A Mixed Initial and Boundary-Value Problem
for the Hamilton-Jacobi Equation in
Several Space Variables II

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To Professor Tokui Satō on the Occasion of His Retirement

Introduction.
This paper is concerned with the existence of generalized solutions which
are defined in the quarter-space

\[ D = \{(t, x) : t \geq 0, \ x_1 \geq 0, \ |x'| < \infty \}, \]

\[ x = (x_1, x_2, \ldots, x_n), \ x' = (x_2, \ldots, x_n), \]

of an \( (n+1) \)-dimensional Euclidean space \( E_{n+1} \), to the mixed initial-boundary
value problem

\[ (1) \quad z_t + f(z_x) = 0, \]

where \( z_x \) denotes the gradient in the variables \( x \), with the initial data

\[ (2) \quad z(0, x) = \varphi(x), \ x_1 \geq 0, |x'| < \infty, \]

and the boundary data

\[ (3) \quad z(t, 0, x') = \psi(t, x'), \ t \geq 0, |x'| < \infty. \]

We call \( z \) a generalized solution of (1) if \( z \) is Lipschitz-continuous and satisfies
(1) at almost all points of \( D \).

We assume that \( f(u) \) is of class \( C^2 \) in \( E_n \) and:

\[ (A) \quad \sum_{i,j=1}^{n} f_{ij}(u) \lambda_i \lambda_j > 0 \]

for all \( u \in E_n \) and all (real) \( \lambda_1, \ldots, \lambda_n \), where we write \( f_{ij} = \partial^2 f / \partial u_i \partial u_j \).

\[ (B) \quad f(u) / |u| \to \infty \quad \text{as} \quad |u| \to \infty. \]

Let us assume that \( \varphi \) and \( \psi \) are Lipschitz-continuous on their respective
domains of definition and that

\[ \varphi(0, x') = \psi(0, x'), \ |x'| < \infty. \]

In a previous paper [1] we constructed a generalized solution to this problem
under the additional assumption that the boundary values \( \varphi(t, x') \) are convex on the half-space \( t \geq 0 \) of \( E_n \) (but under a weaker assumption on \( f \)). The purpose of the present paper is to obtain a generalized solution of (1)–(3) without the convexity condition on \( \varphi \). Our method in this paper is the same as in the previous work [1]. In §§1–2 we consider the Cauchy problem (1), (2) and the pure boundary-value problem (1), (3) respectively. In §3 we obtain a generalized solution of the mixed problem (1)–(3) as \( z = \min (z_1, z_2) \), \( z_1 \) and \( z_2 \) being two generalized solutions constructed in §§1–2 respectively.

It is known from examples that generalized solutions of (1)–(3) are not unique in the class of Lipschitz-continuous solutions. However, it is shown [2] that there is uniqueness in a subclass of generalized solutions which satisfy the so-called semi-concavity condition. (But, to insure uniqueness, some further assumptions on \( f \) are necessary). In §3 we also show that the generalized solution which we construct satisfies this condition.

1. The Cauchy problem.

In this section we consider the Cauchy problem (1), (2) in the quarter-space \( D \).

On account of Hypotheses (A), (B), the Legendre transformation

\[
(1.1) \quad v = f(u), \quad u \in E_n,
\]

is globally one to one from \( E_n \) onto itself. The conjugate function \( g \) defined by

\[
(1.2) \quad g(v) = u \cdot v - f(u), \quad v = f(u),
\]

is of class \( C^2 \) in \( E_n \) and also fulfills Hypotheses (A), (B). The matrices \( (f_{ij}) \) and \( (g_{ij}) \) of second derivatives are related by \( (g_{ij}) = (f_{ij})^{-1} \).

Denote by \( x, y, u, \cdots \) points of the \( n \)-dimensional space \( E_n \), and by \( x', y', u', \cdots \) points of \( E_{n-1} \) with coordinates \( x' = (x_2, \cdots, x_n), \cdots \). \( E_n^+ \) denotes the (closed) halfspace \( x_1 \geq 0 \) of \( E_n \).

To obtain a generalized solution to our Cauchy problem, we take advantage of the complete integral \( z = u + u \cdot x - f(u)t \) for the Hamilton-Jacobi equation (1) and consider a family of solutions containing the parameters \( y \) and \( u \):

\[
(1.3) \quad z = z(t, x; y, u) = \varphi(y) + u \cdot (x - y) - f(u)t,
\]

where \( y \in E_n^+ \) and \( u \in E_n \) are two arbitrary vectors.

Our generalized solution is then given by

\[
(1.4) \quad z(t, x) = \inf_{y \in E_n^+} \sup_{u \in E_n} z(t, x; y, u).
\]

Concerning the properties of the function \( z \) so defined, we have already proved the following theorem ([11], Theorem 3.1, pp. 143–144).
Theorem 1.1. If \( f(u) \) fulfills Hypotheses (A), (B), then the function \( z \) defined by (1.4) is a locally Lipschitzian solution of (1) for all \( t > 0 \) with the property that

\[
\lim_{t \to 0} z(t, x) = \begin{cases} 
\varphi(x) & x_1 \geq 0, |x'| < \infty, \\
\infty & x_1 < 0, |x'| < \infty.
\end{cases}
\]

In the remainder of this section we shall show that \( z \) is Lipschitz-continuous in the quarter-space \( D \) and satisfies a semi-concavity condition.

Lemma 1.1. The function \( z \) defined by (1.4) is Lipschitz-continuous in the quarter-space \( D \).

Proof. Using the conjugate function \( g \), we get from (1.3) and (1.4)

\[
\begin{align*}
z(t, x) &= \min_{y \in E^+_n} z(t, x ; y), \\
z(t, x ; y) &= \sup_{u} z(t, x ; y, u) = \varphi(y) + tg((x-y)/t).
\end{align*}
\]

By Rademacher’s theorem, the locally Lipschitzian function \( z \) is (totally) differentiable almost everywhere in the interior of \( D \). At each point \((t, x)\) where \( z \) is differentiable, the minimum (1.5) is attained at a unique point \( y^0 \) since at each such point there holds

\[
z_x(t, x) = g_x((x-y^0)/t)
\]

and the transformation \( u = g_x(v) \) which is the inverse of (1.1) is globally one to one from \( E_n \) onto itself.

Now let us take an arbitrary point \((t, x)\) in the interior of \( D \) and suppose that \( z \) is differentiable at \((t, x)\). Denote by \( y^0 \) the unique point at which the minimum (1.5) is attained.

We want to show that the gradient \( z_x \) is bounded in \( D \).

To do this we have to consider the following two cases.

First let us consider the case where \( y^0 \) is an interior point of \( E^+_n \). In view of (1.6) this means that

\[
tg(v) - tg(v^0) \geq \varphi(y^0) - \varphi(y)
\]

holds for every \( y \) (sufficiently near to \( y^0 \)), where \( tv = x - y, tv^0 = x - y^0 \). Since \( \varphi(y) \) is Lipschitz-continuous with constant \( L \), this inequality gives that

\[
g(v) - g(v^0) \geq -L|v - v^0|
\]

for every \( v \). Since \( g \) is of class \( C^2 \), it follows that

\[
\xi \cdot g_u(v^0) \geq -L
\]

for every vector \( \xi, |\xi| = 1 \). Hence we get

\[
|g_u(v^0)| \leq L,
\]
which in turn shows

\[(1.7)\quad |z_\varphi(t, x)| = |g_\varphi(\psi)| \leq L.\]

Next consider the case where \(y^0\) lies on the boundary \(y_1 = 0\) of \(E_n^+\). In this case we can proceed exactly in the same way as above and get

\[(1.8)\quad |z_{\varphi'}(t, x)| \leq L, \quad z_{\varphi}(t, x) \leq L,
\]

where \(z_{\varphi'}\) denotes the gradient in the variables \(x' = (x_2, \ldots, x_n)\). It remains to estimate the derivative \(z_{\varphi}\) from below. To do this let

\[(1.9)\quad u^0 = g_\varphi(\psi^0), \quad \psi^0 = (x - y^0)/t,\]

and note that \(\psi^1 = x_1/t \geq 0\) since \(y^0\) lies on the boundary of \(E_n^+\). (1.9) implies \(\psi^0 = f_\psi(u^0)\). Now, for every fixed \(u', |u'| \leq L\), denote by \(u_t(u')\) the value of \(u_t\) for which \(f(u_t, u')\) attains its minimum as a function of the single variable \(u_t\). Let

\[(1.10)\quad -L'_1 = \min_{|u'| \leq L} u_t(u').\]

Then, obviously,

\[u^0_t = g_{\psi_t}(\psi^0) \geq -L'_1\]

which implies

\[z_{\varphi}(t, x) \geq -L'_1.\]

Therefore, if we define

\[(1.11)\quad -L_1 = \min (-L, -L'_1),\]

then we get in either case

\[(1.12)\quad -L_1 \leq z_{\varphi}(t, x) \leq L, \quad |z_{\varphi}(t, x)| \leq L,
\]

\(L\) being the Lipschitz-constant of \(\varphi\). But the validity of (1.12) almost everywhere in \(D\) implies the Lipschitz-continuity of \(z\).

**Lemma 1.2.** The function \(z\) defined by (1.4) is semi-concave in the sense that for each \(t > 0\) and for each triple \((t, x), (t, x + h), (t, x - h)\) belonging to \(D\), there holds the inequality

\[(1.13)\quad z(t, x + h) + z(t, x - h) - 2z(t, x) \leq \frac{C}{t} |h|^2\]

where \(C\) is a constant > 0.

The proof of this lemma is based upon the following two lemmas concerning the properties of semi-concave functions.

A function \(z\), defined on a convex set \(G\) of \(E_n\), is called semi-concave with constant \(k\) if \(z\) satisfies the inequality
\[ z(x+h)+z(x-h)-2z(x) \leq k|h|^2 \]

for any point \( x \in G \) and any vector \( h, x \pm h \in G \).

**Lemma 1.3.** Let \( z_\lambda(x), \lambda \in \Lambda \), be defined on a convex set \( G \) of \( E_n \) and semi-concave with the same constant \( k \) for all \( \lambda \in \Lambda \). Suppose that
\[ z(x) = \min_{\lambda \in \Lambda} z_\lambda(x) \]
exists at every point of \( G \). Then the function \( z \) is semi-concave with the constant \( k \).

**Proof.** This lemma is obvious.

A function \( z \) defined in \( E_n^+ \) is called *locally semi-concave with constant \( k \)* if, for any point \( x \in E_n^+ \), there exists a convex neighborhood of \( x \) where \( z \) is semi-concave with constant \( k \).

**Lemma 1.4.** If \( z \) is continuous in \( E_n^+ \) and locally semi-concave with constant \( k \), then \( z \) is semi-concave with the same constant \( k \).

**Proof.** First suppose that \( z \) is of class \( C^2 \) in \( E_n^+ \). Then the local semi-concavity of \( z \) implies that
\[ \sum_{i,j=1}^n z_{ij}(x)\lambda_i\lambda_j \leq k|\lambda|^2 \]
for all \( x \in \text{Int} E_n^+ \) and all \( \lambda_1, \ldots, \lambda_n \), where \( z_{ij} = \partial^2 z / \partial x_i \partial x_j \). But this inequality means that \( z \) is semi-concave with constant \( k \) on the whole of \( E_n^+ \).

The general case of continuous \( z \) follows from the fact that any continuous, locally semi-concave function can be locally uniformly approximated by a sequence of infinitely differentiable functions also locally semi-concave with the same constant. They can be obtained, for example, by convolving the original function with Friedrichs' mollifying kernels.

**Proof of Lemma 1.2.** For a given point \( (t, x) \) in the interior of \( D \), let \( y^0 \) be a point at which the minimum (1.5) is attained. Then, by repeating the same reasoning as in the proof of Lemma 1.1, we get
\[ -L_1 \leq g_{ij}((x-y^0)/\varepsilon) \leq L, \quad |g_{ij}((x-y^0)/\varepsilon)| \leq L, \]
where, as before, \( L \) is the Lipschitz-constant of the initial values \( \varphi \), and \(-L_1 \) is a constant defined by (1.11).

Let \( U_L \) denote the (open) cylinder in \( u \)-space defined by
\[ U_L = \{ u; -L_1 < u_1 < L, |u'| < L \} \]
and define
\[ a = \inf_{u \in U_L} \sum_{i,j=1}^n f_{ij}(u)\lambda_i\lambda_j. \]
We shall show that the function \( z \) is semi-concave with constant \( 1/\alpha t \) at every level \( t>0 \).

For each fixed \( t>0 \), let us consider the right-hand side of (1.6) as a function of \( z \) containing the parameter \( y \). Then, for a given point \( x^0 \in \text{Int} \ E^*_n \), there exists an open set \( Y \) of points \( y \) such that the image of \( Y \) under the transformation \( u = g_v(x^0-y)/t \) is the open cylinder \( U_{L+\varepsilon} \),

\[
U_{L+\varepsilon} = \{ u; \ -L_1-\varepsilon < u_1 < L_1+\varepsilon, \ |u'| < L+\varepsilon \},
\]

where \( \varepsilon \) is a given positive number. (Recall that the map \( u = g_v(v) \) is globally one to one from \( E_n \) onto itself.) For this set \( Y \) we can find a neighborhood \( X \) of \( x^0 \) such that the transformation \( u = g_v(x-y)/t \) maps \( Y \) onto the closure of \( U_L \) for each fixed \( x \in X \). Hence we infer from (1.14) that

\[
z(t, x) = \min_{y \in Y \cap E^*_1} z(t, x; y)
\]

for every point \( x \in X \).

Now, as a function of \( x \) defined on \( X \), the right-hand side of (1.6) is, for every vector \( y \in Y \), a \( C^0 \)-function semi-concave with constant \( 1/\alpha't \), where \( \alpha' \) is a positive constant defined by

\[
\alpha' = \inf_{u \in U_{L+\varepsilon}} \sum_{i,j=1}^n f_{ij}(u) \lambda_i \lambda_j,
\]

since the matrix \( (g_{ij}(v)) \) is the inverse matrix of \( (f_{ij}(u)) \) with \( u = g_v(v) \).

Thus, by Lemma 1.3, \( z(t, x) \) is locally semi-concave with constant \( 1/\alpha't \) and, by Lemma 1.4, it is semi-concave with the same constant at every level \( t>0 \).

But since \( \varepsilon>0 \) is arbitrary, we conclude that the function (1.4) is semi-concave with constant \( 1/\alpha t \).

**Theorem 1.2.** The function \( z \) defined by (1.4) is a generalized solution to the Cauchy problem (1), (2) and satisfies a semi-concavity condition in the sense (1.13).

2. The boundary-value problem.

We take again advantage of the complete integral \( z = a + u \cdot x - f(u)t \) and consider, for the pure boundary-value problem (1), (3), the solutions

\[
z = z(t, x; y, u) \equiv \psi(y) + u_1 x_1 + u'_1 (x' - y') - f(u)(t - y_1)
\]

containing the parameters \( y \) and \( u \), where \( y \in E^*_n \) and \( u \in E^*_n \) are two arbitrary vectors.

In the sequel we study the properties of the function \( z \) defined by
(2.2) \[ z(t, x) = \inf_{0 \leq y < t} \sup_{y' \in E_{n-1}} z(t, x ; y, u) \]

which can be written by using the conjugate function \( g \)

(2.3) \[ z(t, x) = \min_{y' \in E_{n-1}} z(t, x ; y) \]

(2.4) \[ z(t, x ; y) = \sup_u z(t, x ; y, u) = \psi(y) + (t - y_1)g\left( \frac{x_1}{t - y_1}, \frac{x' - y'}{t - y_1} \right) \]

for every \((t, x)\) in the interior of \(D\).

**Lemma 2.1.** \( \lim_{t \to 0} z(t, x) = \infty, \quad x_1 > 0, |x'| < \infty. \)

**Proof.** This lemma follows easily from the Lipschitz-continuity of \( \psi \) and the growth property of \( g \),

(2.5) \[ g(v)/|v| \to \infty \quad \text{as} \quad |v| \to \infty. \]

**Lemma 2.2.** *The function \( z \) defined by (2.3) is locally Lipschitz-continuous in the interior of \( D \) and satisfies (1) at almost all points of \( D \).*

**Proof.** Take an arbitrary point \((t^0, x^0) \in \text{Int} \, D \) and keep it fixed. By virtue of the Lipschitz-continuity of \( \psi \) and the growth property (2.5) of \( g \), there are an open neighborhood \( N \) of \((t^0, x^0)\) and a positive number \( \eta \) such that i) the closure of \( N \) is contained in the interior of \( D \), ii) \( \eta \) is less than \( \inf \{ t ; (t, x) \in N \} \), and iii) there holds

(2.6) \[ z(t, x) = \min_{0 \leq y \leq \eta} \sup_{y' \in E_{n-1}} z(t, x ; y) \]

for every \((t, x) \in N \), since the right-hand side of (2.4) with \((t, x)\) replaced by \((t^0, x^0)\) tends to infinity, uniformly in \( y' \in E_{n-1} \), as \( y_1 \uparrow t^0 \).

We now apply E. Hopf’s lemma ([4], Lemma 2.2) to \( z(t, x ; y) \). The \( x \) in that lemma is the present \((t, x)\) and \( \xi = y \),

\[ D = N, \quad \mathcal{E} = \{ y ; 0 \leq y_1 \leq \eta, y' \in E_{n-1} \}. \]

It is clear that \( \alpha \) \( z \), \( z_x, z_t = -f(z_x) \) are continuous in \( t, x, y, (t, x) \in N, y \in \mathcal{E}, \)

\( \beta \) \( z(t, x ; y) \to \infty \) holds uniformly with respect to \((t, x) \in N \) and \( y \)

\( z(t, x ; y) \) is, for every fixed \( y \in \mathcal{E} \), a \( C^2 \)-solution of (1) in \( N \). Thus the conclusion of E. Hopf’s lemma applies: (2.6) is Lipschitz-continuous in \( N \) and satisfies (1) almost everywhere in \( N \). But since \((t^0, x^0) \in \text{Int} \, D \) is arbitrary, the proof is complete.

In order to show that the function (2.3) satisfies the boundary condition (3), it is necessary to impose an additional restriction on the upper bound of \( \partial \psi / \partial t \) in connection with the range of \( f \). In the case of one space variable, it was shown ([3], § 6) that \( -\psi'(t) \) must be in the range of \( f \) for almost all
for almost all $t > 0$, where $m$ is the minimum value of $f$.

In our case we have to impose a restriction on the upper bound of the Lipschitz-constant of $\psi$ with respect to $t$. To be more precise, let $\psi(t, x')$ satisfy the following Lipschitz-conditions:

$$|\psi(t, x') - \psi(t, y')| \leq L'|x' - y'|$$

for each fixed $t \geq 0$ and all $x', y' \in E_{n-1}$, and

$$N(t^1 - t^2) \leq \psi(t^1, x^1) - \psi(t^2, x^1) \leq M(t^1 - t^2)$$

for all $t^1 > t^2 \geq 0$ and each fixed $x^1 \in E_{n-1}$, where $L', M$ and $N$ are constants. Then we require that the Lipschitz-constant $M$ in (2.8) satisfy the condition

$$(1) \quad -M \geq \max_{|u'| \leq L'} \left( \min_{-\infty < u_1 < \infty} f'(u) \right),$$

where $L'$ is the Lipschitz-constant figuring in (2.7).

**Lemma 2.3.** If $\psi(t, x')$ is Lipschitz-continuous and fulfills Condition (1) then

$$\lim_{x_1 \to 0} z(t, x) = \psi(t, x'), \quad t > 0, |x'| < \infty.$$

**Remark.** In the case of one space variable, this lemma was proved by E. D. Conway and E. Hopf ([3], Lemma 6.9). However, their method in that paper which is different from our method below does not seem to be applicable to the multi-dimensional case.

Before proving Lemma 2.3 we prove the following two lemmas.

**Lemma 2.4.** If the boundary values $\psi$ and $\bar{\psi}$ satisfy the inequality

$$|\psi(t, x') - \bar{\psi}(t, x')| \leq \varepsilon(T), \quad 0 \leq t \leq T, |x'| < \infty,$$

for any fixed $T > 0$ where $\varepsilon(T)$ is a positive increasing function in $T \geq 0$, then the corresponding functions $z$ and $\bar{z}$ defined by (2.3) satisfy the inequality

$$|z(t, x) - \bar{z}(t, x)| \leq \varepsilon(T), \quad 0 \leq t < T, x_1 > 0, |x'| < \infty,$$

for any fixed $T > 0$. 

**Proof.** This lemma follows easily from the expressions (2.3) and (2.4).

For each point $(t, x)$ in the interior of $D$, denote by $y^0 = y^0(t, x)$ any of those points $y$ for which the minimum (2.3) is attained. (Note that at each point $(t, x)$ where $z$ is differentiable, the minimum (2.3) is attained at a unique point $y^0$.) For the sake of simplicity let

$$v^0_1 = \frac{x_1}{(t - y^0_1)}, \quad v^0_2 = (x' - y^0)/(t - y^0).$$
Lemma 2.5. Suppose that $\phi(t, x')$ fulfills Condition (1). We have:

a) If $y_1^0 > 0$ then

$$L_1(M) \leq g_{v_1}(\phi^0) \leq K_1(N), \quad |g_{v'}(\phi^0)| \leq L',$$

where $L_1(M)$ and $K_1(N)$ are constants given by (2.13) and (2.14) respectively.

b) If $y_1^0 = 0$ then

$$L_1(M) \leq g_{v_1}(\phi^0) \leq K_2(x_1/t), \quad |g_{v'}(\phi^0)| \leq L',$$

where $K_2(x_1/t)$ is a constant given by (2.15).

Proof. The second inequality in both cases a), b) can be proved in the same way as in the proof of Lemma 1.1. It thus remains to prove the first inequality in either case.

a) By the definition of $y^0$

$$z(t, x; y_1, y^0) \geq z(t, x; y^0)$$

holds for any $y_1, y_1^0 < y_1 < t$. From this inequality we infer after simple calculations

$$f(u^0) \geq -M, \quad u^0 = g_v(\phi^0).$$

Now, for each fixed $u', |u'| \leq L'$, let us consider $f(u)$ as a function of the single variable $u_1$ and denote by $u_1(u'; -M)$ the value of $u_1$ for which both $f(u) = -M$ and $f_{u_1}(u) \geq 0$ hold. Let

$$L_1(M) = \min_{|u'| \leq L'} u_1(u'; -M).$$

Then, in view of the second inequality, we get obviously

$$u_1^0 = g_{v_1}(\phi^0) \leq L_1(M).$$

To prove the second half of the inequality we note that (2.11) holds also for any $y_1, 0 < y_1 < y_1^0$. But this fact gives

$$f(u^0) \leq N, \quad u^0 = g_v(\phi^0).$$

Again consider $f(u)$ as a function of $u_1$ and, for each fixed $u', |u'| \leq L'$, denote by $u_1(u'; N)$ the value of $u_1$ for which both $f(u) = N$ and $f_{u_1}(u) \geq 0$ hold. Let

$$K_1(N) = \max_{|u'| \leq L'} u_1(u'; N).$$

Then, obviously,

$$u_1^0 = g_{v_1}(\phi^0) \leq K_1(N).$$

b) The first half of the first inequality has just been proved. To prove the second half of the inequality let us note that $u^0 = g_v(\phi^0)$ implies $\phi^0 = f_v(u^0)$. Since $y_1^0 = 0, \phi^0 = x_1/t$ and consequently $f_v(u^0) = x_1/t$. For each fixed $u', |u'| \leq L'$,
let $u_1(u'; x_1/t)$ denote the value of $u_1$ for which $f_{u_1}(u) = x_1/t$ holds. Let

$$K_2(x_1/t) = \max_{|u'| \leq L'} u_1(u'; x_1/t).$$

We then get

$$u_1^0 = g_{u_1}(\phi^0) \leq K_2(x_1/t).$$

The proof of the lemma is hereby completed.

Remark. More precisely, (2.12) and (2.12)' imply that

$$u_1(u''; -M) \leq u_1^0 \leq u_1(u''; N), \quad |u''| \leq L'.$$

We are now in a position to prove Lemma 2.3.

Proof of Lemma 2.3. First let us consider the case where

$$-M > \max_{|u'| \leq L'} \left( \min_{u_1} f(u) \right).$$

Note that this condition implies

$$L_0(M) = \min_{|u'| \leq L'} f_{u_1}(u_1(u''; -M), u') > 0,$$

where $u_1(u''; -M)$ denotes, as before, the value of $u_1$ for which both $f(u) = -M$ and $f_{u_1}(u) \geq 0$ hold for each fixed $u', |u'| \leq L'$.

Let $u_0 = g_{u_1}(\phi^0)$. By virtue of Lemma 2.5

$$L_1(M) \leq u_1^0 \leq K, \quad |u''| \leq L'$$

for any $(t, x) \in \text{Int } D$ satisfying $x_1 \leq \omega t$, where $\omega$ is a given positive number and

$$K = \max (K_1(N), K_2(\omega)).$$

Further we have

$$v^0 = x_1/(t - y^0) = f_{u_1}(u^0)$$

and hence

$$v^0 = x_1/(t - y^0) \geq L_0(M) > 0$$

by virtue of (2.17), since $u_1^0 \geq u_1(u''; -M), |u''| \leq L'$, by the remark to Lemma 2.5.

By (2.18) $\phi^0 = f_u(u'')$ is bounded for any $(t, x) \in \text{Int } D$ if $x_1/t$ is bounded. Therefore $g(\phi^0)$ is also bounded if $x_1/t$ is bounded.

From (2.19) it follows that $g^0(t, x) \to t$ as $x_1 \to 0$. Consequently, $y^{\theta}(t, x) \to x'$ as $x_1 \to 0$ by virtue of the boundedness of $v^0$ for $x_1 \to 0$. Thus we have showed that $y^\theta(t, x) \to (t, x')$ as $x_1 \to 0$ for each fixed $t > 0$ and $x' \in E_{\mu - 1}$. But this fact proves together with the boundedness of $g(\phi^0)$ for $x_1 \to 0$ that $z(t, x) \to \psi(t, x')$ as $x_1 \to 0$. 


The general case where $\phi(t, x')$ fulfills Condition (1) can be proved by using Lemma 2.4 and taking as $\tilde{\phi}$ in that lemma the boundary values $\psi_\varepsilon$ defined by

$$\psi_\varepsilon(t, x') = \phi(t, x') - \varepsilon t,$$

where $\varepsilon$ is an arbitrary positive number.

The function (2.3) is Lipschitz-continuous and satisfies a semi-concavity condition in any wedge-shaped subset $D_\omega$ of $D$,

$$D_\omega = \{(t, x) : t \geq 0, 0 \leq x_1 \leq \omega t, |x'| < \infty\}.$$

For later use let

$$(2.20) \quad \sigma(N) = \max_{|u'| \leq L'} f_{u_1}(u_1; N, u'),$$

where $u_1(u'; N)$ denotes, as before, the value of $u_1$ for which both $f(u) = N$ and $f_{u_1}(u) \geq 0$ hold for each fixed $u'$, $|u'| \leq L'$.

**Lemma 2.6.** If $\phi(t, x')$ fulfills Condition (1), then $y^i(t, x) = 0$ for any point $(t, x)$ in the interior of $D$ such that $x_i > \sigma(N)t$.

**Proof.** This is an immediate consequence of (2.12)' and (2.20).

**Lemma 2.7.** If $\phi(t, x')$ fulfills Condition (1) then the function $z(t, x)$ satisfies a uniform Lipschitz condition in the wedge-shaped subset $D_\omega$ of $D$ for any positive number $\omega$.

**Proof.** By virtue of Lemma 2.5 there holds

$$L_1(M) \leq g_{\vartheta'}(\vartheta) \leq K, \quad |g_{\vartheta'}(\vartheta)| \leq L'$$

for any $(t, x) \in \text{Int } D_\omega$, where $K = \max (K_1(N), K_2(\omega))$. But this inequality implies

$$L_1(M) \leq z_2(t, x) \leq K, \quad |z_2(t, x)| \leq L'$$

almost everywhere in $D_\omega$, since $z_2(t, x) = g_{\vartheta'}(\vartheta)$ at each $(t, x)$ where $z$ is differentiable. This completes the proof of the lemma.

**Lemma 2.8.** If $\phi(t, x')$ fulfills Condition (1), then the function $z(t, x)$ is semi-concave in the sense that for each $t > 0$ and for each triple $(t, x), (t, x + h), (t, x - h)$ belonging to $D_{\omega, \delta}$,

$$D_{\omega, \delta} = \{(t, x) : t \geq 0, \delta \leq x_1 \leq \omega t, |x'| < \infty\},$$

where $\omega$ and $\delta, \omega > \delta$, are arbitrary positive numbers, there holds the inequality

$$(2.21) \quad z(t, x + h) + z(t, x - h) - 2z(t, x) \leq C_{\omega, \delta} |h|^2,$$

where $C_{\omega, \delta}$ is a constant depending on $\omega$ and $\delta$.

**Proof.** By using Lemma 2.5 and noting that the second derivatives of $z(t, x ; y)$ as a function of $x$ are
\[
 z_{ij}(t, x ; y) = \frac{1}{t-y_1} g_{ij} \left( \frac{x_1}{t-y_1}, \frac{x'-y'}{t-y_1} \right)
\]

and

\[
 v_i = x_i(t-y_i^0) \leq \max(x_i/L, \sigma(N))
\]

where \(\sigma(N)\) is defined by (2.20), this lemma can be proved in the same way as in the proof of Lemma 1.2.

Summarizing the main results in this section, we have the following:

**Theorem 2.1.** Suppose that \(\phi(t, x')\) fulfills Condition (1). Then the function \(z\) defined by (2.3) is a locally Lipschitzian solution of (1) in the quarts-space \(D\) with the property that

\[
\lim_{t \to 0} z(t, x) = \phi(t, x'), \quad t > 0, |x'| < \infty,
\]

\[
\lim_{t \to 0} z(t, x) = \infty, \quad x_1 > 0, |x'| < \infty.
\]

Moreover, \(z\) is uniformly Lipschitz-continuous in any wedge-shaped subset \(D_w\) of \(D\) and satisfies a semi-concavity condition in the sense of Lemma 2.8.

3. The mixed initial-boundary value problem.

Let \(z_1\) be the Lipschitz-continuous solution (1.4) to the Cauchy problem (1), (2) and \(z_2\) be the locally Lipschitzian solution (2.3) to the boundary-value problem (1), (3). If we define the function \(z\) as

\[
z(t, x) = \min(z_1(t, x), z_2(t, x)),
\]

then \(z\) is again a locally Lipschitzian solution of (1) in the quarter-space \(D\). From Theorems 1.1 and 2.1 it follows that

\[
\lim_{t \to 0} z(t, x) = \lim_{t \to 0} z_1(t, x) = \phi(t, x'), \quad x_1 > 0, |x'| < \infty.
\]

But it is not necessarily true that

(3.1) \[
\lim_{t \to 0} z(t, x) = \lim_{t \to 0} z_2(t, x) = \phi(t, x'), \quad t > 0, |x'| < \infty.
\]

On the other hand, by virtue of (1.4) and (1.6),

\[
z_1(t, 0, x') = \inf_{y \in E_{z-1}} (\phi(y) + t g(-y_1/t, (x'-y')/t))
\]

for all \(t > 0\) and all \(x' \in E_{n-1}\). Therefore, for (3.1) to be valid, it is necessary and sufficient that \(\phi(x)\) and \(\phi(t, x')\) satisfy the following compatibility condition:

(II) \[
\phi(t, x') \leq \inf_{y \in E_{z-1}} (\phi(y) + t g(-y_1/t, (x'-y')/t))
\]

for all \(t > 0\) and all \(x' \in E_{n-1}\).
Under this compatibility condition the function \( z \) becomes a locally Lipschitzian solution to the mixed initial-boundary value problem (1)–(3).

Moreover, \( z \) is Lipschitz-continuous in the quarter-space \( D \) and satisfies a semi-concavity condition.

Lemma 3.1. The function \( z \) is Lipschitz-continuous in the quarter-space \( D \).

Proof. Since \( z_1 \) is Lipschitz-continuous in \( D \) (Lemma 1.1) and \( z_2 \) is Lipschitz-continuous in the wedge-shaped sebset \( D_\omega \) for any positive \( \omega \) (Lemma 2.7), it is sufficient to prove that there is a positive number \( \omega \) such that there holds

\[
z(t, x) = z_1(t, x) < z_2(t, x)
\]

for any \((t, x) \in \text{Int } D\) satisfying \( x_1 > \omega t \). But this can be proved as follows: If not, then we could find a sequence \((t^n, x^n)\) such that

\[
z_2(t^n, x^n) \leq z_1(t^n, x^n), \quad x_1^n = n t^n, \quad n = 1, 2, \ldots
\]

Let \( y^n(t^n, x^n) \) be a point at which the minimum (2.3) for \((t^n, x^n)\) is attained.

It then follows from Lemma 2.6 that \( y^n(t^n, x^n) = 0 \) for all sufficiently large \( n \) so that

\[
z_2(t^n, x^n) = \varphi(0, y^n') + t^n g(x^n_1/t^n, (x^n' - y^n')/t^n)
\]

\[
= \varphi(0, y^n') + t^n g(x^n_1/t^n, (x^n' - y^n')/t^n) \leq z_1(t^n, x^n),
\]

since \( \varphi(0, x') = \varphi(0, x') \). But the last inequality implies that \( y^n(t^n, x^n) \) is also a point for which the minimum (1.5) for \((t^n, x^n)\) is attained, which contradicts the boundedness of \( \varphi_1 \) for (1.5) (see (1.14)).

Therefore, for some positive \( \omega, z_1(t, x) < z_2(t, x) \) if \( x_1 > \omega t \) and the proof of the lemma is completed.

Lemma 3.2. The function \( z \) is semi-concave in the sense that for each \( t > 0 \) and for each triple \((t, x), (t, x + h), (t, x - h)\) belonging to \( D_\delta \),

\[
D_\delta = \{(t, x) ; t \geq \delta, x_1 \geq \delta, |x'| < \infty\},
\]

where \( \delta \) is an arbitrary positive number, there holds the inequality

\[
z(t, x + h) + z(t, x - h) - 2 z(t, x) \leq C_\delta |h|^2,
\]

where \( C_\delta \) is a constant depending on \( \delta \).

Proof. As in the proof of the preceding lemma, choose a positive number \( \omega \) such that \( z_1(t, x) < z_2(t, x) \) for any \((t, x) \in \text{Int } D\) satisfying \( x_1 > \omega t \) and keep it fixed. Then for any positive \( \delta, z_1 \) is semi-concave in \( D_\delta \) (Lemma 1.2) and \( z_2 \) is semi-concave in \( D_\delta \cap D_{\omega, \delta} \) (Lemma 2.8) so that the lemma is proved by using Lemma 1.3.

The main results in this section is summarized as follows.
Theorem 3.1. Under Hypotheses (A), (B) on \( f \), let \( \varphi(x) \) and \( \psi(t, x') \) be Lipschitz-continuous on their respective half-spaces \( E^+_n \) of definition, \( \varphi(0, x') = \psi(0, x') \) for all \( x' \in E_{n-1} \), and satisfy Conditions (I), (II). Then the function \( z \) defined by

\[
z(t, x) = \min\{z_1(t, x), z_2(t, x)\},
\]

where \( z_1 \) and \( z_2 \) are defined by (1.4) and (2.3) respectively, is a generalized solution to the mixed initial-boundary value problem (1)–(3) in the quarter-space \( D \) and satisfies a semi-concavity condition in the sense of Lemma 3.2.

References


(Ricevita la 16-an de septembro, 1969)