Funkcialaj Ekvacioj **13** (1970), 1–18

On the Asymptotic Solution of Two First Order Linear Differential Equations with Large Parameter

By Frank STENGER

(The University of Utah)

A system of two first order linear differential equations with coefficient matrix whose elements may become large as a parameter u becomes large or as the independent variable z becomes large is brought to a canonical form by a finite sequence of explicit transformations. The coefficient matrix of the canonical form consists of a diagonal dominant part plus a 2×2 non-dominant matrix whose elements are "small" with respect to both large z and large u relative to the diagonal matrix. Error bounds are given for the difference between an actual solution vector of the system in canonical form and a partial sum of a formal solution vector. Solutions uniformly valid for large u and/or z are thus obtained. For the second order equation the method resembles the WKB method.

1. Introduction

Let us establish our notatation. To refer to the $(i, j)^{\text{th}}$ element of the matrix A we shall write $\{A\}_{ij}$, while $\{A\}_i$ refers to the i^{th} column of A. The paths $\mathcal{P}$ we shall use consist of a finite number of Jordan arcs $t=t(s)$ $(a\leqq s\leqq b)$ on each of which $\dfrac{dt}{ds}$ is continuous and non-vanishing.

Let $c(z, u)$ be a holomorphic function of two independent variables z and u with z in a domain $\mathcal{D}$ extending to infinity and u in a domain $\mathcal{S}$ extending to infinity. We use the notation

$$c(z, u)\approx(\omega, \mu) \tag{1.1}$$

to imply that ω and μ are real numbers defined as follows.

$$\begin{aligned} \omega^*(u)&=\inf\{t\in R:\ \overline{\lim}_{(z\in\mathcal{P})}|c(z, u)z^{-t}|=0\}\\ \omega&=\sup_{(u\in\mathcal{S})}\omega^*(u)\\ \mu^*(z)&=\inf\{t\in R:\ \overline{\lim}_{(u\in\mathcal{P})}|c(z, u)u^{-t}|=0\}\\ \mu&=\sup_{(z\in\mathcal{D})}\mu^*(z). \end{aligned} \tag{1.2}$$

Here R denotes the real line and the limits $(\overline{\lim})$ are taken along paths $\mathcal{P}$ extending to infinity.

The relation $(\omega, \mu)\leqq(\omega_1, \mu_1)$ will then imply that $\omega\leqq\omega_1$ and $\mu\leqq\mu_1$, the relation $(\omega, \mu)=(\omega_1, \mu_1)$ that $\omega=\omega_1$ and $\mu=\mu_1$, and the relation $(\omega, \mu)<(\omega_1, \mu_1)$ that $\omega<\omega_1$ and $\mu<\mu_1$.

The system of two first order linear differential equations we shall study in this paper takes the form

$$\frac{dX}{dz}=A(z,\ u)X. \tag{1.3}$$

Here,

$$A(z,u)=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\alpha(z,\ u) & \beta(z,\ u) \\ \gamma(z,\ u) & \delta(z,\ u)\end{pmatrix} \tag{1.4}$$

where α, β, γ and δ are holomorphic functions of two independent variables z and u for all $(z,\ u)$ in $\mathcal{D}\times\mathcal{S}$. $\mathcal{D}$ and $\mathcal{S}$ are domains in the z and u planes respectively, each containing a sector of positive angle extending to infinity. Furthermore, we assume that α, β, γ and δ have representations of the form

$$\begin{aligned}\rho_i&=\sum_{k=0}^{r-1} a_k^{(i)}(u)z^{\omega_i-k}+a_r^{(i)}(z,u)z^{\omega_i-r}\\ &=\sum_{k=0}^{s-1} b_k^{(i)}(z)u^{\mu_i-k}+b_s^{(i)}(z,u)u^{\mu_i-s},\quad r,s=1,2,\cdots \\ i&=1,\ 2,\ 3,\ 4\end{aligned} \tag{1.5}$$

in $\mathcal{D}\times\mathcal{S}$, where the ω_i and μ_i are integers, $\rho_1=\alpha$, $\rho_2=\beta$, $\rho_3=\gamma$, $\rho_4=\delta$, the functions $a_r^{(i)}(z,u)$ are holomorphic in $\mathcal{D}\times\mathcal{S}$ and remain bounded as $z\to\infty$ in $\mathcal{D}$, and the $b_s^{(i)}(z,u)$ are holomorphic in $\mathcal{D}\times\mathcal{S}$ and remain bounded as $u\to\infty$ in $\mathcal{S}$. By allowing the introduction of isolated singularities we shall assume that after taking a finite number of sums, products and quotients (in the case of quotients we do not allow the denominator to vanish identically) we are again left a function of type, which has expansions of this form valid in almost all of $\mathcal{D}\times\mathcal{S}$ *).

Our approach is modelled along the lines of previous authors (Turrittin, [3]; Wasow, [4]; and Kiyek, [5]) in that we begin by transforming (1.3) to canonical form. We also differ in that our approach is based on successively taking square roots, and for the second order equation the method resembles the WKB method. The leading terms of the resulting coefficient matrix are diagonal and either dominant in z or u (or both) or else zero; in the latter case the norm of the integral of the resulting off-diagonal coefficient matrix remains bounded as $u\to\infty$ in $\mathcal{S}$ and as $z\to\infty$ in $\mathcal{D}$. Paralleling Olver [1], [2] we obtain uniform asymptotic expansions together with error bounds. The results obtained are valid in any subset $\mathcal{D}'\times\mathcal{S}'$ of $\mathcal{D}\times\mathcal{S}$ in which the sign of $\mathrm{Re}\,\nu$ remains unchanged, $\nu\approx(\omega,\mu)$ and $(\omega,\mu)=(\omega^*(u),\mu^*(z))$, where ν is the difference between the eigenvalues of the coefficient matrix.

*) With the exception of a set of measure zero in $\mathcal{D}\times\mathcal{S}$.

2. Transformation to Canonical Form

The following theorem describes the canonical form. In this theorem, the *unitary, zero inducing* and *shearing transformations* are defined in the process of the proof.

Theorem 2.1. *The system* (1.3) *can be transformed by a finite sequence of unitary, zero inducing and shearing transformations to the canonical system*

$$\frac{dW}{dz}=C(z,\ u)W \tag{2.1}$$

where

$$C(z,u)=\left\{\begin{pmatrix} q_1(z,u) & 0 \\ 0 & q_2(z,u)\end{pmatrix}+C^*(z,u)\right\} \tag{2.2}$$

is holomorphic in a domain $\mathcal{D}'\times\mathcal{S}'\subset\mathcal{D}\times\mathcal{S}$. *In* (2.2) $C^*(z,u)$ *is such that*

$\pi(z,u)=\left\|\int_z^\infty C^*(t,u)dt\right\|$ *is bounded for all* (z,u) *in* $\mathcal{D}'\times\mathcal{S}'$.

If $\nu(z,u)=\frac{1}{2}(q_1(z,u)-q_2(z,u))\not\equiv 0$ *then either* $\int^z \nu(z,u)dz\to\infty$ *as* $z\to\infty$ *in* $\mathcal{D}'$ *or else* $\int^z \nu(z,u)dz\to\infty$ *as* $u\to\infty$ *in* $\mathcal{S}'$ *(or both).*

If $\nu(z,u)\to\infty$ *as* $u\to\infty$ *in* $\mathcal{S}'$ *then* $\pi(z,u)\to 0$ *as* $u\to\infty$ *in* $\mathcal{S}'$.

Proof. The flow chart of Figure 1 serves as a visual aid in our proof. It is assumed in the flow chart after each transformation the notation of (1.3) and (1.4) is again used, and $\mathcal{D}\times\mathcal{S}$ is redefined to exclude new singularities introduced. The process is complete upon arrival at any F_i, $i=1, 2, 3$. The numbers (r, s) where $\|A(z,u)\|\approx(r,s)$ also depend on $A(z,u)$, and may change as $A(z,u)$ is changed.

Let us put

$$A(z,u)=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \tag{2.3}$$

and let us define

$$\nu=\nu(z,u)=\pm\sqrt{\left(\frac{\alpha-\delta}{2}\right)^2+\beta\gamma}\ . \tag{2.4}$$

In box No. 2 of the flow chart we test whether or not $\left\|\int_z^\infty A(t,u)dt\right\|$ remains bounded as $z\to\infty$ in $\mathcal{D}$ and as $u\to\infty$ in $\mathcal{S}$. If so, then the reduction to canonical form is complete. If not, we proceed on to box No. 3.

In boxes No. 3 and 4 we test whether or not the coefficient matrix is tri-

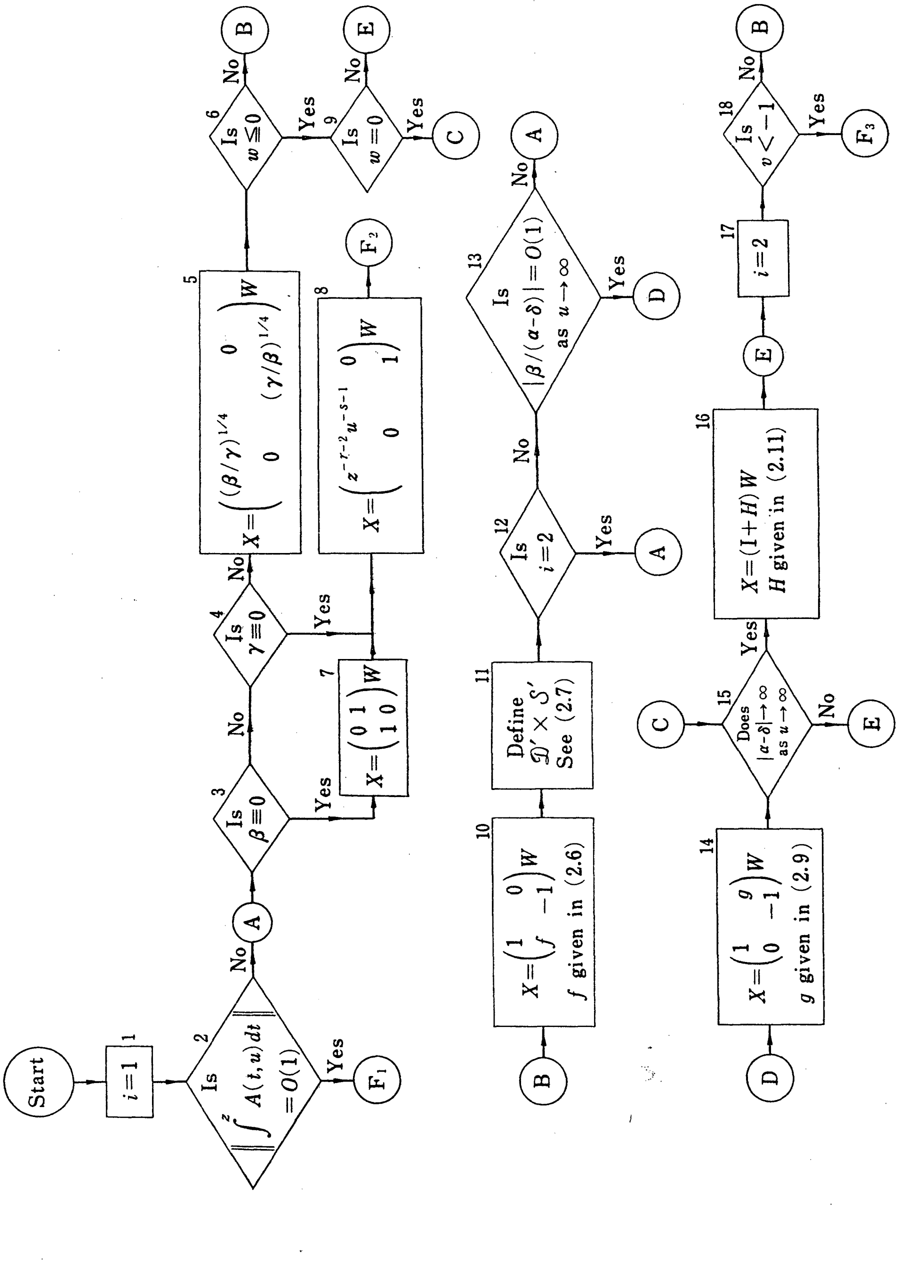

Fig. 1 The Transformation of (1.3) to Canonical Form.

angular. In this case (1.3) can be solved explicitly, the explicit solution in the case when $\gamma\equiv 0$ being

$$X=\begin{pmatrix} \exp\left(\int^z \alpha dt\right) & \exp\left(\int^z \alpha dt\right)\int^z \beta(t)\exp\left(\int^t (\delta-\alpha)\,dt\right)dt \\ 0 & \exp\left(\int^z \delta dt\right) \end{pmatrix}. \tag{2.5}$$

The transformation in box No. 7 replaces $A(z,u)$ by its transpose. It is actually unnecessary——we have used it merely to simplify the flow chart. From (2.5) we see that the transformation in box No. 8 is unnecessary; we have included it solely to simplify the statement of Theorem 2.1.

Both transformations, that in box No. 8 and that in box No. 5, are *shearing transformations*. The shearing transformation in box No. 5 has the effect of making the off-diagonal elements of $A(z,u)$ equal.*)

The pair of numbers (v,w) in boxes Nos. 6, 9 and 18 are defined by $\sqrt{\beta\gamma}\approx(v,w)$.
In these boxes we test whether or not the off-diagonal elements of $A(z,u)$ are sufficiently small to meet the statement of Theorem 2.1.

In box No. 10 we make a *unitary transformation*, choosing

$$f=\frac{\nu-(\alpha-\delta)/2}{\beta} \tag{2.6}$$

where ν is defined in (2.4). We assume here that the particular square root is taken in (2.4) which makes f bounded as either z or u become large. In box No. 11 we define

$$\mathcal{D}'\times\mathcal{S}'=\{(z,u)\in\mathcal{D}\times\mathcal{S}\colon\ \alpha(z,u)-\delta(z,u)\approx(\omega,\mu)\ \text{and}\ (\omega,\mu)=(\omega^*(u),\mu^*(z))\} \tag{2.7}$$

In box No. 13 we inquire whether or not $\alpha-\delta$ is sufficiently large relative to β. More precisely, we want to know whether or not the limit

$$\varlimsup_{u\to\infty}\left|\frac{\beta}{(\alpha-\delta)}\right| \tag{2.8}$$

is finite in $\mathcal{D}'\times\mathcal{S}'$. If the limit in (2.8) is not finite we return to Ⓐ. Otherwise, we proceed to Ⓓ.

In box No. 14 we make another unitary transformation, taking

$$g=\frac{\beta}{(\alpha-\delta)}. \tag{2.9}$$

*) The transformation in box No. 4 may be replaced by $X=\begin{pmatrix}\sqrt{\beta/\gamma} & 0\\ 0 & 1\end{pmatrix}W$; the effect of this is equivalent to that in box No. 4.

Since $\dfrac{\beta}{(\alpha-\delta)}$ is bounded, it follows that after leaving box No. 13 β and γ satisfy

$$\left|\int_z^\infty \begin{Bmatrix}\beta\\ \gamma\end{Bmatrix} dt\right| = o(1) \quad \text{as } z\to\infty\,; \qquad (2.10)$$
$$= O(1) \quad \text{as } u\to\infty.$$

We make the transformation in box No. 16 only if $|\alpha-\delta|\to\infty$ as $u\to\infty$. In this case, by choosing

$$H=\begin{pmatrix} 0 & \dfrac{\beta}{\delta-\alpha} \\ \dfrac{\gamma}{\alpha-\delta} & 0 \end{pmatrix} \qquad (2.11)$$

we alter β and γ such that $O(1)$ may be replaced by $o(1)$ in (2.10).

In boxes Nos. 1, 12 and 17 we have employed a "counter" to ensure that the reduction with respect to u is sufficiently complete before proceeding to that with respect to z.

The test in box No. 9 ensures that the transformation in box No. 16 is carried out only once.

We have yet to show that the process of the above flow chart terminates in all cases. To this end we prove the following lemma.

Lemma 2.1. *If each transformation is carried out as described above it is impossible to arrive at any point* Ⓐ *an infinite number of times.*

Proof of Lemma. Let us first examine the reduction of order of the off-diagonal elements of $A(z, u)$ with respect to u as we traverse the loop Ⓐ-Ⓑ-Ⓐ.

On arriving at the point Ⓑ in the chart we have

$$A(z, u)=\begin{pmatrix} \alpha_k & \sqrt{\beta_k\gamma_k} \\ \sqrt{\beta_k\gamma_k} & \delta_k \end{pmatrix} \qquad (2.12)$$

where the integer k denotes the number of times we have passed box No. 11. We assume that $A(z, u)$ may be expanded in powers of u^ω where $\omega=\omega_k>0$ and where the coefficients of these powers of u^ω are functions of z only. We thus write

$$\beta_k\gamma_k \approx (\,\cdot\,, 2w_k)\,;$$
$$f_k=\sqrt{\frac{(\alpha_k-\delta_k)^2}{4\beta_k\gamma_k}+1}-\sqrt{\frac{(\alpha_k-\delta_k)^2}{4\beta_k\gamma_k}} \approx (\cdot, t_k)\,; \quad \frac{df_k}{dz}\approx(\cdot, \tau_k). \qquad (2.13)$$

We observe at the outset that $t_k\leqq 0$ and so $\tau_k\leqq 0$. From

$$\beta_{k+1}\gamma_{k+1}=-\sqrt{\beta_k\gamma_k}\,\frac{df_k}{dz} \qquad (2.14)$$

it follows that

$$w_{k+1}=\frac{1}{2}(w_k+\tau_k)\leqq\frac{1}{2}w_k\;; \tag{2.15}$$

consequently, if $w_k\leqq 0 (w_k<0)$ for some finite k then $w_{k+j}\leqq 0$ $(w_{k+j}<0)$ for all finite $j\geqq 0$.

If the test in box No. 13 is satisfied, the process will clearly terminate with respect to u. Let us assume then, that the test in box No. 13 is not satisfied. In this case we must have

$$\beta_k\gamma_k\Big/\left[\left(\frac{\alpha_k-\delta_k}{2}\right)^2+\beta_k\gamma_k\right]\to\infty \qquad \text{as } u\to\infty, \text{ i.e.}$$

$$\frac{(\alpha_k-\delta_k)^2}{\beta_k\gamma_k}\to -4 \qquad \text{as } u\to\infty,\ z\in\mathscr{D}' \tag{2.16}$$

and so $\tau_k\leqq -\frac{1}{2}\omega_k$. Thus by (2.15)

$$w_{k+1}\leqq\frac{1}{2}\left(w_k-\frac{1}{2}\omega_k\right);\ \omega_k=\frac{\omega_0}{2^k} \tag{2.17}$$

On applying induction to (2.17) we easily verify that

$$w_k\leqq\frac{w_0-\dfrac{k\omega_0}{2}}{2^k}\ . \tag{2.18}$$

Hence, if

$$k\geqq\left[\frac{2w_0}{\omega_0}\right] \tag{2.19}$$

then $w_k\leqq 0$.

We next examine the reduction of order of the off-diagonal elements of $A(z, u)$ with respect to z. We again use the notation (2.12) where the integer k denotes the number of times we have passed box No. 11. Furthermore, we assume that each term on the right of (2.12) can be expanded in powers z^σ where $\sigma=\sigma_k>0$ and where the coefficients of these powers of z^σ are functions of u only. As in (2.13) we again write

$$\beta_k\gamma_k\approx(2\,v_k, \cdot)\;;$$

$$f_k=\frac{2\nu_k-(\alpha_k-\delta_k)}{2\sqrt{\beta_k\gamma_k}}\approx(t_k, \cdot)\;;\ \frac{df_k}{dz}\approx(\tau_k, \cdot). \tag{2.20}$$

We shall show that $v<-1$ $(v=v_k)$ after traversing the loop Ⓐ-Ⓑ-Ⓐ a finite number of times.

Let us note by (2.14) that

$$v_{k+1}=\frac{1}{2}(v_k+\tau_k). \tag{2.21}$$

Also, since $t_k \leqq 0$,

$$\tau_k \leqq -1-\frac{1}{2}\sigma_k\ ;\ \sigma_{k+1}=\frac{1}{2}\sigma_k. \tag{2.22}$$

Hence, if $v_k<-1$ then $v_{k+j}<-1$ for every finite integer $j \geqq 0$.

Combining (2.21) and (2.22) we have

$$v_{k+1} \leqq \frac{1}{2}\left(v_k-1-\frac{1}{2}\sigma_k\right). \tag{2.23}$$

Using induction we then easily verify that

$$v_k \leqq \frac{v_0-(2^k-1)-\dfrac{k\sigma_0}{2}}{2^k}\ ; \tag{2.24}$$

from this inequality we find that if

$$k>\left[\frac{2(v_0+1)}{\sigma_0}\right] \tag{2.25}$$

then $v_k<-1$.

This completes the proof of Lemma 2.1.

We finally take

$$\begin{aligned} g_1(z,\ u)&=\alpha(z,\ u)-C_{11}^*(z,\ u) \\ g_2(z,\ u)&=\delta(z,\ u)-C_{22}^*(z,\ u) \end{aligned} \tag{2.26}$$

where $C_{11}^*(z,u) \equiv \alpha(z,u)$ and $C_{22}^*(z,u) \equiv \delta(z,u)$ if both $\int^z \alpha dz$ and $\int^z \sigma dz$ remain bounded as z and/or u approach infinity ; otherwise C_{11}^* and C_{22}^* are arbitrary functions of the same type as*) $\beta(z,u)$ and $\gamma(z,u)$.

This completes the proof of Theorem 2.1.

In the majority of cases in applications it will be necessary to traverse any loop in the flow chart at most once. Let us suppose that this is the case for the system

$$\begin{pmatrix} W' \\ W'' \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ G & 0 \end{pmatrix}\begin{pmatrix} W \\ W' \end{pmatrix} \tag{2.27}$$

which is equivalent to the second order equation

$$W''=GW \tag{2.28}$$

*) Here and in (2.26) we refer to the present transformed system using the original notation.

If we drop the off-diagonal part of the coefficient matrix of the transformed system and solve the resulting system with diagonal coefficient matrix we obtain the approximate solution

$$W=G^{-1/4}\exp\int(\nu+f'g) \tag{2.29}$$

where $\nu=\sqrt{G+(G'/4G)^2}$, $f=G^{-1/2}[\nu-G'/4G]$, $g=-G^{1/2}/(2\nu)$, which shows the similarity of the transformation and the WKB method.

3. Formal Solution

Let us replace u by u^σ, z by z^ω in the coefficient matrix obtained in the previous section, where σ and ω are the smallest positive integers such that the resulting coefficient matrix can be expanded in integral powers of u and z. We also replace $\mathcal{D}'$ and $\mathcal{S}'$ by the corresponding new domain. The resulting system takes the form

$$\frac{dW}{dz}=[Q'(z,u)+C(z,u)]W \tag{3.1}$$

where $Q'(z,u)$ is a diagonal matrix with i^{th} diagonal element

$$q_i'(z,u)=\sum_{s=0}^{r}q_{is}'(z,u)u^{r-s},\quad i=1,2, \tag{3.2}$$

and where $C(z,u)$ is just $\omega z^{\omega-1}C^*(z^\omega,u^\sigma)$ where $C^*(z,u)$ is given in (2.2). It follows that $C(z,u)$ may be expanded in the form

$$C(z,u)=\sum_{k=0}^{s-1}C_k(z,u)u^{-k}+C_s{}^*(z,u)u^{-s} \tag{3.3}$$

for every positive integer s, where the elements of the matrices

$$\int^z C_k(t,u)dt,\quad k=0,1,2,\cdots$$

$$\int^z C_s^*(t,u)dt,\quad s=0,1,2,\cdots \tag{3.4}$$

are holomorphic in $\mathcal{D}'\times\mathcal{S}'$ and remain bounded as z and/or u approach infinity in $\mathcal{D}'$ and/or $\mathcal{S}'$ respectively,

For purposes of obtaining formal solutions we assume that once a particular form of expansion of the type (3.3) has been chosen it is fixed. The $C_k(z,u)$ are thus independent of s and the sequences $\{C_k(z,u)\}$, $\{C_s^*(z,u)\}$ are unique.

The elements $q_{is}'(z,u)$ in (3.2) are defined as follows:

(i) The $q_{is}'(z,u)$ are holomorphic functions of z and u in $\mathcal{D}'\times\mathcal{S}'$;

(ii) $\overline{\lim_{u\to\infty}} q_{is}(z, u)$ is finite for all finite z;

(iii) If $\nu\equiv 0$ in $\mathcal{D}'\times\mathcal{S}'$ where

$$\nu=\nu(z, u)=\frac{1}{2}(q_1'(z, u)-q_2'(z, u)) \tag{3.5}$$

we restrict the q_{is} subject only to (i) and (ii). If $\nu\not\equiv 0$ in $\mathcal{D}'\times\mathcal{S}'$ we require in addition to (i) and (ii) that

$$\begin{aligned}
&\overline{\lim_{z\to\infty}}\frac{\nu_s}{\nu_0}=O(1) \quad \text{as } u\to\infty \text{ in } \mathcal{S}'\\
&\overline{\lim_{u\to\infty}}\frac{\nu_s}{\nu_0}=O(1) \quad \text{as } z\to\infty \text{ in } \mathcal{D}'\\
&\overline{\lim_{u\to\infty}}\nu_0=h(z)\not\equiv 0 \quad \text{in } \mathcal{D}'\\
&\nu(z, u)\approx(\omega, \mu)\geqq(-2, 1)
\end{aligned} \tag{3.6}$$

where

$$\nu_s=\nu_s(z, u)=\frac{1}{2}(q_{1s}'(z, u)-q_{2s}'(z, u)). \tag{3.7}$$

It may be necessary to precede (3.1) by an *exponential transformation* in order that the last two requirements in (3.6) are satisfied. The transformation

$$W=\exp\left(\int^z\frac{\alpha(t, u)+\delta(t, u)}{2}dt\right)X \tag{3.8}$$

suffices for this purpose.

Also, it may be convenient to replace $\mathcal{D}'\times\mathcal{S}'$ by a dense subset of itself if (3.6) cannot be conveniently made to be valid for all of $\mathcal{D}'\times\mathcal{S}'$.

We next separate the two different cases: (a) that when all of the elements of $C(z, u)$ approach zero as $u\to\infty$, and (b) that when not all of the elements of $C(z, u)$ approach zero as $u\to\infty$. We apply a different method of solution in each case.

3.1 The Case When $r>0$.

For this case we take $C_0(z, u)\equiv 0$ in (3.3). We have the following theorem.

Theorem 3.1. *The system* (3.1) *possesses a formal independent series solution matrix of the form*

$$\tilde{W}(z, u)\triangleq U(z, u)\exp Q(z, u) \tag{3.9}$$

*where**)

*) The Symbol "$\triangleq$" here and henceforth denotes a formal equality.

$$U(z,u)\triangleq I+\sum_{k=1}^{\infty}U_k(z,u)u^{-k} \tag{3.10}$$

$$Q(z,u)=\sum_{s=0}^{r}Q_s(z,u)u^{r-s} \tag{3.11}$$

$$Q_s(z,u)=\begin{pmatrix} q_{1s}(z,u) & 0 \\ 0 & q_{2s}(z,u)\end{pmatrix}=\int_{a_s}^{z}Q_s'(t,u)dt \tag{3.12}$$

and where U_k and Q_k are holomorphic in $\mathcal{D}'\times\mathcal{S}'$.

Proof. We substitute (3.7) into (3.1) treating Q_s, U_k and C_k as if they were matrices of functions z only and then equate equal powers of u to get*)

$$\sum_{k=0}^{\mu}(Q_k'U_{s-k}-U_{s-k}Q_k')+\sum_{k=1}^{s-r}C_kU_{s-r-k}-U_{s-r}'=0 \tag{3.13}$$

where $\mu=\min(s,r)$, $U_0=I$ and $U_k=0$ if $k<0$.

A detailed analysis of (3.13) similar to that in Turittin [3] and Kiyek [5] leads us to the equations

$$U_0=I,\ \{U_s\}_{ij}=\delta_{ij}\int_{b_s}^{z}\sum_{k=1}^{s}\{C_kU_{s-k}\}_{ij}dt \quad s=1,2,\cdots,r \tag{3.14}$$

$$\{U_s\}_{ij}=\frac{(j-i)}{2\nu_0}\left[\left\{\sum_{k=1}^{r}(Q_k'U_{s-k}-U_{s-k}Q_k')+\sum_{k=1}^{s-r}C_kU_{s-r-k}-U_{s-r}'\right\}_{ij}\right]$$
$$+\delta_{ij}\int_{b_s}^{z}\left\{\sum_{k=1}^{s}C_kU_{s-k}\right\}_{ij}dt \quad i,j=1,2.$$

Q_s is obtained directly by integration of Q_s'. The points a_s and b_s are suitably chosen in $\mathcal{D}'$ or on the boundary of $\mathcal{D}'$ such that all of the integrals exist.

3.2 The Case When $r\leqq 0$.

In this case we take $Q'(z,u)\equiv 0$ in (3.1). Note that this amounts to redefining $C_0(z,u)$ in (3.1)——the elements of $C_0(z,u)$ may now become infinite as $z\to\infty$. For $s\geqq 1$, the elements of $C_s(z,u)$ are defined as above.

We have the following theorem.

Theorem 3.2. *The system* (3.1) *possesses a formal independent series solution matrix of the form*

$$\tilde{W}(z,u)\triangleq\sum_{k=0}^{\infty}U_k(z,u)u^{-k} \tag{3.15}$$

where each matrix $U_k(z,u)$ is holomorphic in $\mathcal{D}'\times\mathcal{S}'$.

Proof. Substituting (3.15) into (3.1) and equating equal powers of u we get the equations

*) Here and henceforth any sum $\sum_{j=k}^{r}$ is to be replaced by zero if $k>r$.

$$U_0' = C_0 U_0 \tag{3.16}$$

$$U_k' = C_0 U_k + \sum_{s=1}^{k} C_s U_{k-s}, \quad k=1,2,3,\cdots \tag{3.17}$$

At this point we assume that (3.16) can be solved explicitly. A method of solution is given in Stenger [6] for the case when C_0 is a function of z only. In all but one case C_0 will then also be in proper canonical form for formal solution as described in Stenger [6]. The exception occurs when the difference between the two eigenvalues C_0 is of the form ρ/z where $\rho \neq 0$ is an integer. In this case we make an adjustment in the canonical form. This adjustment starts at the point Ⓑ, Stenger [6] (p. 205). Thus, to solve (3.16) when C_0 is a function of u as well as z we separate out the part C_0^{**} of C_0 that is a function of z only, solve $X' = C_0^{**} X$ and then repeat the algorithm of this section.

With U_0 in (3.16) known, (3.17) can be solved explicitly——we obtain

$$U_k = U_0 \int_{c_k}^{z} U_0^{-1} \sum_{s=1}^{k} C_s U_{k-s} dt, \quad k=1,2\cdots, \tag{3.18}$$

where c_k is an arbitrary interior or boundary point of $\mathcal{D}'$ chosen so that each integral exists.

4. The Differential Equation for an Approximation

We shall obtain an actual solution vector $W_1(z, u)$ corresponding to the first formal solution vector of one of the formal solutions obtained in Section 3.

4.1 The Case When $r>0$.

Let $\Phi_m = \Phi_m(z, u)$ be defined by

$$\Phi_m = \left(\sum_{k=0}^{m-1} U_{1k} u^{-k} \right) e^{q_1} \tag{4.1}$$

where U_{1k} denotes the first column of U_k defined in Section 3. We then define a vector $R_m = R_m(z, u)$ by the differential equation

$$\frac{d\Phi_m}{dz} - [Q' + C]\Phi_m = R_m e^{q_1}. \tag{4.2}$$

Expanding (4.2) we obtain

$$R_m = \sum_{s=0}^{r+m} \left[U_{1,s-r}^{+\prime} - \sum_{k=0}^{\mu} (Q_k' - I q_{1k}') U_{s-k}^{+} + \sum_{k=1}^{s-r} C_k U_{1,s-r-k}^{+} \right] u^{r-s}$$
$$- \sum_{k=0}^{m-1} C_{m+1-k}^{*} U_{1k} u^{-m-1} \tag{4.3}$$

where $\mu = \min(r, s)$ and $U_{1k}^{+} = \{U_{1k}$ if $0 \leqq k \leqq m-1$; 0 otherwise$\}$.

On expanding the first sum in (4.3) in view of (3.13) we obtain

$$\{R_m\}_1=\{U_{1m}'\}_1 u^{-m}-\left\{\sum_{k=0}^{m-1} C_{m+1-k}^* U_{1k}\right\}_1 u^{-m-1} \tag{4.4}$$

$$\begin{aligned}\{R_m\}_2=&-2\nu\{U_{1m}\}_2 u^{-m}+2(\nu-u^r\nu_0)u\{U_{1m}\}_2 u^{-m-1}\\&+\sum_{s=m+1}^{m+r+1}\left\{U_{1,s-r}^{+}-\sum_{k=0}^{m-1}(C_{s-r-k}-Iq_{1,s-k})U_{1k}\right\}_2 u^{r-s}\\&-\left\{\sum_{k=0}^{m-1} C_{m+1-k}^* U_{1k}\right\}_2 u^{-m-1}\end{aligned} \tag{4.5}$$

where $C_k=\{C_k \text{ if } k>0;\ Q'_{-k} \text{ if } k\leqq 0\}$.

4.2 The Case When $r\leqq 0$.

We again start with

$$\Phi_m=\sum_{k=0}^{m-1} U_{1k}u^{-k} \tag{4.6}$$

and define R_m by the differential equation

$$\frac{d\Phi_m}{dz}-C\Phi_m=R_m. \tag{4.7}$$

Expanding (4.7) and using (3.16) and (3.17) we obtain

$$R_m=U_0\frac{d}{dz}(U_0^{-1}U_{1m})u^{-m}-\sum_{k=0}^{m-1}C_{m+1-k}^* U_{1k}u^{-m-1}. \tag{4.8}$$

5. Definition of Domains

Let us define a region $\mathcal{D}''\times\mathcal{S}''$ in which we can obtain an actual solution vector of the Eq. (3.1) corresponding to a formal solution vector defined in Section 3.

Let u be a fixed point in $\mathcal{S}'$. Corresponding to a fixed point*) ζ in $\bar{\mathcal{D}}'$ (the closure of $\mathcal{D}'$) we define a region $\mathcal{D}''(u)$ consisting of all points z in $\mathcal{D}'$ such that there exists a path $\mathcal{P}$ joining ζ and z with the following properties:

(1) Except perhaps for ζ if ζ is a boundary point of $\mathcal{D}'$, $\mathcal{P}$ lies wholly in $\mathcal{D}'$;

(2) Let t, τ be points on $\mathcal{P}$ in the order ζ, t, τ, z.

(a) If $r>0$, let $\exp\left\{\int_t^\tau \nu(\xi,u)d\xi\right\}$ be bounded for all such t, τ.

(b) If $r\leqq 0$, let $\|U_0(\tau,u)U_0^{-1}(t,u)\|$ be bounded for all such t, τ.

*) The author expects that ζ may in fact depend upon u. The full extent of this dependence has not been investigated.

(3) $\mathcal{V}_{\mathcal{P}}(t^{-1})$ is bounded.

In (3) above the variation symbol is defined as follows for every holomorphic function $f(t)$:

$$\mathcal{V}_{\mathcal{P}}(f)=\int_{\mathcal{P}}|df|=\int_{\mathcal{P}}|f'(t)\,dt|. \tag{5.1}$$

It is shown in Stenger [6], [7] that if $\mathcal{D}'$ is non-empty a suitable choice of ζ can always be made such that $\mathcal{D}''(u)$ is non-empty. In this case it is shown in Stenger [6] p. 198 that $\mathcal{D}''(u)$ is a domain.

Let $\mathcal{S}''\subset\mathcal{S}'$ be a closed set of points ($\mathcal{S}''$ may possibly become a sector extending to ∞ in the complex u plane for all u sufficiently large) such that corresponding to the fixed point ζ the above conditions define a domain $\mathcal{D}''(u)$ for each u in $\mathcal{S}''$. We then define a region $\mathcal{D}''$ by

$$\mathcal{D}''=\bigcap_{u\in\mathcal{S}''}\mathcal{D}''(u). \tag{5.2}$$

We shall assume that $\mathcal{S}''$ is sufficiently small so that $\mathcal{D}''$ is not empty.

6. Boundedness of the Coefficients

Let us first consider the case $r>0$.

The coefficients of the system (3.1) satisfy

$$\|C_k(z,u)\|=O(z^{-2}u^0) \quad \text{as } z \text{ and/or } u\to\infty. \tag{6.1}$$

Hence, if paths of integration in (3.14) are chosen to satisfy the conditions in Section 5 then $\{U_1\}_{ii}=O(z^{-1}u^0)$ if $|b_1|=\infty$ while $\{U_1\}_{ii}=O(z^0u^0)$ if b_1 is bounded. By (3.6), (6.1) and the fact that $\{U_0\}_{ij}=\delta_{ij}$, it follows that $\|U_1\|=O(z^{-1}u^0)$ if $|b_1|=\infty$ while $\|U_1\|=O(z^0u^0)$ if b_1 is bounded. A similar argument using induction proves

Lemma 6.1. *Let*

$$|b_1|=|b_2|=\cdots=|b_k|=\infty,\quad k\geqq 1 \tag{6.2}$$

in (3.14). *Then the coefficients* U_s *defined in* (3.14) *satisfy*

$$\|U_s\|=O(z^{-1}u^0) \tag{6.3}$$

as z *and/or* $u\to\infty$ *in* $\mathcal{D}''\times\mathcal{S}''$.

If some of the b_s $(1\leqq s\leqq k)$ *are bounded then*

$$\|U_s\|=O(z^0u^0) \tag{6.4}$$

as z *and/or* $u\to\infty$ *in* $\mathcal{D}''\times\mathcal{S}''$.

Similarly, we have

Lemma 6.2. *Let each path of integration in* (3.18) *satisfy the conditions of Section* 5, *and let* $|\zeta|=\infty$. *If*

$$c_k=\infty, \tag{6.5}$$

in (3.18), *then the coefficient* U_k *definition* (3.18) *satisfies*

$$||U_k||=O(z^{-1}u^0) \tag{6.6}$$

as z *and/or* $u\to\infty$ *in* $\mathcal{D}''\times\mathcal{S}''$. *If* c_k *is bounded, then*

$$||U_k||=O(z^0u^0) \tag{6.7}$$

as z *and/or* $u\to\infty$ *in* $\mathcal{D}''\times\mathcal{S}''$.

7. The Integral Equations for the Error

Let a vector $W_1=W_1(z, w)$ of holomorphic functions satisfy the differential equation (3.1). Then, by (4.2) the error vector

$$\varepsilon_m=\varepsilon_m(z, u)=(W_1-\Phi_m)e^{-q_1} \tag{7.1}$$

satisfies the differential equation

$$\frac{d}{dz}\varepsilon_m-\left[D'+\frac{1}{u}C_1^*\right]\varepsilon_m=-R_m \tag{7.2}$$

where R_m is defined in Section 4, and

$$D'=D'(z, u)=Q'+C_0-Iq_1. \tag{7.3}$$

Note that D' is diagonal if $r>0$.

If $r>0$ and ε_m satisfies

$$\varepsilon_m(z, u)=\frac{1}{u}\int_\zeta^z e^{D(z,u)-D(t,u)}C_1^*(t, u)\varepsilon_m(t, u)dt+R_m^*(z, u) \tag{7.4}$$

where the path of integration satisfies the conditions of Section 5, then $\varepsilon_m(z, u)$ simultaneously satisfies (7.2).

Similarly, if $r\leqq 0$ and ε_m satisfies

$$\varepsilon_m(z, u)=\frac{1}{u}\int_\zeta^z U_0(z, u)U_0^{-1}(t, u)C_1^*(t, u)\varepsilon_m(t, u)dt+R_m^*(z, u) \tag{7.5}$$

where the path of integration satisfies the conditions of Section 5, then $\varepsilon_m(z, u)$ simultaneously satisfies (7.2). In (7.4)

$$R_m^*(z, u)=-\int_\zeta^z e^{D(z,u)-D(t,u)}R_m(t, u)dt \tag{7.6}$$

where $R_m(t, u)$ is defined in Section 4.1 while in (7.5)

$$R_m^*(z, u)=-\int_\zeta^z U_0(z, u)U_0^{-1}(t, u)R_m(t, u)dt \tag{7.7}$$

where $R_m(t, u)$ is defined in Section 4.2.

8. Error Bounds

8.1 The Case When $r>0$.

Substituting (4.4) into (7.5) we obtain

$$\{R_m^*(z,u)\}_1=\phi_m^{(1)}(z,u)u^{-m}+\psi_m^{(1)}(z,u)u^{-m-1} \tag{8.1}$$

where

$$\phi_m^{(1)}(z,u)=\{U_{1m}(z,u)-U_{1m}(\zeta,u)\}_1 \tag{8.2}$$

$$\psi_m^{(1)}(z,u)=-\int_\zeta^z\left\{\sum_{k=0}^{m-1}C_{m+1-k}^*(t,u)\,U_{1k}(t,u)\right\}_1 dt. \tag{8.3}$$

In order to more explicitly obtain the second element of R_m^*, we first define $P_2=P_2(z,u)$ in view of (4.5) by

$$\{R_m\}_2=-2\nu\{U_{1m}\}_2u^{-m}+P_2-\left\{\sum_{k=0}^{m-1}C_{m+1-k}^*U_{1k}\right\}_2u^{-m-1}. \tag{8.4}$$

It follows then by the assumptions in Section 3 that $\frac{P_2}{2\nu}\approx(\omega,\mu)\leqq(-1,-m-1)$. Substituting (8.4) into (7.5) and integrating by parts we obtain

$$\{R_m^*\}_2=\phi_m^{(2)}(z,u)u^{-m}+\psi_m^{(2)}(z,u)u^{-m-1} \tag{8.5}$$

where

$$\phi_m^{(2)}(z,u)=\left\{U_{1m}(z,u)-e^{\int_\zeta^z 2\nu d\xi}\,U_{1m}(\zeta,u)\right\}_2 \tag{8.6}$$

$$\psi_m^{(2)}(z,u)=\frac{[P_2(z,u)u^{m+1}-\{U_{1m}'(z,u)\}_2u]}{2\,\nu(z,u)}+e^{\int_\zeta^z 2\nu d\xi}\frac{[P_2(\zeta,u)u^{m+1}-\{U_{1m}'(\zeta,u)\}_2u]}{2\,\nu(\zeta,u)}$$
$$+\int_\zeta^z e^{\int_t^z 2\nu d\xi}\frac{\frac{d}{dt}[P_2(t,u)u^{m+1}+\{U_{1m}'(t,u)\}_2u]}{2\,\nu(t,u)}+\left\{\sum_{k=0}^{m-1}C_{m+1-k}^*(t,u)\,U_{1k}(t,u)\right\}_2 dt. \tag{8.7}$$

For purposes of obtaining a norm bound we put

$$U_{1m}^*(\tau,u)=\sup_{t\in\mathscr{P}}\|(\phi_m^{(1)}(t,u),\phi_m^{(2)}(t,u))\| \tag{8.8}$$

$$V_{1m}^*(\tau,u)=\sup_{t\in\mathscr{P}}\|\psi_m^{(1)}(t,u),\psi_m^{(2)}(t,u))\| \tag{8.9}$$

$$C_1^{**}(\tau,u)=\sup_{t\in\mathscr{P}}\|e^{D(\tau,u)-D(t,u)}C_1^*(t,u)\| \tag{8.10}$$

where in each of (8.8), (8.9) and (8.10) t and τ are points on $\mathscr{P}$ in the order ζ, t, τ, z and t ranges between ζ and τ.

Let us assume that $\mathscr{D}''$ contains a domain. Then the existence of a unique vector ε_m of holomorphic functions of z which satisfies (7.4) readily follows by application of the contraction mapping principle. Taking norms of both sides of (7.4) and applying Bellman's inequality (see e.g. [7], p. 181) we have

Theorem 8.1. *If corresponding to the formal vector solution $W_1(z,u)$ of Eq.* (3.1) *obtained in Section* 3.1 *we can define a region $\mathcal{D}''\times\mathcal{S}''$ by the conditions of Section* 5 *such that $\mathcal{D}''$ contains a domain then the Eq.* (3.1) *possesses an actual solution vector of functions holomorphic in $\mathcal{D}''$ given by*

$$W_{1m}(z,u)=\left[\sum_{k=0}^{m-1} U_{1k}(z,u)u^{-k}+\varepsilon_m(z,u)\right]e^{q_1(z,u)} \tag{8.11}$$

for all u in $\mathcal{S}''$, where

$$\|\varepsilon_m(z,u)\|\leqq\exp\left\{\mathcal{V}_{\mathcal{P}}\left(\frac{1}{u}C_1^{**}\right)\right\}\{\mathcal{V}_{\mathcal{P}}(U_{1m}^*u^{-m})+\mathcal{V}_{\mathcal{P}}(V_{1m}^*u^{-m-1})\} \tag{8.12}$$

for all (z,u) in $\mathcal{D}''\times\mathcal{S}''$.

A vector bound can also be easily obtained using the above definitions. We omit this, since the results are similar to those of Stenger [6] (p. 209).

8.2 The Case When $r\leqq 0$.

Substituting (4.8) into (7.6) yields

$$R_m^*(z,u)=U_{1m}(z,u)u^{-m}+\int_\zeta^z U_0(z,u)U_0^{-1}(t,u)\sum_{k=0}^{m-1}C_{m+1-k}^*(t,u)U_{1k}(t,u)u^{-m-1}dt \tag{8.13}$$

where we have taken $c_m=\zeta$. Setting

$$U_{1m}^*(\tau,u)=\sup_{t\in\mathcal{P}}\|U_{1m}(t,u)\| \tag{8.14}$$

$$V_{1m}^*(\tau,u)=\sup_{t\in\mathcal{P}}\left\|\int_\zeta^t U_0(t)U_0^{-1}(\xi)\cdot\sum_{k=0}^{m-1}C_{m+1-k}^*(\xi,u)U_{1k}(\xi,u)d\xi\right\| \tag{8.15}$$

$$C_1^{**}(\tau,u)=\sup_{t\in\mathcal{P}}\|U_0(\tau,u)U_0^{-1}(t,u)C_1^*(t,u)\| \tag{8.16}$$

where in (8.14), (8.15) and (8.16) t and τ are located on $\mathcal{P}$ in the order ζ, t, τ, z and t ranges between ζ and τ.

We again assume that the region $\mathcal{D}''$ defined in Section 5.2 contains a domain. Then we have

Theorem 8.2. *If corresponding to the formal vector solution $\tilde{W}_1(z,u)$ of Eq.* (3.1) *obtained in Section* 3.2 *we can define a region $\mathcal{D}''\times\mathcal{S}''$ by the conditions of Section* 5 *such that $\mathcal{D}''$ contains a domain then the Eq.* (3.1) *possesses an actual solution vector of functions holomorphic in $\mathcal{D}''$ given by*

$$W_{1m}(z,u)=\sum_{k=0}^{m-1} U_{1k}(z,u)u^{-k}+\varepsilon_m(z,u) \tag{8.17}$$

for all u *in* $\mathcal{S}''$, *where*

$$\|\varepsilon_m(z, u)\| \leqq \exp\left\{\mathcal{V}_{\mathcal{P}}\left(\frac{1}{u}C_1^{**}\right)\right\}\{\mathcal{V}_{\mathcal{P}}(U_{1m}^{*}u^{-m})+\mathcal{V}_{\mathcal{P}}(V_{1m}^{*}u^{-m-1})\} \tag{8.18}$$

for all (z, u) *in* $\mathcal{D}''\times\mathcal{S}''$.

9. Conclusion

If $\mathcal{D}'$ is sufficiently large and $\mathcal{S}''$ is taken to be a sufficiently small sector extending to infinity, it is always possible in the case when $z=\infty$ is an ordinary point or an irregular singular point of the transformed system to choose ζ such that $|\zeta|=\infty$. This is no longer always possible if $z=\infty$ is a regular singular point of the transformed system. For example,

$$\exp\left[\int_t^\tau \nu(\xi, u)d\xi\right]$$

may have the form $(\tau/t)^\rho$, where $\mathrm{Re}\rho=\mathrm{Re}\rho(u)<0$. In this case, we can put $\varepsilon_m(z, u)=z^\rho\eta_m(z, u)$ in (7.2). This replaces

$$\exp\left[\int_t^\tau \nu(\xi, u)d\xi\right] \quad \text{by} \quad \left(\frac{t}{\tau}\right)^\rho \exp\left[\int_t^\tau \nu(\xi, u)d\xi\right]$$

and so enables us to choose $|\zeta|=\infty$ and obtain a bound for $\eta_m(z, u)$ by proceeding as for $\varepsilon_m(z, u)$ above.

References

[1] F. W. J. Olver, Error Bounds for Asymptotic Expansions, with an Application to Cylinder Functions of Large Argument, chapter in Asymptotic Solution of Differential Equations and their Applications, C. H. Wilcox, ed., John Wiley and Sons, New York, 1964.

[2] F. W. J. Olver, Error Bounds for First Approximations in Turning Point Problems, J. Soc. Indust, Appl. Math., **11** (1963), 748–772.

[3] H. L. Turrittin, Asymptotic Expansions of Solutions of Systems of Ordinary Linear Differential Equations Containing a Parameter, Contributions to the Theory of Nonlinear Oscillations II, in Annals of Math. Studies, 29, Princeton Univ. Press, New Jersey, 1952.

[4] W. Wasow, On Turning Point Problems for Systems with Almost Diagonal Coefficient Matrix, MRC Technical Summary Report No. 612, 1966.

[5] K. H. Kiyek, Zur Theorie der linearen Differential-gleichungssysteme mit einem grossen Parameter, Doctoral Thesis, University of Würzburg, 1963.

[6] F. Stenger, Error Bounds for Asymptotic Solutions of Differential Equations, II The General Case, J. Res. NBS, 70 B, Math. and Math. Phys., No. 3 (1976) 187–210.

[7] F. Stenger, Error Bounds for Asymptotic Solutions of Differential Equations, I. The Distinct Eigenvalue Case, J. Res. NBS, 70 B, Math. and Math. Phys., No. 3 (1966), 167–186.

(Ricevita la 23-an de aprilo, 1969)