=0} \\ &= \frac{1}{\delta} (f(x\pi \delta) - f(x)), \end{aligned}$$

so that ∂g is continuous. Let $\omega(\cdot)$ be a common modulus of continuity for the $f \circ_x \pi$; thus $\omega(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0+$, and

$$|f(x\pi t) - f(x)| \leq \omega(\lambda) \quad \text{for all } x \in X, |t| \leq \lambda.$$

Then from (6), $|\partial g| \leq \frac{1}{\delta} \omega(\delta)$, proving (5). For (3) note that

$$g(x) - f(x) = \frac{1}{\delta} \int_0^\delta (f(x\pi s) - f(x)) ds,$$

so that

$$|g - f| \leq \frac{1}{\delta} \cdot \delta \cdot \omega(\delta) < \varepsilon$$

if δ is taken sufficiently small.

10. Corollary. *In Lemma 9, if*

$$(7) \quad f(x) \rightarrow \infty \text{ as } x \rightarrow \infty \text{ in } X,$$

$$(8) \quad \text{the collection } f \circ_x \pi \text{ (} x \in X \text{) is uniformly equicontinuous,}$$

then the function g also has these properties.

Proof (The first follows from (3), the second from boundedness of ∂g .)

The following result is, essentially, a re-working of part of the construction of local sections [4, p. 142].

11. Proposition. *In a Tichonov phase space X , let S be a compact local section, and U a neighborhood of S . There exists a continuous function $f: X \rightarrow R^1$ with continuous $\partial f: X \rightarrow R^1$ such that*

$$|f| \leq 1, \quad f = 0 \text{ on } S \text{ and in } X - U,$$

$$|\partial f| \leq 1, \quad \partial f = 1 \text{ on } S.$$

Proof. Assume from the outset that U is open. There exists δ , $0 < \delta \leq 1$, such that

$$(9) \quad S\pi[-\delta, \delta] \cap (X - U)\pi[-\delta, \delta] = \emptyset$$

and that δ is a time extent of S . Thus the mapping taking $(x, t) \in S \times [-\delta, \delta]$ to $x\pi t$ is a homeomorphism; therefore g ,

$$(10) \quad g(x\pi t) = t \text{ for } x \in S, |t| \leq \delta$$

is a well-defined continuous mapping, with $|g| \leq \delta$. Let $g = 0$ in $(X - U)\pi[-\delta, \delta]$, and extend g to a continuous map $X \rightarrow [-\delta, \delta]$ (recall that in (9), the terms are compact and closed, respectively). Set

$$f(x) = \frac{1}{2\delta} \int_{-\delta}^\delta g(x\pi s) ds.$$

This defines a continuous map $f: X \rightarrow R^1$. Obviously

$$|f| \leq \frac{1}{2\delta} \cdot 2\delta \cdot \delta \leq 1;$$

from (10) we obtain $f(x) = 0$ for $x \in S$; since $g = 0$ in $(X - U)\pi[-\delta, \delta]$, we have $f = 0$ in $X - U$. Next,

$$\begin{aligned} \partial f(x) &= \left[\frac{\partial}{\partial t} f(x\pi t) \right]_{t=0} = \left[\frac{\partial}{\partial t} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} g(x\pi s) ds \right]_{t=0} \\ &= \frac{1}{2\delta} (g(x\pi\delta) - g(x\pi(-\delta))); \end{aligned}$$

thus ∂f is continuous, $|\partial f| \leq \frac{1}{2\delta}(\delta + \delta) = 1$, and, for $x \in S$,

$$\partial f(x) = \frac{1}{2\delta}(\delta - (-\delta)) = 1$$

(according to (10)).

12. Lemma. (*Existence of perturbations*). *Every point (x_0, y_0) in $X \times X$ with $y_0 \neq x_0 \in Q$ has compact neighborhoods $U \times V$ which, for some δ , $0 < \delta \leq 1$, have the following property: for arbitrarily small $\varepsilon > 0$ and large $\eta > 0$ there exists a continuous function $p: X \rightarrow R^1$ with continuous $\partial p: X \rightarrow R^1$ such that*

$$|p| \leq \varepsilon, \quad p = 0 \quad \text{on } Q, \quad |\partial p| \leq \eta,$$

and that, for every $(x, y) \in U \times V$,

$$\partial p(x\pi t) = \eta \quad \text{for some } |t| \leq \delta$$

$$\partial p(y\pi t) = 0 \quad \text{for all } |t| \leq \delta.$$

Proof. There exist open neighborhoods $U' \times V'$ such that $U' \cap (V' \cup Q) = \emptyset$. For V take a compact neighborhood of y_0 within V' ; let $U = S\pi[-\delta, \delta]$ where S is a compact local section generating neighborhoods of x_0 , δ is a time-extent of S chosen so small that

$$U = S\pi[-\delta, \delta] \subset U', \quad V\pi[-\delta, \delta] \cap U' = \emptyset.$$

Let f be a function as described in Proposition 11, corresponding to S and U' , and

$$p(x) = \varepsilon \cdot \sin \frac{\eta}{\varepsilon} f(x).$$

Thus $|p| \leq \varepsilon$, $p = 0$ outside Q (since $f = 0$ outside U' , and $U' \cap Q = \emptyset$); next,

$$\partial p(x) = \eta \cdot \cos \frac{\eta}{\varepsilon} f(x) \cdot \partial f(x),$$

so that ∂p is continuous, $|\partial p| \leq \eta$, $\partial p = 0$ in $V\pi[-\delta, \delta]$ (since $f = 0$ outside U'); finally, for any $x \in U = S\pi[-\delta, \delta]$, we have $x\pi t \in S$ for suitable $t \in [-\delta, \delta]$, whereupon

$$\partial p(x\pi t) = \eta \cdot 1 \cdot 1.$$

The main theorems.

13. Theorem. *If X is locally compact, metrisable and separable, and if its critical point set Q is homeomorphic to a closed subset of R^N , $N \geq 1$, then π can be embedded within the Bebutov system over R^N , with X mapping onto a closed invariant nowhere dense subset of $C(R^1)^N$.*

That a closed image of X in $C(R^1)^N$ is nowhere dense is implied by local compactness of X . We will actually prove a stronger result (that it is stronger follows from Lemma 1):

14. Theorem. *Let π be a dynamical system on X , with critical point set*

Q ; let $N \geq 1$. Then the following are equivalent :

- A. X is locally compact metrisable and separable, Q is homeomorphic to a closed subset of R^N .
- B. There exists a continuous mapping $f: X \rightarrow R^N$ such that $x \mapsto f \circ_x \pi$ defines a homeomorphism taking X onto a closed nowhere dense subset of $C(R^1)^N$, and that the collection of mappings $f \circ_x \pi: R^1 \rightarrow R^N$, $x \in X$, is uniformly equicontinuous.

Proof of $B \Rightarrow A$: X is homeomorphic to a closed subset X' of $C(R^1)^N$; thus it is metrisable and separable. The critical points are mapped onto the set of constants in X' ; since X' is closed, Q is homeomorphic to a closed subset of R^N . Local compactness of X' follows from the equicontinuity condition and the Arzelà-Ascoli theorem.

The proof of $A \Rightarrow B$ will be interrupted by several lemmas. Consider the set $\mathcal{F} = f_0 + \mathcal{F}_0$ of functions $X \rightarrow R^N$, where f_0 is the function constructed in Lemmas 6 to 8, and \mathcal{F}_0 consists of all continuous functions $g: X \rightarrow R^N$ with the following three properties: g is bounded, $g|_Q = 0$, the collection $g \circ_x \pi$, $x \in X$, is uniformly equicontinuous. Endow this set with the uniform topology (not with the compact-open topology).

15. Lemma. \mathcal{F} is a non-void Baire space. Each $f \in \mathcal{F}$ has the following properties: $f: X \rightarrow R^N$ is continuous, f is an extension of h , $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, the collection $f \circ_x \pi$, $x \in X$, is uniformly equicontinuous.

Proof. \mathcal{F} is non-void since $f_0 \in \mathcal{F}$ (equicontinuity follows from boundedness of ∂f_0). \mathcal{F} is a Baire space since, as is easily shown, \mathcal{F}_0 is a closed subset of the complete metric space $C^*(X)^N$. The properties of $f = f_0 + g \in \mathcal{F}$ follow from those of f_0 and \mathcal{F} ; e.g., $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ since f_0 has the property and g is bounded.

(Proof of $A \Rightarrow B$ continued): Given arbitrary compact subsets M, P of X , let $W(M \times P)$ consist of all $f \in \mathcal{F}$ such that, for any $x \in M$ and $y \in P$, there exists $t \in [-1, 1]$ with $f(x\pi t) \neq f(y\pi t)$.

16. Lemma. $W(M \times P)$ is open in \mathcal{F} .

Proof. If

$$f_i \rightarrow f, \quad f \in W(M \times P) \ni f_i,$$

then there exist $x_i \in M$, $y_i \in P$ such that

$$f_i(x_i\pi t) = f_i(y_i\pi t) \quad \text{for all } t \in [-1, 1].$$

Take convergent subnet $x_i \rightarrow x \in M$, $y_i \rightarrow y \in P$. Since $f_i \rightarrow f$ in \mathcal{F} , we conclude $f(x\pi t) = f(y\pi t)$ for all $t \in [-1, 1]$, contradicting $f \in W(M \times P)$.

17. Lemma. Every point (x, y) , $y \neq x \in Q$, has a neighborhood $U \times V$ such that $W(U \times V)$ is dense in \mathcal{F} .

Proof. Take $U \times V$ as described in Lemma 12 (ϵ and η are still left free

to vary). We wish to show that, for any $f \in \mathcal{F}$ and $2\varepsilon > 0$, the set $W(U \times V)$ contains a member uniformly 2ε -close to f . Take a smooth ε -approximation g to f as described in Lemma 9 and choose $\eta > 2|\partial g|$. To these ε, η find a perturbation p as described in Lemma 12 (with $N-1$ coordinates chosen as 0). We assert that $g+p$ is in $W(U \times V)$ and is 2ε -close to f . The latter follows, of course, from

$$|f-g| < \varepsilon, \quad |p| \leq \varepsilon.$$

For the former proceed by contradiction. Assume that, for some $(x, y) \in U \times V$ and all $|t| \leq 1$,

$$g(x\pi t) + p(x\pi t) = g(y\pi t) + p(y\pi t).$$

Restrict t to $[-\delta, \delta]$, whereupon $p(y\pi t) = 0$, and take time-derivatives:

$$\partial p(x\pi t) = \partial g(y\pi t) - \partial g(x\pi t).$$

However, $\eta > 2|\partial g|$ and $\partial p(x\pi t) = \eta$ for some $t \in [-\delta, \delta]$: this contradiction completes the proof of the lemma.

(Proof of $A \Rightarrow B$ continued): Take a countable cover of the set $\{(x, y) \in X \times X: x \neq y, x \in Q\}$ by sets $U_n \times V_n$ as described in the preceding lemma (the space in question is metric separable). The corresponding sets $W(U_n \times V_n)$ are then open and dense in \mathcal{F} by the two lemmas above. Since \mathcal{F} is a non-void Baire space, there exists $f \in \mathcal{F}$ belonging to all of them. In other words, f has, in addition to the properties described in Lemma 15, the following: for any $x \neq y$ in X ,

$$f(x\pi t) \neq f(y\pi t), \text{ for some } t \in [-1, 1].$$

Indeed, if both x, y are in Q , then this follows from $f|_Q = h$; and if e.g. $x \in Q$, then (x, y) is in some $U_n \times V_n$.

To show that $x \mapsto f \circ_x \pi$ defines a homeomorphism onto a closed subset of $C(R^1)^N$, merely note that it is one-to-one, and that

$$(f \circ_x \pi)(0) = f(x) \rightarrow \infty \text{ as } x \rightarrow \infty,$$

which is the condition needed in Corollary 3. This concludes the proof of Theorem 14.

18. Theorem. *Let π be a dynamical system on a locally compact separable metrisable space X , whose set of critical points is homeomorphic to a closed subset of some Euclidean space. There exists a homeomorphism between X and a closed subset Y of $C[-1, 1]$, and a bounded continuous map $F: C[-1, 1] \rightarrow C[-1, 1]$, such that all solutions of π are mapped into solutions of the differential equation*

$$\frac{dy}{dt} = F(y), \quad y \in Y.$$

Proof. Let f be the mapping described in Theorem 14, $B; N$ is determined by the assumptions on the critical point set. To recapitulate, $f: X \rightarrow R^N$ is continuous, $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, the collection $f \circ_x \pi, x \in X$, is uniformly equi-

continuous, for any $x \neq y$ there exists $t \in [-1, 1]$ with $f(x\pi t) \neq f(y\pi t)$.

Define $h : X \rightarrow C[-1, 1]^N$ by

$$h(x)(t) = \int_0^t f(x\pi s) ds \quad \text{for } x \in X, |t| \leq 1.$$

Obviously h is continuous and one-to-one. From equicontinuity, $x \rightarrow \infty$ in X implies $f(x\pi t) \rightarrow \infty$ uniformly for $|t| \leq 1$, so that some k -th coordinate has $|f_k(x\pi t)| \rightarrow +\infty$ uniformly for $|t| \leq 1$; thus $h(x) \rightarrow \infty$. From Corollary 3, h maps X homeomorphically onto a closed subset Y of $C[-1, 1]^N$.

Next, take $x \in X$ and $\theta \neq 0$; then

$$(h \circ \pi)(\theta)(t) = h(x\pi\theta)(t) = \int_\theta^{\theta+t} f(x\pi s) ds,$$

so that

$$\begin{aligned} \frac{1}{\theta}(h \circ \pi(\theta) - h \circ \pi(0))(t) &= \frac{1}{\theta} \left(\int_\theta^{\theta+t} - \int_0^t \right) \\ &= \frac{1}{\theta} \int_0^t (f(x\pi(\theta+s)) - f(x\pi s)) ds, \end{aligned}$$

and therefore, at $\theta=0$,

$$\left(\frac{d}{d\theta} h \circ \pi(\theta) \right)(t) = f(x\pi t) - f(x).$$

Thus, if we define $G : Y \rightarrow C[-1, 1]^N$ by

$$G(h(x))(t) = f(x\pi t) - f(x) \quad (x \in X, t \in [-1, 1]),$$

then G is continuous, since $h : X \approx Y$ and f, π are continuous; G is bounded, $|G| \leq \omega(1)$, where $\omega(\cdot)$ is a common modulus of continuity for the $f \circ \pi$; and the solutions π of $\dot{x} = \pi$ transform into the mappings $h \circ \pi : R^1 \rightarrow C[-1, 1]^N$ which satisfy $\frac{dy}{dt} = G(y)$.

In the next step we reduce the N in $C[-1, 1]^N$ to $N=1$, in the obvious fashion. E.g. for $N=2$, associate with every $(x, y) \in C[-1, 1]^2$ the element $z \in C[-1, 1]$ defined by

$$z(t) = \begin{cases} x(4t+3) & \text{for } -1 \leq t \leq -\frac{1}{2} \\ \text{linear in } t & \text{for } |t| \leq \frac{1}{2} \\ y(4t-3) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Thus there exists a linear isometry mapping $C[-1, 1]^N$ onto a closed set in $C[-1, 1]$.

Finally, extend $G : Y \rightarrow C[-1, 1]$ to a bounded continuous mapping $C[-1, 1] \rightarrow C[-1, 1]$ (Dugundji's extension theorem, [1]).

Concluding Remarks.

It seems very probable that the main results can be extended, *mutatis mutandis*, to local dynamical systems. It is even possible that they might be applied to semi-dynamical systems; the lacking step is a semidynamical version of Proposition 11, or, more specifically, a theorem for semi-systems on existence of local sections (at this stage it is not even clear what these local sections for semisystems should be).

Another possibility would be to replace R^1 as phase group by a Lie group. Here the required existence of local sections is supplied in [7].

It would be interesting to know whether local compactness may be dropped in Theorem 13. The equivalence in Theorem 14 suggests that a proof would probably not be obtained by a refinement of the present techniques.

References

- [1] J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.*, **1** (1951) 353-367.
- [2] J. Dugundji, *Topology*, Allyn and Bacon, 1966.
- [3] M. Grabar, The mapping of dynamical systems into the system of solutions of differential equations (in Russian), *Doklady AN SSR* **61** (1948) 433-436.
- [4] O. Hájek, *Dynamical Systems in the Plane*, Academic Press, New York and London, 1968.
- [5] S. Kakutani, A proof of Bebutov's theorem, *Journal of Differential Equations*, **4** (1968) 194-201.
- [6] L. Markus, *The global theory of differential equations*, Lecture Notes, University of Minnesota, 1964-1965.
- [7] R. McCann, Local sections for simultaneous dynamical systems, to appear.
- [8] V. V. Niemyckii, *Topological problems of the theory of dynamical systems*, AMS Translation no. 103, Providence, 1954.

(Ricevila la 8-an de julio, 1970)