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On Some Obvious Theorems in Functional Analysis

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Introduction.

The purpose of this note is to formulate some obvious facts which were used in the notes [1–4] of the present author. The first theorem is concerned with the identity of solutions of equations in two different Banach spaces. The second and the third are concerned with the generalization of the Perron–Frobenius theorem in matrix theory. This generalization is done in complex Banach lattices.

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1. Identity of Two Types of Solutions.

Let B_1 and B_2 be two Banach spaces such that $B_1 \subset B_2$ as sets and also as topological spaces: This implies the continuity of the inclusion map i: $B_1 \rightarrow B_2$ defined by $i(x)=x$. Moreover, we require that B_1 is dense in B_2 with respect to the topology of B_2. And let $T_i (i=1,2)$ be compact linear operators in B_i. We further assume that the restriction of T_2 in B_1 coincides with T_1: $T_2 x = T_1 x$ for all $x \in B_1$. Then we have the following theorem.

Theorem 1. *Every solution u of the equation*[1]

(¥) $$(I+\lambda T_2)u=f \quad \text{in } B_2$$

belongs to B_1 provided that $f \in B_1$ and that λ is any (complex) number.

Corollary. *If $f \in B_1$, then every solution u of the equation*

($) $$(I+\lambda T_1)u=f \quad \text{in } B_1$$

is a solution of the equation (¥) and, conversely, every solution u of (¥) is a solution of ($). Hence the equations (¥) and ($) are equivalent if $f \in B_1$.

Proof of Theorem 1. The proof proceeds quite similarly as in the note [1]. But, for the sake of completeness we give a proof.

It is clear that every solution u of ($) is also a solution of (¥). So, if N is the number of independent solutions of (¥) and M is that for ($), then

1) We always denote the identity operator by I.

we have $0 \leqq M \leqq N < \infty$.

Now, let B_i^* $(i=1,2)$ be the dual space of B_i and let T_i^* $(i=1,2)$ be the dual operator of T_i. Then the equation

$$(I+\lambda T_i^*)u^*=0 \quad \text{in } B_i^* \ (i=1,2)$$

has just M independent solutions for $i=1$ and N independent solutions for $i=2$. We may regard that B_2^* is a linear subspace of B_1^*, for, if $u_1^* \in B_2^*$ and $u_2^* \in B_2^*$ are such that $u_1^*(u)=u_2^*(u)$ for all $u \in B_1$, then $u_1^*=u_2^*$ since B_1 is dense in B_2. Thus, to each $u^* \in B_2^*$ there corresponds $v^* \in B_1^*$ in the one-to-one fashion and that v^* is the restriction of u^* on B_1. Furthermore, the restriction of T_1^* on B_2^* coincides with T_2^*. For, if $u^* \in B_2^*$ and if $u \in B_1$, then

$$(T_1^*u^*)(u)=u^*(T_1u)=u^*(T_2u)=(T_2^*u^*)(u).$$

And hence, the restriction of T_1^* on B_2^* is T_2^*, since B_1 is dense in B_2. Hence we have $N \leqq M$. There follows the equality $M=N$ and we have the theorem.

2. Some Definitions.

In the next section we shall formulate two theorems concerning non-negative compact operators in complex Banach lattices. In order to do this we need introduce some definitions.

Definition 1. A real linear space X is called a *vector lattice* if X is a lattice[2)] by a partial order relation $x \leqq y$ satisfying the conditions:

(1) $$x \leqq y \text{ implies } x+z \leqq y+z$$

(2) $x \leqq y$ implies $\alpha x \leqq \alpha y$ (or $\alpha x \geqq \alpha y$) for every $\alpha \geqq 0$ (or $\alpha \leqq 0$).

In a vector lattice X we define

(3) $$x^+=x \vee 0, \quad x^-=x \wedge 0, \quad |x|=x \vee (-x).$$

Then

(4) $$x=x^+ + x^-, \quad |x|=x^+ - x^-, \quad x^+ \geqq 0, \quad x^- \leqq 0.$$

Definition 2. A (real) *Banach lattice* X is a Banach space which is a vector lattice possessing the property that

(5) $$|x| \leqq |y| \text{ implies } \|x\| \leqq \|y\|.$$

Here $\|x\|$ is the norm of $x \in B$.

2) A *lattice* L is a partially ordered set such that for each pair (x,y) of L the supremum $x \vee y$ as well as the infimum $x \wedge y$ exists. The meaning of the supremum $x \vee y$ is: $x \leqq x \vee y$, $y \leqq x \vee y$ and, if $x \leqq z$ and $y \leqq z$ then $x \vee y \leqq z$. The infimum $x \wedge y$ is defined by replacing $\vee$ and $\leqq$ by $\wedge$ and $\geqq$ respectively.

Definition 3. A complex Banach space B is called a *complex Banach lattice* if $B=X+iX$ with a real subspace X of B which is a Banach lattice: I. e., each $f\in B$ can be uniquely represented as $f=f_1+if_2$, where $f_1, f_2\in X$.

Definition 4. A linear operator T in a (real or complex) Banach lattice B is called *non-negative* if $T\,f\geqq 0$ for all $f\geqq 0$.

Definition 5. A (real or complex) Banach lattice B is said to *have the relation* $<$ if the following requirements hold: there exists an $h\geqq 0, \neq 0$, such that for every $f>0$ we have $h\leqq C(f)f$ with a positive constant $C(f)$. And moreover, if $f\in X$ and if $f^+(=f\vee 0)>0$, then $f^-=0$. (The existence of a certain element $k>0$ is required.)

Definition 6. A linear operator T in a (real or complex) Banach lattice B possessing the relation $<$ is called *positive* if $Tf>0$ for all $f\geqq 0, \neq 0$.

3. Generalization of the Perron-Frobenius Theorem.

By Krein and Rutman [5] we have

Theorem 2. *Let B be a (real or complex) Banach lattice*[3] *and let T be a non-negative linear compact operator in B. If $Tf\geqq cf$ for some constant $c>0$ and for some $f\geqq 0, \neq 0$, then we have the following consequences:*

i) *There exist a positive number ρ and a $g\geqq 0, \neq 0$, such that $Tg=\rho g$.*

ii) $(\lambda I-T)^{-1}$ *exists and is continuous for $|\lambda|>\rho$.*

iii) *There exists a $\mu\in B^*$ (the dual space of B), $\mu\geqq 0$*[4], *$\neq 0$, such that $T^*\mu=\rho\mu$.*

iv) *$Tf\geqq\lambda f$, $f\geqq 0, \neq 0$, has no solution if $\lambda>\rho$.*

v) $\rho=\lim_{n\to\infty}||T^n||^{1/n}$.

Corollary. *The same conclusions hold if the assumptions imposed on T is satisfied by a certain power T^m $(m>1)$ provided that T is a bounded linear, non-negative opeorator.*

The proof is omitted. For the proof see Akô [4] and Krein and Rutman [5].

For positive compact operators we have the following theorem.

Theorem 3. *Let B be a (real or complex) Banach lattice possessing the realtion $<$ and T be a positive compact operator in B. If $Tf\geqq cf$ for some constant $c>0$ and for some $f\geqq 0, \neq 0$, then we have the following consequences:*

i) *There exist a positive number ρ and a $g>0$ such that $Tg=\rho g$.*

ii) $(\lambda I-T)^{-1}$ *is continuous for $|\lambda|\geqq\rho$, $\lambda\neq\rho$.*

iii) *There exists a $\mu\in B^*$, $\mu>0$*[5], *such that $T^*\mu=\rho\mu$.*

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3) In this theorem B need not be a Banach lattice since the relation (5) is not required.

4) This means that $\mu(f)\geqq 0$ for all $f\geqq 0$.

5) This means that $\mu(f)>0$ for all $f\geqq 0, \neq 0$.

iv) *If* $\lambda \geqq \rho$ *and if the relation* $Tf \geqq \lambda f \geqq 0$ *holds, then either* $f=0$ *or* f *is a constant multiple of* g *and* $\lambda=\rho$.

v) *The eigenspace of* T *associated with* ρ (*i.e.* $\{f:\ Tf=\rho f\}$) *is one-dimensional* (*see* iv)).

vi) $(\lambda I-T)^{-1}$ *is positive for* $\lambda>\rho$.

vii) $\rho=\lim_{n\to\infty}\|T^n\|^{1/n}$.

Corollary. *If the conditions imposed on* T *are satisfied by a certain power* $T^m (m>1)$ *instead of* T, *then the conclusions* i)-vii) *hold provided that* T *is a non-negative bounded linear operator.*

Proof of Theorem 3. The proof goes quite similarly as in the note [4]. The only point essentially different from the proof in [4] is that, in [4], we required the assumption: if $0<f<g$, then $\|f\|<\|g\|$ in order to establish the statement v) of Theorem 3. The above assumption was only used to verify the fact that if $f\geqq 0$ satirfies the relation $Tf=\rho f$, the $\|g\|f=\|f\|g$. So, we have only to show that

(6) if $Tf=\rho f\geqq 0$, then $\|g\|f=\|f\|g$.

Let $\varepsilon>0$ be a positive number. We set $F=[\|g\|f-(\|f\|+\varepsilon)g]^+ (\geqq 0)$. By the argument of the note [4] we have $TF=\rho F$. Hence, if $F\neq 0$, then $F=\rho^{-1}TF>0$ by the positivity of T. So, we have $[\|g\|f-(\|f\|+\varepsilon)g]^-=0$ and $F=\|g\|f-(\|f\|+\varepsilon)g$. But then, we have a *contradiction*

$$\|g\|\,\|f\|=\|(\|g\|f)\|=\|F+(\|f\|+\varepsilon)g\|\geqq\|(\|f\|+\varepsilon)g\|=(\|f\|+\varepsilon)\|g\|.$$

Thus we have shown that $F=0$ and hence $\|g\|f\leqq(\|f\|+\varepsilon)g$. Letting $\varepsilon\to 0$, we have $\|g\|f\leqq\|f\|g$[6]. Interchanging the roles of f and g, we have another inequality $\|f\|g\leqq\|g\|f$. This implies $\|g\|f=\|f\|g$, as asserted.

4. A Remark.

For the validity of Theorem 3 the condition (5) of Definition 2 is too heavy. As is seen from the proof we only need the following weaker condition:

(5)′ If $0\leqq f\leqq g$, then $\|f\|\leqq\|g\|$.

References

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6) It is clear that $\|f\|g\leqq(\|g\|f)\vee(\|f\|g)\leqq(\|f\|+\varepsilon)g$ and hence $0\leqq(\|g\|f)\vee(\|f\|g)-\|f\|g\leqq\varepsilon g\to 0$ $(\varepsilon\to 0)$. Thus $(\|g\|f)\vee(\|f\|g)-\|f\|g=0$, or equivalently $\|g\|f\leqq\|f\|g$.

to appear.
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