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On Riemann's Equations which are Solvable by Quadratures

By Tosihusa KIMURA

(University of Tokyo)

Dedicated to Professor Tokui Satō on the Occasion of His Retirement

§1. Introduction and notation.

In this paper we are concerned with Riemann's equations whose solutions are expressible in terms of elementary functions and their integrals. By definition, the Riemann's equation is a homogeneous linear ordinary differential equation of the second order with at most three singularities which are of the regular type. If a Riemann's equation has regular singular points at a, b and c, then the equation may be written in the form

$$\text{(E)}\qquad \frac{d^2y}{dx^2}+\left(\frac{1-\rho-\rho'}{x-a}+\frac{1-\sigma-\sigma'}{x-b}+\frac{1-\tau-\tau'}{x-c}\right)\frac{dy}{dx}$$
$$+\left\{\frac{\rho\rho'(a-b)(a-c)}{x-a}+\frac{\sigma\sigma'(b-c)(b-a)}{x-b}+\frac{\tau\tau'(c-a)(c-b)}{x-c}\right\}$$
$$\times\frac{y}{(x-a)(x-b)(x-c)}=0.$$

Here $\rho, \rho'; \sigma, \sigma'$ and τ, τ' are the exponents of (E) respectively belonging to a, b and c, and satisfy Fuchs's relation

$$\rho+\rho'+\sigma+\sigma'+\tau+\tau'=1.$$

It is permitted that one of a, b and c is at infinity and then the equation is obtained from (E) by an appropriate limiting process. The complete set of solutions of (E) is denoted by the symbol

$$\text{(F)}\qquad y=P\left\{\begin{matrix} a & b & c & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{matrix}\right\}$$

and is called Riemann's P-function.

The hypergeometric differential equation

$$x(1-x)y''+\{\gamma-(\alpha+\beta+1)x\}y'-\alpha\beta y=0$$

is a special case of Riemann's equation and the set of solutions is given by

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\}.$$

In his famous paper, H. A. Schwarz determined the class of hypergeometric differential equations which have algebraic solutions only. His result will immediately lead us to the determination of the class of Riemann's equations all solutions of which are algebraic. We shall call this class of Riemann's equations the *class of Schwarz* [Cf. Poole [4]]. On the other hand, in 1950 and 1956, M. Hukuhara and S. Ōhasi obtained a class of Riemann's equations whose solutions are all expressible by the use of elementary functions and their integrals. Although there are a lot of overlaps between those two classes, neither of them includes the other.

The object of the present paper is to show that if solutions of a Riemann's equation can be expressed in terms of elementary functions and their integrals, then the equation belongs to the class of either Schwarz or Hukuhara-Ōhasi. In order to state this fact in a more rigorous form, we have to give the phrase "expressible in terms of elementary functions and their integrals" a precise definition. For this purpose we shall use the Picard-Vessiot theory for linear differential equations, which was developed by E. R. Kolchin in an elegant and abstract form.

Throughout this paper, we shall use the following notation. Let $\boldsymbol{C}$ denote the set of all complex numbers and $\boldsymbol{C}(x)$ the set of all rational functions of x over $\boldsymbol{C}$. The extended complex plane (Riemann's sphere) is denoted by $\bar{\boldsymbol{C}}$. We define X to be $\bar{\boldsymbol{C}}-\{a, b, c\}$. The quantities $\rho'-\rho, \sigma'-\sigma$ and $\tau'-\tau$ are called the *exponent differences* of (E) at a, b and c respectively and are denoted by λ, μ and ν:

$$\lambda=\rho'-\rho, \quad \mu=\sigma'-\sigma, \quad \nu=\tau'-\tau.$$

§ 2. Transformations.

In this section we list several well-known transformation formulas for Riemann's P-functions.

If $x'=\dfrac{\alpha x+\beta}{\gamma x+\delta}$ is a homographic substitution which maps $x=a, b, c$ to $x'=a', b', c'$ respectively, we have

$$(2.1) \qquad P\left\{\begin{matrix} a & b & c & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{matrix}\right\}=P\left\{\begin{matrix} a' & b' & c' & \\ \rho & \sigma & \tau & x' \\ \rho' & \sigma' & \tau' & \end{matrix}\right\}.$$

If some of the exponents take special values, then there exist substitutions of higher degree which map one P-function into another. We state the following two basic substitutions.

$$(2.2)\qquad P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & \sigma & \tau & x \\ \frac{1}{2} & \sigma' & \tau' & \end{Bmatrix} = P\begin{Bmatrix} -1 & 1 & \infty & \\ \sigma & \sigma & 2\tau & \sqrt{x} \\ \sigma' & \sigma' & 2\tau' & \end{Bmatrix}$$

$$(2.3)\qquad P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & \sigma & 0 & x \\ \frac{1}{3} & \sigma' & \frac{1}{3} & \end{Bmatrix} = P\begin{Bmatrix} 1 & \omega & \omega^2 & \\ \sigma & \sigma & \sigma & \sqrt[3]{x} \\ \sigma' & \sigma' & \sigma' & \end{Bmatrix} \qquad \left(\omega=\frac{-1+\sqrt{-3}}{2}\right)$$

Among linear transformations of the form

$$z=p(x)y+q(x)y'$$

which transform Riemann's equations into other Riemann's equations without changing the position of the singularities, the following are fundamental.

$$(2.4)\qquad \left(\frac{x-a}{x-c}\right)^r\left(\frac{x-b}{x-c}\right)^s P\begin{Bmatrix} a & b & c & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \rho+r & \sigma+s & \tau-r-s & x \\ \rho'+r & \sigma'+s & \tau'-r-s & \end{Bmatrix}$$

(If $c=\infty$, then $x-c$ must be replaced by unity).

$$(2.5)\qquad \frac{d}{dx}P\begin{Bmatrix} a & b & \infty & \\ \rho & \sigma & \tau & x \\ 0 & \sigma' & \tau' & \end{Bmatrix} = P\begin{Bmatrix} a & b & \infty & \\ \rho-1 & \sigma-1 & \tau+1 & x \\ 1 & \sigma'-1 & \tau'+1 & \end{Bmatrix} \qquad (\sigma\sigma'=\tau\tau'\neq 0)$$

$$(2.6)\qquad \frac{d}{dx}P\begin{Bmatrix} a & b & \infty & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & 0 & \end{Bmatrix} = P\begin{Bmatrix} a & b & \infty & \\ \rho-1 & \sigma-1 & \tau+1 & x \\ \rho'-1 & \sigma'-1 & 3 & \end{Bmatrix} \qquad (\rho\rho'=\sigma\sigma'\neq 0)$$

$$(2.7)\qquad \frac{d}{dx}P\begin{Bmatrix} a & b & \infty & \\ \rho & \sigma & \tau & x \\ 0 & 0 & \tau' & \end{Bmatrix} = P\begin{Bmatrix} a & b & \infty & \\ \rho-1 & \sigma-1 & \tau+1 & x \\ 0 & 0 & \tau'+1 & \end{Bmatrix} \qquad (\tau\tau'\neq 0)$$

$$(2.8)\qquad \frac{d}{dx}P\begin{Bmatrix} a & b & \infty & \\ \rho & \sigma & \tau & x \\ 0 & \sigma' & 0 & \end{Bmatrix} = P\begin{Bmatrix} a & b & \infty & \\ \rho-1 & \sigma-1 & \tau+1 & x \\ 0 & \sigma'-1 & 2 & \end{Bmatrix} \qquad (\sigma\sigma'\neq 0)$$

§ 3. Reducibility.

A non-trivial solution of a Riemann's equation is called *degenerate* if it satisfies a differential equation of the form

$$y'+r(x)y=0,$$

where $r(x)$ is a rational function: $r(x) \in \boldsymbol{C}(x)$. A Riemann's equation is said to be *reducible* if it admits a degenerate solution, and otherwise, it is said to be *irreducible*. Reducibility of a Riemann's equation implies solvability of the equation by quadratures.

It is well-known that Equation (E) is reducible if and only if at least one of the eight sums of exponents $\rho+\sigma+\tau$, $\rho'+\sigma+\tau$, $\rho+\sigma'+\tau$, $\rho+\sigma+\tau'$, $\rho'+\sigma'+\tau$, $\rho'+\sigma+\tau'$, $\rho+\sigma'+\tau'$, $\rho'+\sigma'+\tau'$ is an integer. The results can be restated as follows: Equation (E) is reducible if and only if at least one of the eight constants $\pm\lambda\pm\mu\pm\nu$ is an odd integer. It follows from Fuchs's relation that it suffices to examine the sums $\rho+\sigma+\tau$, $\rho'+\sigma+\tau$, $\rho+\sigma'+\tau$, $\rho+\sigma+\tau'$ or the constants $\lambda+\mu+\nu$, $-\lambda+\mu+\nu$, $\lambda-\mu+\nu$, $\lambda+\mu-\nu$ in order to see whether or not (E) is reducible. We note that if at least three of $\lambda+\mu+\nu$, $-\lambda+\mu+\nu$, $\lambda-\mu+\nu$, $\lambda+\mu-\nu$ are integers, then all of them are odd integers.

If $\rho+\sigma+\tau$ is equal to a non-positive integer $-n$ and if none of $\rho'+\sigma+\tau$, $\rho+\sigma'+\tau$, $\rho+\sigma+\tau'$ is equal to $0, -1, \cdots, -n+1$, then we have a degenerate solution of the form

$$(x-a)^{\rho}(x-b)^{\sigma}(x-c)^{\tau}p_n(x),$$

where $p_n(x)$ is a polynomial of degree n. (If $c=\infty$, then $x-c$ is replaced by unity).

§ 4. Monodromy group.

Let x_0 be an arbitrary but fixed point in $X=\overline{\boldsymbol{C}}-\{a,b,c\}$ and let φ, ψ be two linearly independent branches of solutions of (E) defined in the neighborhood of x_0. If we continue φ, ψ analytically along any closed path in X, beginning and ending at x_0, we obtain new branches $\hat{\varphi}, \hat{\psi}$ of solutions which are connected by a linear transformation:

$$(\hat{\varphi}, \hat{\psi})=(\varphi, \psi)\Gamma$$

where $\Gamma \in \mathrm{GL}(2, \boldsymbol{C})$. We take three simple and positive loops at x_0, $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, the first enclosing the singularity a only, the second b only and the third c only in such a way that the composite loop of them in this order is null-homotopic. Then the analytic continuation of φ, ψ along $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ gives rise to linear transformations acting on (φ, ψ) in the way indicated by the arrows:

$$(\varphi, \psi) \rightarrow (\varphi, \psi)A,$$
$$(\varphi, \psi) \rightarrow (\varphi, \psi)B,$$
$$(\varphi, \psi) \rightarrow (\varphi, \psi)C,$$

where $A, B, C \in \mathrm{GL}(2, \boldsymbol{C})$. By the hypothesis on $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ we have the relation

$$CBA=I,$$

where I denotes the identity matrix in $\mathrm{GL}(2, \boldsymbol{C})$. The subgroup M of $\mathrm{GL}(2, \boldsymbol{C})$ generated by A, B and C is, by definition, the monodromy group of (E) with respect to (φ, ψ). It is easy to see that any two monodromy groups of (E) are equivalent to each other.

If (E) is reducible, then we can choose (φ, ψ) so that two of A, B and C, say A and B, take either the form

$$A=\begin{bmatrix} e^{2\pi i\rho} & \alpha \\ 0 & e^{2\pi i\rho'} \end{bmatrix}, \quad B=\begin{bmatrix} e^{2\pi i\sigma} & \beta \\ 0 & e^{2\pi i\sigma'} \end{bmatrix}$$

or the form

$$A=\begin{bmatrix} e^{2\pi i\rho'} & \alpha \\ 0 & e^{2\pi i\rho} \end{bmatrix}, \quad B=\begin{bmatrix} e^{2\pi i\sigma} & \beta \\ 0 & e^{2\pi i\sigma'} \end{bmatrix},$$

where $\alpha\beta=0$. The existence of such a system (φ, ψ) shows that reducibility of a Riemann's equation implies reducibility of monodromy groups of the equation to triangular form. The converse also is true, that is, if a monodromy group of a Riemann's equation is reducible to triangular form, then the equation is reducible. The case that $\alpha=\beta=0$ occurs if and only if exactly two or four of $\lambda+\mu+\nu$, $-\lambda+\mu+\nu$, $\lambda-\mu+\nu$, $\lambda+\mu-\nu$ are odd integers and none of the singularities is logarithmic

If (E) is irreducible, then we can choose (φ, ψ) so that two of A, B and C, say A and B, take the form:

$$A=\begin{bmatrix} e^{2\pi i\rho} & \alpha \\ 0 & e^{2\pi i\rho'} \end{bmatrix}, \quad B=\begin{bmatrix} e^{2\pi i\sigma} & 0 \\ \beta & e^{2\pi i\sigma'} \end{bmatrix},$$

where $\alpha\beta\neq 0$. In this case, every associated P-function

$$\text{(F*)} \qquad y^*=P\begin{Bmatrix} a & b & c & \\ \rho+r & \sigma+s & \tau+t & x \\ \rho'+r' & \sigma'+s' & \tau'+t' & \end{Bmatrix},$$

where r, r', s, s', t, t' are integers such that $r+r'+s+s'+t+t'=0$, contains two linearly independent branches φ^*, ψ^* of solutions such that the effect upon (φ^*, ψ^*) of the analytic continuation along $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ is the same as upon (φ, ψ). Moreover (F) and (F*) are connected by linear transformations

$$y^*=p(x)y+q(x)y', \quad y=p^*(x)y^*+q^*(x)y^{*\prime},$$

where $p(x), q(x), p^*(x), q^*(x) \in \boldsymbol{C}(x)$. We note that if we set

$$\lambda^*=\rho+r-\rho'-r', \quad \mu^*=\sigma+s-\sigma'-s', \quad \nu^*=\tau+t-\tau'-t',$$

then we have

$$\pm\lambda^*\pm\mu^*\pm\nu^*\equiv\pm\lambda\pm\mu\pm\nu \pmod 2,$$

where the same combination of signs should be taken on both sides.

§ 5. Schwarz's table.

The following theorem is an immediate consequence of Schwarz's result for hypergeometric differential equations.

THEOREM. *In order that* (E) *has algebraic solutions only, it is necessary and sufficient that all the exponents of* (E) *are rational numbers and that either*

(A) *Exactly two or four of* $\lambda+\mu+\nu$, $-\lambda+\mu+\nu$, $\lambda-\mu+\nu$, $\lambda+\mu-\nu$ *are odd integers and none of the singularities* a, b *and* c *is logarithmic, or*

(B) *The quantities* λ *or* $-\lambda$, μ *or* $-\mu$ *and* ν *or* $-\nu$ *take, in an arbitrary order, values given in the following table of Schwarz.*

1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	rational number	
2	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	
3	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
4	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	
5	$\frac{2}{3}+l$	$\frac{1}{4}+m$	$\frac{1}{4}+n$	$l+m+n=$even
6	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	
7	$\frac{2}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
8	$\frac{2}{3}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	〃
9	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{5}+n$	〃
10	$\frac{3}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	〃
11	$\frac{2}{5}+l$	$\frac{2}{5}+m$	$\frac{2}{5}+n$	〃
12	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	〃
13	$\frac{4}{5}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	〃
14	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	〃

15	$\frac{3}{5}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n=$even

Here l, m, n *stand for integrers.*

§ 6. Hukuhara-Ōhasi table.

The result of Hukuhara-Ōhasi is stated as follows.

THEOREM. *Equation* (E) *can be transformed into a reducible Riemann's equation by the use of formulas* (2.1)~(2.8), *if either*

(A) *At least one of* $\lambda+\mu+\nu$, $-\lambda+\mu+\nu$, $\lambda-\mu+\nu$ *and* $\lambda+\mu-\nu$ *is an odd integer, or*

(B) *The quantities* λ *or* $-\lambda$, μ *or* $-\mu$ *and* ν *or* $-\nu$ *take, in an arbitrary order, values given in the following table of Hukuhara-Ōhasi:*

1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	complex number	
2	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	
3	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$even
4	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	
5	$\frac{2}{3}+l$	$\frac{1}{4}+m$	$\frac{1}{4}+n$	$l+m+n=$even

Here l, m, n *stand for integers.*

In case (A), Equation (E) itself is reducible. In case (B), explicit expressions of solutions are given by Hukuhara and Ōhasi for special values of l, m, n as follows.

Case 1. $\lambda=\mu=\frac{1}{2}$, $l=m=0$.

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & -\frac{\nu}{2} & x \\ \frac{1}{2} & \frac{1}{2} & \frac{\nu}{2} & \end{Bmatrix} \text{ contains } (\sqrt{x}\pm\sqrt{x-1})^{\nu}.$$

Case 2. $\lambda=\frac{1}{3}$, $\mu=\frac{1}{2}$, $\nu=\frac{1}{3}$, $l=m=n=0$.

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & -\frac{1}{12} & 0 & x \\ \frac{1}{3} & \frac{5}{12} & \frac{1}{3} & \end{Bmatrix} \text{ contains } (x-1)^{-1/12}\{\sqrt{3}(x^{1/3}+1)\pm2\sqrt{x^{2/3}+x^{1/3}+1}\}^{1/4}.$$

Case 3. $\lambda=\frac{1}{3}$, $\mu=\frac{1}{3}$, $\nu=\frac{2}{3}$, $l=m=n=0$.

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & -\frac{1}{6} & x \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \end{matrix}\right\} \text{ contains } \left[\sqrt{3}\{4^{1/3}x^{1/3}(1-x)^{2/3}+1\} \pm 2\sqrt{4^{2/3}x^{2/3}(1-x)^{2/3}+4^{1/3}x^{1/3}(1-x)^{1/3}+1}\right]^{1/4}.$$

Case 4. $\lambda=\frac{1}{2}$, $\mu=\frac{1}{3}$, $\nu=\frac{1}{4}$, $l=m=n=0$.

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & -\frac{1}{24} & 0 & x \\ \frac{1}{2} & \frac{7}{24} & \frac{1}{4} & \end{matrix}\right\} \text{ contains } (x-1)^{-1/24}\left[\sqrt{3}\{(\sqrt{x}-1)^{1/3}+(\sqrt{x}+1)^{1/3}\}^{1/3} \pm 2\sqrt{(\sqrt{x}-1)^{2/3}+(x-1)^{1/3}+(\sqrt{x}+1)^{2/3}}\right]^{1/4}.$$

Case 5. $\lambda=\frac{1}{4}$, $\mu=\frac{1}{4}$, $\nu=\frac{2}{3}$, $l=m=n=0$.

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & -\frac{1}{24} & x \\ \frac{1}{4} & \frac{1}{4} & \frac{7}{24} & \end{matrix}\right\} \text{ contains } \left[\sqrt{3}\{(\sqrt{x}-\sqrt{x-1})^{2/3}+(\sqrt{x}+\sqrt{x-1})^{2/3}\} \pm 2\sqrt{(\sqrt{x}-\sqrt{x-1})^{4/3}+1+(\sqrt{x}+\sqrt{x-1})^{4/3}}\right]^{1/4}.$$

§ 7. Liouville extension and generalized Liouville extension.

In this section we shall state basic notions in the Picard-Vessiot theory for linear ordinary differential equations.

For our purpose it suffices to start with $K=\boldsymbol{C}(x)$, the set of all rational functions over $\boldsymbol{C}$. K is not only a field but also a differential field with field of constants $\boldsymbol{C}$. We denote by $K<\xi, \eta, \cdots>$ the differential field extension obtained from K by the differential adjunctions of the functions $\xi, \eta, \cdots$. If (φ, ψ) is a fundamental system of solutions of Equation (E), then $L=K<\varphi, \psi>$ is the smallest differential field extension which contains all solutions of (E), and hence L is independent of the choice of a fundamental system of solutions. It is clear that the constant field of L is the same as that of K, namely $\boldsymbol{C}$. Therefore L is, by definition, a *Picard-Vessiot extension* of K.

The differential Galois group of L over K is defined to be the group of all differential automorphisms of L leaving K elementwise fixed. Given a fundamental system of solutions (φ, ψ), we obtain a subgroup G of $\mathrm{GL}(2, \boldsymbol{C})$ by assigning to each automorphism γ in the differential Galois group a matrix Γ satisfying

$$(\gamma\varphi, \gamma\psi)=(\varphi, \psi)\Gamma.$$

We call the subgroup G the *differential Galois group of L over K with respect to* (φ, ψ).

It is well-known that G is a linear algebraic group and that the monodromy group of (E) with respect to (φ, ψ) is a dense subgroup of G in the sense of Zariski topology.

The differential field L is said to be a *Liouville extension* of K if there exists a chain of intermediate differential fields:

$$K=K_0\subset K_1\subset\cdots\subset K_n=L$$

such that each K_i is an extension of K_{i-1} by either an integral or an exponential of an integral, i.e. $K_i=K_{i-1}\langle u_i\rangle$ with either $u_i'\in K_{i-1}$ or $u_i'/u_i\in K_{i-1}$. In order for L to be a Liouville extension of K, it is necessary and sufficient that the differential Galois group of L over K is solvable.

We say that L is a *generalized Liouville extension* of K if L can be obtained from K by a number of steps, each of which is either a finite algebraic extension or the adjunction of an integral or the adjunction of an exponential of an integral. It is well-known that L is a generalized Liouville extension of K if and only if the differential Galois group of L over K with respect to a system (φ, ψ) has the solvable connected component of the identity.

We see by the expressions of the solutions given by Hukuhara-Ōhasi that L is a Liouville extension of K if $\pm\lambda$, $\pm\mu$ and $\pm\nu$ take values given in their table.

§ 8. Statement of our theorems.

We are now in a position to state our theorems.

THEOREM I. *Let L be the Picard-Vessiot extension of K for Riemann's equation* (E). *In order for L to be a generalized Liouville extension of K, it is necessary and sufficient that either*

(A) *At least one of $\lambda+\mu+\nu$, $-\lambda+\mu+\nu$, $\lambda-\mu+\nu$ and $\lambda+\mu-\nu$ is an odd integer, or*

(B) *The quantities λ or $-\lambda$, μ or $-\mu$ and ν or $-\nu$ take values given in the union of the tables of Schwarz and of Hukuhara-Ōhasi*:

1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	arbitrary complex number
2 ≀ 15	the same as in the table of Schwarz		

THEOREM II. *A necessary and sufficient condition that the Picard-Vessiot* *extension L of K is a Liouville extension is that the quantities* λ, μ *and* ν *satisfy the condition stated in the theorem of Hukuhara-Ōhasi.*

To prove the theorems above, we shall use the following proposition (see [2]):

Let G be a linear algebraic group of 2×2 *matrices with determinant* 1, *over an algebraically closed field. Suppose that the component of the identity* G_0 *in G is solvable. Then either*

(i) *G can be put in simultaneous triangular form, or*

(ii) G_0 *can be put in diagonal form and* $[G:G_0]=2$, *or*

(iii) *G is finite.*

§ 9. Proof of Theorem I.

The sufficiency is an immediate consequence of the results of Schwarz and Hukuhara-Ōhasi. Therefore we have only to prove the necessity.

Suppose that L is a generalized Liouville extension of K. We consider instead of (F) the following P-function

$$(\hat{\mathrm{F}})\qquad \hat{y}=P\begin{Bmatrix} a & b & c & \\ \frac{1-\lambda}{2} & \frac{1-\mu}{2} & \frac{-1-\nu}{2} & x \\ \frac{1+\lambda}{2} & \frac{1+\mu}{2} & \frac{-1+\nu}{2} & \end{Bmatrix}$$

which has the same exponent differences as (F). The equation for $(\hat{\mathrm{F}})$ is written in the form

$$(\hat{\mathrm{E}})\qquad \frac{d^2\hat{y}}{dx^2}+\frac{2}{x-c}\frac{d\hat{y}}{dx}+\frac{1}{4}\left\{(1-\lambda^2)\frac{(a-b)(a-c)}{x-a}+\cdots\right\}\frac{\hat{y}}{(x-a)(x-b)(x-c)}=0.$$

We denote by $\hat{L}$ the Picard-Vessiot extension of K for $(\hat{\mathrm{E}})$. Let $(\hat{\varphi},\hat{\psi})$ be a fundamental system of solutions of $(\hat{\mathrm{E}})$ and we denote by $\hat{G}$ and $\hat{M}$ the differential Galois group and the monodromy group of $(\hat{\mathrm{E}})$ with respect to $(\hat{\varphi},\hat{\psi})$.

By (2.4) we have

$$P\begin{Bmatrix} a & b & c & \\ \frac{1-\lambda}{2} & \frac{1-\mu}{2} & \frac{-1-\nu}{2} & x \\ \frac{1+\lambda}{2} & \frac{1+\mu}{2} & \frac{-1+\nu}{2} & \end{Bmatrix}=\phi(x)P\begin{Bmatrix} a & b & c & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{Bmatrix},$$

where

$$\phi(x)=\left(\frac{x-a}{x-c}\right)^{\frac{1-\rho-\rho'}{2}}\left(\frac{x-b}{x-c}\right)^{\frac{1-\sigma-\sigma'}{2}}.$$

It follows that

$$\hat{L} \subset L<\phi>.$$

Since $\phi'/\phi \in K, L<\phi>$ is a generalized Liouville extension of K, whence $\hat{L}$ is so. Consequently, the component of the identity $\hat{G}_0$ in $\hat{G}$ is solvable.

Next we shall prove that $\hat{G}$ is contained in $\mathrm{SL}(2, \boldsymbol{C})$. Consider the Wronskian $W(\hat{\varphi}, \hat{\psi})$ of $\hat{\varphi}, \hat{\psi}$. We have

$$W(\hat{\varphi}, \hat{\psi}) = \frac{\text{const.}}{(x-c)^2},$$

from which $W(\hat{\varphi}, \hat{\psi}) \in K$. Hence the Wronskian $W(\hat{\varphi}, \hat{\psi})$ is invariant under every differential automorphism $\hat{\gamma}$ of $\hat{L}$ over K:

$$\hat{\gamma} W(\hat{\varphi}, \hat{\psi}) = W(\hat{\varphi}, \hat{\psi}). \tag{9.1}$$

If $\hat{\Gamma}$ is the element in $\hat{G}$ corresponding to $\hat{\gamma}$, we have

$$(\hat{\gamma}\hat{\varphi}, \hat{\gamma}\hat{\psi}) = (\hat{\varphi}, \hat{\psi})\hat{\Gamma}, \quad (\hat{\gamma}\hat{\varphi}', \hat{\gamma}\hat{\psi}') = (\hat{\varphi}', \hat{\psi}')\hat{\Gamma}.$$

It follows that

$$\hat{\gamma} W(\hat{\varphi}, \hat{\psi}) = W(\hat{\varphi}, \hat{\psi}) \det \hat{\Gamma}. \tag{9.2}$$

Combining (9.1) and (9.2), we obtain

$$\det \hat{\Gamma} = 1,$$

which proves that $\hat{G} \subset \mathrm{SL}(2, \boldsymbol{C})$.

By the proposition stated in § 8, the following possibilities arise:

(i) $\hat{G}$ is reducible to triangular form,

(ii) $\hat{G}_0$ is reducible to diagonal form and $[\hat{G}:\hat{G}_0]=2$,

(iii) $\hat{G}$ is finite.

In case (i), Equation $(\hat{\mathrm{E}})$ itself is reducible and in case (iii), Equation $(\hat{\mathrm{E}})$ admits algebraic solutions only. Recalling that $(\hat{\mathrm{E}})$ has the exponent differences λ, μ and ν, we can conclude that in case (i) Condition (A) of THEOREM I is satisfied and in case (iii) the quantities λ or $-\lambda, \mu$ or $-\mu, \nu$ or $-\nu$ take values given in the Schwarz's table.

Next we consider case (ii). W may suppose without loss of generality that $\hat{G}$ is neither reducible to triangular form nor finite. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be the paths stated in § 4 and $\hat{A}, \hat{B}, \hat{C}$ the matrices in $\mathrm{GL}(2, \boldsymbol{C})$ which describe the effect upon $(\hat{\varphi}, \hat{\psi})$ of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$.

We shall show that $\hat{G}_0$ cannot contain two of $\hat{A}, \hat{B}$ and $\hat{C}$. In fact, it is obvious that at least one of $\hat{A}, \hat{B}, \hat{C}$ does not belong to $\hat{G}_0$, because of $[\hat{G}:\hat{G}_0]=2$. If, for example, $\hat{A}, \hat{B}$ belong to $\hat{G}_0$, then the relation $\hat{C}\hat{B}\hat{A}=I$ implies that $\hat{C}$ necessarily belongs to $\hat{G}_0$, which is a contradiction.

By the symmetry for $\hat{A}, \hat{B}, \hat{C}$ it suffices to examine the case when $\hat{A}\notin\hat{G}_0$ and $\hat{B}\notin\hat{G}_0$. We can suppose that the fundamental system $(\hat{\varphi}, \hat{\psi})$ has been chosen so that $\hat{A}$ and $\hat{B}$ take the form:

$$\hat{A}=\begin{bmatrix} -e^{-\pi i\lambda} & \hat{\alpha} \\ 0 & -e^{\pi i\lambda} \end{bmatrix}, \quad \hat{B}=\begin{bmatrix} -e^{-\pi i\mu} & 0 \\ \hat{\beta} & -e^{\pi i\mu} \end{bmatrix}.$$

Since $[\hat{G}:\hat{G}_0]=2$, both $\hat{A}^2$ and $\hat{B}\hat{A}$ belong to $\hat{G}_0$. Therefore the matrices

$$\hat{A}^2=\begin{bmatrix} e^{-2\pi i\lambda} & -\hat{\alpha}(e^{-\pi i\lambda}+e^{\pi i\lambda}) \\ 0 & e^{2\pi i\lambda} \end{bmatrix},$$

$$\hat{B}\hat{A}=\begin{bmatrix} e^{-\pi i(\lambda+\mu)} & -\hat{\alpha}e^{-\pi i\mu} \\ -\hat{\beta}e^{-\pi i\lambda} & e^{\pi i(\lambda+\mu)}+\hat{\alpha}\hat{\beta} \end{bmatrix}$$

can be simultaneously put into diagonal form. If $e^{-2\pi i\lambda}=e^{2\pi i\lambda}$, then $e^{-\pi i\lambda}+e^{\pi i\lambda}=0$. This occurs precisely when $\pm\lambda=1/2+l$, where l is an integer.

Next suppose that $e^{-2\pi i\lambda}\neq e^{2\pi i\lambda}$. Then $\hat{A}^2$ can be transformed by matrices

$$\hat{P}=\begin{bmatrix} p & \hat{\alpha}\dfrac{e^{-\pi i\lambda}+e^{\pi i\lambda}}{e^{-2\pi i\lambda}-e^{2\pi i\lambda}}q \\ 0 & q \end{bmatrix} \text{ and } \hat{Q}=\begin{bmatrix} \hat{\alpha}\dfrac{e^{-\pi i\lambda}+e^{\pi i\lambda}}{e^{-2\pi i\lambda}-e^{2\pi i\lambda}}q & q \\ p & 0 \end{bmatrix}$$

into diagonal matrices:

$$\hat{P}^{-1}\hat{A}^2\hat{P}=\begin{bmatrix} e^{-2\pi i\lambda} & 0 \\ 0 & e^{2\pi i\lambda} \end{bmatrix}, \quad \hat{Q}^{-1}\hat{A}^2\hat{Q}=\begin{bmatrix} e^{2\pi i\lambda} & 0 \\ 0 & e^{-2\pi i\lambda} \end{bmatrix},$$

where p and q are arbitrary complex numbers $\neq 0$. The matrix $\hat{B}\hat{A}$ is transformed by $\hat{P}$ and $\hat{Q}$ as follows:

$$\hat{P}^{-1}\hat{B}\hat{A}\hat{P}=\begin{bmatrix} * & * \\ -pq^{-1}\hat{\beta}e^{-\pi i\lambda} & * \end{bmatrix} \text{ and } \hat{Q}^{-1}\hat{B}\hat{A}\hat{Q}\begin{bmatrix} * & -pq^{-1}\hat{\beta}e^{-\pi i\lambda} \\ * & * \end{bmatrix}.$$

In order that either $\hat{P}^{-1}\hat{B}\hat{A}\hat{P}$ or $\hat{Q}^{-1}\hat{B}\hat{A}\hat{Q}$ can be a diagonal matrix, it is necessary that

$$\hat{\beta}=0,$$

which contradicts $\hat{\alpha}\hat{\beta}\neq 0$. Therefore we obtain

$$\pm\lambda=\frac{1}{2}+l \quad (l\text{: integer}).$$

By considering $\hat{A}\hat{B}$ and $\hat{B}^2$, we shall conclude that

$$\pm\mu=\frac{1}{2}+m \quad (m\text{: integer}).$$

§ 10. Proof of Theorem II.

The sufficiency is obvious. We shall prove the necessity. For this purpose it is sufficient to show that in the cases 6~15 of Schwarz's table the differential Galois group of L over K is not solvable. Let M, G be the monodromy group and the differential Galois group of (E) with respect to a fundamental system of solutions (φ, ψ). Consider the mapping which assigns to every matrix $\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}$ in M the homographic substitution $\frac{\alpha z+\beta}{\gamma z+\delta}$. It is clear that this mapping is a homomorphism from M into the group of homographic substitutions and its kernel M_0 consists of matrices of the form εI, $\varepsilon \neq 0$. Schwarz showed that in the only cases 6~15 the quotient group M/M_0 is isomorphic to the icosahedral group.

It follows from the fact that G is the smallest algebraic group containing M that the component of the identity G_0 in G is either M_0 or the group $\{\varepsilon I;\ \varepsilon\in C, \varepsilon\neq 0\}$ and that

$$G/G_0 \cong M/M_0 .$$

Since the icosahedral group is not solvable, G is not solvable. This completes the proof.

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