Funkcialaj Ekvacioj, 12 (1969), 205-220

On the Dirichlet Problem for Second Order Elliptic Differential Equations

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Dedicated to Professor Tokui Satō on the Occasion of His Retirement

Introduction.

In the notes [1, 2] the author has discussed the identity of the modern and the classical solutions of the Dirichlet problem with vanishing boundary value

$$\begin{cases} Lu=f & \text{in } G, \\ u=0 & \text{on } \partial G,^{1)} \end{cases}$$

where L is a linear elliptic differential operator of the form

$$Lu=-\partial_i(a_{ij}(x)\partial_j u)+b_k(x)\partial_k u+c(x)u^{2)}$$

and where G is a bounded subdomain of the Euclidean n-space $E_n (n\geqq 2)$ which is regular for the operator $L+\lambda_0$ for some constant λ_0.

The purpose of this note is to extend the results of [1, 2] to the case where the boundary value is not zero. This can be done in quite an elementary manner as seen in the sequel. Moreover, for another topic, we shall remark in the "Appendix" placed before the "References" that our method yields a new result about the regularity of (classical) solutions of second order elliptic differential equations near the boundary.

Finally, the author wishes to express his sincere gratitude to his former teacher Prof. Masuo Hukuhara for his kind stimulation.

1. Fundamental assumptions.

In the sequel we shall discuss our problem under the following assumptions.

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†) This paper was written in February, 1967, but was not published. The author is very glad to have an opportunity of dedicating it to Prof. Tokui Satō who helped him to get started on his mathematical survey about twelve years ago.

*) This work was partly supported by the Sakkokai Foundation.

1) We denote by ∂G the boundary of G.

2) Here ∂_i denote the partial differentiations $\partial/\partial x_i$ $(i=1, 2, \cdots, n)$. Throughout this note we use the summation convention for doubly appearing indices so that $\partial_i(a_{ij}(x)\partial_j u)$ means $\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$, etc.

Assumptions on the operator L and the domain G.

I. G is a bounded subdomain of the n-dimensional Euclidean space $E_n (n \geqq 2)$.

II. L is a differential operator of the form

$$Lu = -\partial_i(a_{ij}(x)\partial_j u) + b_k(x)\partial_k u + c(x)u, \tag{1}$$

where the coefficients a_{ij}, b_k, and c satisfy the following two conditions:

a) $a_{ij}, \partial_k a_{ij}, b_k$ $(i, j, k = 1, \cdots, n)$, and c are bounded and (locally) Hölder-continuous in G.

b) There exists a constant $A > 0$ such that

$$a_{ij}(x)\xi_i\xi_j \geqq A\xi_i\xi_i \tag{2}$$

for any $x \in G$ and for any n-tuple $\xi = (\xi_1, \cdots, \xi_n)$.

III. G is regular for the operator $L + \lambda_0$ with $\lambda_0 \equiv \sup_{x \in G} |c(x)|$, i.e. a strong barrier function for the operator $L + \lambda_0$ exists at each boundary point of G.

Remark. A strong barrier function ω for $L + \lambda_0$ at the point $s \in \partial G$ is defined by the following properties:

i) There exists an open neighborhood U of s such that $\omega \in C^2(U \cap G) \cap C(\overline{U \cap G})$.

ii) $(L + \lambda_0)\omega \geqq 1$ in $U \cap G$.

iii) $\omega(s) = 0$ and $\omega(x) > 0$ for all $x \in U \cap G, x \neq s$.

2. Function spaces W_1, H_1 and $\overset{\circ}{H}_1$.

To formulate our problem it is necessary to introduce a few function spaces.

Space $W_1(G)$: The space $W_1(G)$ is the set of all real-valued functions in $L^2(G)$ each member of which has (distributional) derivatives $\partial_i f \in L^2(G)$ $(i = 1, \cdots, n)$. The *norm* $\|f\|_{W_1(G)}$ and the *inner product* $(f, g)_{W_1(G)}$ are defined respectively by

$$\|f\|_{W_1(G)} \equiv \left\{ \|f\|^2_{L^2(G)} + \sum_{i=1}^{n} \|\partial_i f\|^2_{L^2(G)} \right\}^{1/2}, \tag{3}$$

and

$$(f, g)_{W_1(G)} = <f, g> + <\partial_i f, \partial_i g>, \tag{4}$$

where

$$<f, g> = \int_G f(x)g(x)dx \tag{5}$$

is the inner product of $L^2(G)$.

It is well known that $W_1(G)$ is a Hilbert space under the norm (3) and the inner product (4).

Space $H_1(G)$: $H_1(G)$ is the closure of $C^\infty(G)\cap W_1(G)$ in the space $W_1(G)$. $H_1(G)$ may be defined as the completion of $C^\infty(G)\cap W_1(G)$ under the norm (3).

Space $\overset{\circ}{H}_1(G)$: $\overset{\circ}{H}_1(G)$ is the closure of $C_0^\infty(G)$ in the space $W_1(G)$, or equivalently, $\overset{\circ}{H}_1(G)$ is the completion of $C_0^\infty(G)$ under the norm (3). Here $C_0^\infty(G)$ denotes the set of all functions in $C^\infty(G)$ with compact supports in G.

It is clear that

$$\overset{\circ}{H}_1(G)\subseteqq H_1(G)\subseteqq W_1(G). \tag{6}$$

3. Modern solutions and classical solutions.

Definition 1. A *classical solution* u of the Dirichlet problem

$$\begin{cases} Lu=f & \text{in } G, \\ u=\varphi & \text{on } \partial G, \end{cases} \tag{7}$$

is a function in $C^2(G)\cap C(\bar{G})$ which satisfies both the differential equation $Lu=f$ $(\in C(G))$ in G and the boundary condition $u=\varphi(\in C(\partial G))$ on ∂G.

Difinition 2. A *modern solution* u of the Dirichlet problem (7) is a function in $W_1(G)$ which is of the form $v+\varphi$ for some $v\in\overset{\circ}{H}_1(G)$ and which satisfies the functional equation

$$B[u,\psi]\equiv\langle a_{ij}\partial_j u,\partial_i\psi\rangle+\langle b_k\partial_k u,\psi\rangle+\langle cu,\psi\rangle=\langle f,\psi\rangle \tag{8}$$

for every $\psi\in C_0^\infty(G)$ (and hence for every $\psi\in\overset{\circ}{H}_1(G)$). Here we assume that

$$f\in L^2(G) \quad\text{and}\quad \varphi\in W_1(G). \tag{9}$$

Therefore, in proving the identity of these two types of solutions, we should require that

i) $\varphi\in W_1(G)\cap C(\bar{G})$, and

ii) f is bounded and (locally) Hölder-continuous in G. In later sections we shall discuss our problem under the restrictions i) and ii).

For later use we introduce the notion of generalized classical solutions.

Let

$$F[u]\equiv a_{ij}(x)\partial_i\partial_j u-g(x,u,\operatorname{grad} u)=0 \tag{10}$$

be a (semilinear) elliptic differential equation. We assume that

i) $\|a_{ij}\|$ is continuous in G, and

ii) g is defined and continuous in $\mathfrak{D}$:

$$x\in G,\ |u|<\infty,\ |p|=\{p_1^2+\cdots+p_n^2\}^{1/2}<\infty.$$

Difinition 3. A function u in $C^1(G)\cap C(\bar{G})$ is a *generalized classical solution* of the Dirichlet problem

$$(11)\qquad \begin{cases} F[u]=0 & \text{in } G, \\ u=\varphi & \text{on } \partial G, \end{cases}$$

where $\varphi\in C(\partial G)$, if $u(x)=\varphi(x)$ for all $x\in\partial G$ and if for each point x_0 of G there exist a neighborhood U of x_0 and a sequence of functions $\{u_k\}_{k=1}^{\infty}$ in $C^2(U)$ ($U\subset G$) such that uniformly in U

$$u_k\to u,\ \text{grad}\, u_k\to \text{grad}\, u, \text{ and } F[u_k]\to 0 \text{ as } k\to\infty.$$

Remark. This definition of generalized classical solutions would seem different from that given in K. Akô [1, 2] in the linear case $F[u]=Lu-f$. But M. Nagumo [12] has shown that our two definitions are the same in this case.

4. Lax-Milgram's theorem. Gårding's inequality.

Lemma 1 (*a special case of the Lax-Milgram theorem*). *Let $\boldsymbol{B}[u,v]$ be a bilinear form in a real Hilbert space $\boldsymbol{H}$ with norm $\|\cdot\|$, i.e. $\boldsymbol{B}[u,v]$ is a real-valued functional defined on $\boldsymbol{H}\times\boldsymbol{H}$ such that*

$$\boldsymbol{B}[\alpha_1u_1+\alpha_2u_2, v]=\alpha_1\boldsymbol{B}[u_1,v]+\alpha_2\boldsymbol{B}[u_2,v],$$
$$\boldsymbol{B}[u,\beta_1v_1+\beta_2v_2]=\beta_1\boldsymbol{B}[u,v_1]+\beta_2\boldsymbol{B}[u,v_2],$$

where the u's and v's are arbitrarily chosen elements of $\boldsymbol{H}$ and where the α's and β's are arbitrary real numbers. We assume further that there exist two positive constants γ and δ such that

$$(12)\qquad \begin{cases} |\boldsymbol{B}[u,v]|\leqq\gamma\|u\|\,\|v\| & \text{for any } u,v\in\boldsymbol{H}, \\ \boldsymbol{B}[u,u]\geqq\delta\|u\|^2 & \text{for any } u\in\boldsymbol{H}. \end{cases}$$

If $\boldsymbol{L}$ is a continuous linear functional on $\boldsymbol{H}$, then there exists one and only one element $u\in\boldsymbol{H}$ such that

$$\boldsymbol{B}[u,v]=\boldsymbol{L}[v]\quad \text{for all } v\in\boldsymbol{H}.$$

Moreover, the norm $\|u\|$ of this u is estimated by

$$(13)\qquad \|u\|\leqq\delta^{-1}\|\boldsymbol{L}\|.$$

Brief sketch of the proof. By virture of F. Riesz' representation theorem there exists one and only one element $w\in\boldsymbol{H}$ such that

$$\boldsymbol{L}[v]=(w,v)_{\boldsymbol{H}}\quad \text{for all } v\in\boldsymbol{H},$$

where $(w,v)_{\boldsymbol{H}}$ denotes the inner product of w and v. Thus we can apply the Lax-Milgram theorem to our case and we have the lemma. (As to the Lax-Milgram theorem we refer the reader to the book [15] of K. Yosida, p. 92, the book [11] of M. Nagumo, Chapter 3, and the famous paper of Lax and Milgram cited therein.)

Lemma 2 (*a special case of Gårding's inequality*). *Let $B[u, v]$ be the from defined by* (8). *Then the new form*

$$\mathbf{B}[u, v] \equiv B[u, v] + \lambda^* \langle u, v \rangle \quad \textit{for } u, v \in \overset{\circ}{H}_1(G), \tag{14}$$

satisfies the requirements in Lemma 1 *for some $\lambda^*(>\lambda_0 \geqq 0)$. The constants γ and δ are dependent only on the bounds of a_{ij}, b_k, c, the diameter of G, the the constant A in* (2), *and the constant λ^*.*

As to the proof see especially M. Nagumo [11], Chap. 3. See also K. Yosida [15], p. 175.

5. Mollifiers.

We define a function j in $C_0^\infty(E_n)$ by

$$j(x) \equiv \begin{cases} C \exp\{-(1-|x|^2)^{-1}\} & \text{for } |x|<1, \\ 0 \quad \text{for } |x| \geqq 1, \end{cases} \tag{15}$$

where the constant $C>0$ is chosen so that

$$\int_{E_n} j(x)dx = 1. \tag{16}$$

For each δ, $0<\delta<1$, we define an integral operator J_δ, called a *mollifier*, by

$$(J_\delta \psi)(x) = \int_G j_\delta(x-y)\psi(y)dy \tag{17}$$

for $\psi \in C(\bar{G}) \cap W_1(G)$, where

$$j_\delta(x) = \delta^{-n} j\left(\frac{x}{\delta}\right). \tag{18}$$

Then we readily observe that

$$J_\delta \psi \in C^\infty(G) \text{ and } \sup_{\operatorname{dist}(x,\partial G)>\delta} |(J_\delta\psi)(x) - \psi(x)| \leqq \sup_{x,y\in G,\, |x-y|\leqq\delta} |\psi(x)-\psi(y)|. \tag{19}$$

and that

$$\begin{cases} \partial_i(J_\delta\psi)(x) = J_\delta(\partial_i\psi)(x) & \text{for } x\in G,\ \operatorname{dist}(x,\partial G)>\delta, \\ \lim\limits_{\delta\to 0} \|J_\delta(J_i\psi) - \partial_i\psi\|_{L^2(K)} = 0 & \text{for every compact subset } K \text{ of } G, \end{cases} \tag{20}$$

since $j_\delta(x-y) \in C_0^\infty(G)$ as a function of y if $\operatorname{dist}(x, \partial G)>\delta$.

6. The case where $\varphi=0$.

In the note [2] we have proved the following result.

Lemma 3. *If f is bounded and (locally) Hölder-continuous in G and if φ*

vanishes, then we have the following result: Every classical solution of the Dirchlet problem (7) *is also a modern solution and, conversely, every modern solution of the problem* (7) *is also a classical solution.*

7. The main theorem.

We are now in a position to state our main theorem.

Theorem 1. *If f is bounded and (locally) Hölder-continuous in G and if φ is in $C(\bar{G})\cap W_1(G)$, then the following statement is valid: Every classical solution of the Dirichlet problem*

$$\left\{\begin{array}{l} Lu=f \quad in \quad G, \\ u=\varphi \quad on \quad \partial G, \end{array}\right. \tag{7}$$

is also a modern solution and, conversely, every modern solution of the problem (7) *is also a classical solution.*

8. Proof of the main theorem.

Let u be classical solution of (7) and let Φ be the (unique) solution in $C^2(G)\cap C(\bar{G})$ of the Dirichlet problem

$$\left\{\begin{array}{r} (L+\lambda^*)\Phi=0 \quad \text{in} \quad G, \\ \Phi=\varphi \quad \text{on} \quad \partial G, \end{array}\right. \tag{21}$$

where λ^* is the constant in Lemma 2.

We set

$$W=u-\Phi. \tag{22}$$

Then W is a solution in $C^2(G)\cap C(\bar{G})$ of the Dirichlet problem

$$\left\{\begin{array}{l} LW=\lambda^*\Phi+f \quad \text{in} \quad G, \\ W=0 \quad \text{on} \quad \partial G. \end{array}\right. \tag{23}$$

By virtue of Lemma 3 we see that $W\in\overset{\circ}{H}_1(G)$ and that

$$B[W,\psi]=<\lambda^*\Phi+f,\psi> \tag{24}$$

for all $\psi\in\overset{\circ}{H}_1(G)$.

Thus, in order to prove that u is a modern solution, it is sufficient to show that $\Phi-\varphi\in\overset{\circ}{H}_1(G)$. For, if $\Phi-\varphi\in\overset{\circ}{H}_1(G)$, then, by defining V by

$$V=W+(\Phi-\varphi) \tag{25}$$

we have

$$u=V+\varphi \ (=W+\Phi), \quad V\in\overset{\circ}{H}_1(G) \tag{26}$$

and we have for any $\psi \in \overset{\circ}{H}_1(G)$

$$\begin{aligned} B[u,\psi] &= B[W,\psi]+B[\Phi-\varphi,\psi]+B[\varphi,\psi] \\ &= <\lambda^*\Phi+f,\psi>-<\lambda^*\Phi,\psi>-B[\varphi,\psi]+B[\varphi,\psi]=<f,\psi>, \end{aligned}$$

which means that u is a modern solution of the problem (7). In the above calculation we made use of the obvious fact that for all $\psi \in C_0^\infty(G)$ we have

$$B[\Phi,\psi]+<\lambda^*\Phi,\psi>=<(L+\lambda^*)\Phi,\psi>=0$$

and hence

$$B[\Phi,\psi]=-<\lambda^*\Phi,\psi> \quad \text{for all } \psi \in \overset{\circ}{H}_1(G),$$

since $\Phi=(\Phi-\varphi)+\varphi \in W_1(G)$ and since $C_0^\infty(G)$ is dense in $\overset{\circ}{H}_1(G)$.

We shall postpone the proof of the fact $\Phi-\varphi \in \overset{\circ}{H}_1(G)$ in the next section and prove the converse part of Theorem 1: Every modern solution u of the problem (7) is also a classical solution.

Let Φ be the classical solution of the problem (21) as above, and set

$$W=(u-\varphi)-(\Phi-\varphi). \tag{27}$$

Then, since $u-\varphi \in \overset{\circ}{H}_1(G)$ and since $\Phi-\varphi \in \overset{\circ}{H}_1(G)$, we have $W \in \overset{\circ}{H}_1(G)$ and hence for all $\psi \in \overset{\circ}{H}_1(G)$ we have

$$B[W,\psi]=B[u,\psi]-B[\Phi,\psi]=<f,\psi>+<\lambda^*\Phi,\psi>$$

as before. Thus W is a modern solution of the Dirichlet problem (23) and hence, according to Lemma 3, W is a classical solution in $C^2(G) \cap C(\bar{G})$ of the problem (23). Hence we immediately observe that $u=W+\Phi$ is a classical solution of the problem (7).

In proving $\Phi-\varphi \in \overset{\circ}{H}_1(G)$ we need two more lemmas.

Lemma 4. *Let X be a reflexive Banach space, and let $\{z_m\}_{m=1}^\infty$ be any sequence which is bounded in norm. Then we can choose a subsequence $\{z_{m'}\}$ which converges weakly to an element of X.*

For the proof of Lemma 4, see, e.g., K. Yosida [15], p. 126. We remark here that a Hilbert space is a reflexive Banach space.

Lemma 5. *Let X be a Hilbert space. If a sequence $\{z_m\}_{m=1}^\infty$ of X converges weakly to $z_\infty \in X$, then $\lim_{m\to\infty} \|z_m-z_\infty\|=0$ if and only if $\lim_{m\to\infty} \|z_m\|=\|z_\infty\|$.*

Proof. If $\lim_{m\to\infty} \|z_m\|=\|z_\infty\|$, then

$$\begin{aligned} \|z_m-z_\infty\|^2 &= (z_m-z_\infty, z_m-z_\infty)=\|z_m\|^2-(z_m,z_\infty)-(z_\infty,z_m)+\|z_\infty\|^2 \\ &\to \|z_\infty\|^2-\|z_\infty\|^2-\|z_\infty\|^2+\|z_\infty\|^2=0. \end{aligned}$$

And if $\lim_{m\to\infty} \|z_m-z_\infty\|=0$, then $\lim_{m\to\infty} \|z_m\|=\|z\|$ because of the continuity of the norm. (Adapted from K. Yosida [15], p. 124.)

9. Proof of the main theorem (continued).

It remains only to show that $\Phi-\varphi\in\overset{\circ}{H}_1(G)$.

First, we choose a sequence of domains $\{G_m\}_{m=1}^{\infty}$ such that

$$\bar{G}_m\subset G_{m+1} \text{ and } \bigcup_{m=1}^{\infty}G_m=G, \tag{28}$$

and that each G_m is the interior of the union of a finite number of cubes. Then G_m is regular for the operator $L+\lambda^*$ (see, e. g., K. Akô [3] or M. Nagumo [10]).

We denote by $\Phi^{(m)}$ the (unique) solution in $C^2(G_m)\cap C(\bar{G}_m)$ of the Dirichlet problem

$$\begin{cases}(L+\lambda^*)z=0 & \text{in } G_m,\\ \qquad z=\varphi & \text{on } \partial G_m,\end{cases} \tag{29}$$

and set

$$\tilde{\Phi}^{(m)}=\begin{cases}\Phi^{(m)} & \text{for } x\in G_m,\\ \varphi & \textit{for } x\in\bar{G}-G_m.\end{cases} \tag{30}$$

For later use we put

$$\delta_m=\operatorname{dist}(G_m, \partial G)/2 \ (>0). \tag{31}$$

Next, we set

$$\varphi_h(x)=(J_h\varphi)(x) \quad \text{in} \quad \bar{G}, \tag{32}$$

where J_h is a mollifier. Then

$$\|\varphi_h-\varphi\|_{W_1(G_m)}\to 0 \quad \text{as} \quad h\to 0 \tag{33}$$

and

$$\varphi_h\to\varphi \quad \text{uniformly in } \bar{G}_m \text{ as } h\to 0 \tag{34}$$

by virtue of (19). We denote by $\Phi_h^{(m)}$ the unique solution in $C^2(G_m)\cap C(\bar{G}_m)$ of the problem

$$\begin{cases}(L+\lambda^*)z=0 & \text{in } G_m,\\ \qquad z=\varphi_h & \text{on } \partial G_m,\end{cases} \tag{35}$$

and set

$$Y_h^{(m)}=\Phi_h^{(m)}-\varphi_h \quad \text{in} \quad \bar{G}_m. \tag{36}$$

Then $Y_h^{(m)}$ satisfies the conditions

$$\begin{cases} (L+\lambda^*)Y_h^{(m)} = -(L+\lambda^*)\varphi_h & \text{in } G_m, \\ Y_h^{(m)} = 0 & \text{on } \partial G_m. \end{cases}$$

Since $-(L+\lambda^*)\varphi_h$ is Hölder-continuous in G_m we see from Lemma 3 that $Y_h^{(m)} \in \mathring{H}_1(G_m)$.

Now, let us define

$$\mathbf{B}_m[v, w] \equiv \mathbf{B}[v, w] = B[v, w] + <\lambda^* v, w> \quad (m = 1, 2, \cdots) \tag{37}$$

for $v \in W_1(G_m)$ and $w \in \mathring{H}_1(G_m)$. Then it is clear that

$$\mathbf{B}_m[Y_h^{(m)}, w] = -<(L+\lambda^*)\varphi_h, w> = -\mathbf{B}_m[\varphi_h, w] \tag{38}$$

for all $w \in \mathring{H}_1(G_m)$. Since for $w \in \mathring{H}_1(G_m)$

$$|\mathbf{B}_m[\varphi_h, w] \leqq \gamma^* \|\varphi_h\|_{W_1(G_m)} \|w\|_{\mathring{H}_1(G_m)}$$

for some positive constant γ^*, depending only on the bounds of a_{ij}, b_k, c in G, the diameter of G and the constant λ^* and being independent of any choice of (m, h), we have

$$\|Y_h^{(m)}\|_{\mathring{H}_1(G_m)} \leqq \delta^{-1} \gamma^* \|\varphi_h\|_{W_1(G_m)} \tag{39}$$

from Lemma 1.

Since $\|\varphi_h\|_{W_1(G_m)} \to \|\varphi\|_{W_1(G_m)}$, the family $\{Y_h^{(m)}\}$ is bounded in norm in the space $\mathring{H}_1(G_m)$ as $h \to 0$. In what follows we shall prove that for some sequence $\{Y_{h_j}^{(m)}\}_{j=1}^{\infty} (h_j \to 0$ as $j \to \infty)$

$$\lim_{j\to\infty} \|Y_{h_j}^{(m)} - Y^{(m)}\|_{\mathring{H}_1(G_m)} = 0, \text{ where } Y^{(m)} = \Phi^{(m)} - \varphi \text{ in } \bar{G}_m. \tag{40}$$

We introduce in $\mathring{H}_1(G_m)$ another inner product by

$$(v, w)_a = \frac{1}{2}\{\mathbf{B}_m[v, w] + \mathbf{B}_m[w, v]\} \quad \text{for} \quad v, w \in \mathring{H}_1(G_m) \tag{41}$$

and another norm associated with (41) by

$$\|v\|_a = \sqrt{\mathbf{B}_m[v, v]} \quad \text{for} \quad v \in \mathring{H}_1(G_m). \tag{42}$$

The bilinearity of $\mathbf{B}_m[v, w]$ and the inequality (43) below ensure that $(v, w)_a$ fulfills the axioms of an inner product.

$$\sqrt{\delta}\, \|v\|_{\mathring{H}_1(G_m)} \leqq \|v\|_a = \{\mathbf{B}_m[v, v]\}^{1/2} \leqq \sqrt{\gamma}\, \|v\|_{\mathring{H}_1(G_m)}. \tag{43}$$

By (43) the norms $\|v\|_a$ and $\|v\|_{\mathring{H}_1(G_m)}$ are equivalent and hence $\mathring{H}_1(G_m)$ as a linear space forms a Hilbert space $\mathfrak{H}$ with inner product $(\cdot, \cdot)_a$ and norm $\|\cdot\|_a$. Since $\{Y_h^{(m)}\}$ $(\subset \mathfrak{H})$ is bounded in norm in the space $\mathfrak{H}$ as $h \to 0$ we can choose a sequence $\{Y_{h_j}^{(m)}\}_{j=1}^{\infty}$ $(h_j \to 0$ as $j \to \infty)$ converging weakly to $Y'^{(m)} \in \mathfrak{H}$.

By (36) we have

$$(44) \qquad (Y_h^{(m)}, w)_a = -\boldsymbol{B}_m[\varphi_h, w] \quad \text{for all } w \in \mathfrak{H}.$$

Since $\lim_{h\to 0} \|\varphi - \varphi_h\|_{W_1(G_m)} = 0$, we have

$$(45) \qquad |\boldsymbol{B}_m[\varphi_h, Y_h^{(m)}] - \boldsymbol{B}_m[\varphi, Y_h^{(m)}]| \leqq \gamma^* \|\varphi_h - \varphi\|_{W_1(G_m)} \cdot \|Y_h^{(m)}\|_{\overset{\circ}{H}_1(G_m)} \to 0 \ (\text{as } h \to 0).$$

Furthermore,

$$\boldsymbol{B}_m[\varphi, Y_{h_j}^{(m)} - Y'^{(m)}] \to 0 \quad \text{as } j \to \infty,$$

since $\boldsymbol{L}[w] = \boldsymbol{B}_m[\varphi, w]$ (for $w \in \mathfrak{H}$) is a continuous linear functional on $\mathfrak{H}$ by (43). Thus we have proved that

$$(46) \qquad \lim_{j\to\infty} \boldsymbol{B}_m[\varphi_{h_j}, Y_{h_j}^{(m)}] = \boldsymbol{B}_m[\varphi, Y'^{(m)}].$$

But since, by (38),

$$\boldsymbol{B}_m[Y_h^{(m)}, w] = -\boldsymbol{B}_m[\varphi_h, w] \quad \text{for all } w \in \mathfrak{H},$$

we have

$$(47) \qquad \|Y'^{(m)}\|_a^2 = \boldsymbol{B}_m[Y'^{(m)}, Y'^{(m)}] = \lim_{j\to\infty} \boldsymbol{B}_m[Y_{h_j}^{(m)}, Y'^{(m)}] = -\lim_{j\to\infty} \boldsymbol{B}_m[\varphi_{h_j}, Y'^{(m)}] = -\boldsymbol{B}_m[\varphi, Y'^{(m)}]$$

by (45). Here we use the fact that $\boldsymbol{L}_1[v] = \boldsymbol{B}_m[v, Y^{(m)}]$ is a continuous linear functional on $\mathfrak{H}$ as is easily seen from Lemma 2 and (43).

On the other hand, from (44) we have $\|Y_h^{(m)}\|_a^2 = -\boldsymbol{B}_m[\varphi_h, Y_h^{(m)}]$ and hence from (44) and (47) we obtain

$$(48) \qquad \lim_{j\to\infty} \|Y_{h_j}^{(m)}\|_a^2 = \|Y'^{(m)}\|_a^2.$$

By Lemma 5 we have

$$(49) \qquad \lim_{j\to\infty} \|Y_{h_j}^{(m)} - Y'^{(m)}\|_a^2 = 0$$

and hence, by (43),

$$(50) \qquad \lim_{j\to\infty} \|Y_{h_j}^{(m)} - Y'^{(m)}\|_{\overset{\circ}{H}_1(G_m)}^2 = 0.$$

But, since $Y_h^{(m)} = \Phi_h^{(m)} - \varphi_h \to \Phi^{(m)} - \varphi = Y^{(m)}$ uniformly in $\bar{G}_m$ as $h \to 0$, we see that $Y'^{(m)} = Y^{(m)}$ and thus

$$(51) \qquad \lim_{j\to\infty} \|Y_{h_j}^{(m)} - Y^{(m)}\|_{\overset{\circ}{H}_1(G_m)}^2 = 0 \quad \text{and} \quad Y^{(m)} \in H_1(G_m) \cap C(\bar{G}_m).$$

Now, we have, by (39)

$$\|Y^{(m)}\|_{\overset{\circ}{H}_1(G_m)} \leqq \delta^{-1}\gamma^*\|\varphi\|_{W_1(G_m)} \leqq \delta^{-1}\gamma^*\|\varphi\|_{W_1(\mathrm{G}_m)}. \tag{52}$$

Therefore, if we set

$$\widetilde{Y}^{(m)} = \begin{cases} Y^{(m)} & \text{in } \bar{G}_m, \\ 0 & \text{in } \bar{G}-\bar{G}_m, \end{cases} \tag{53}$$

then, by (27), $\widetilde{Y}^{(m)} = \widetilde{\varPhi}^{(m)} - \varphi \in \overset{\circ}{H}_1(G) \cap C(\bar{G})$, and

$$\|\widetilde{Y}^{(m)}\|_{\overset{\circ}{H}_1(G)} \leqq \delta^{-1}\gamma^*\|\varphi\|_{W_1(G)}. \tag{54}$$

As before, we introduce in $\overset{\circ}{H}_1(G)$ another inner product by

$$(v, w)_b = \frac{1}{2}\{\boldsymbol{B}[v, w] + \boldsymbol{B}[w, v]\} \quad \text{for} \quad v, w \in \overset{\circ}{H}_1(G), \tag{55}$$

and define the norm associated with it by

$$\|v\|_b = \sqrt{(v, v)_b} = \sqrt{\boldsymbol{B}[v, v]} \quad \text{for} \quad v \in \overset{\circ}{H}_1(G) \tag{56}$$

Just as before we have

$$\sqrt{\delta}\,\|v\|_{\overset{\circ}{H}_1(G)} \leqq \|v\|_b \leqq \sqrt{\gamma}\,\|v\|_{\overset{\circ}{H}_1(G)} \tag{57}$$

for all $v \in \overset{\circ}{H}_1(G)$. In what follows we shall adopt the convention: whenever we consider the linear space $\overset{\circ}{H}_1(G)$ with the new inner product (55) and the new norm (56) we write it $\boldsymbol{H}$. According to (57) $\boldsymbol{H}$ is again a Hilbert space.

Now, by (54) and (57), there exists a subsequence $\{\widetilde{Y}^{(m')}\}$ converging weakly in the space $\boldsymbol{H}$ to $Y \in \boldsymbol{H}$. Let $\psi \in C_0^\infty(G)$. The since $\boldsymbol{L}_2[v] = \boldsymbol{B}[v, \psi]$ (for $v \in \boldsymbol{H}$) is a continuous linear functional on $\boldsymbol{H}$, and since $\boldsymbol{B}[\widetilde{Y}^{(m)}, \psi] = -\boldsymbol{B}[\varphi, \psi]$ for large m (i.e. m is so large that supp $(\psi) \subset G_m$), we have

$$\boldsymbol{B}[Y, \psi] = -\boldsymbol{B}[\varphi, \psi],$$

and hence we have

$$\boldsymbol{B}[Y, w] = -\boldsymbol{B}[\varphi, w] \quad \text{for all} \quad w \in \overset{\circ}{H}_1(G). \tag{58}$$

Thus we obtain

$$\|Y\|_b^2 = \boldsymbol{B}[Y, Y] = -\boldsymbol{B}[\varphi, Y]. \tag{59}$$

On the other hand, we have by (47),

$$\|\widetilde{Y}^{(m)}\|_b^2 = \boldsymbol{B}_m[Y^{(m)}, Y^{(m)}] = -\boldsymbol{B}_m[\varphi, Y^{(m)}] = -\boldsymbol{B}[\varphi, \widetilde{Y}^{(m)}] \tag{60}$$

since $\widetilde{Y}^{(m)} = 0$ in $\bar{G} - \bar{G}_m$. But since $\boldsymbol{L}_3[w] = \boldsymbol{B}[\varphi, w]$ (for $w \in \boldsymbol{H}$) is a continuous linear functional on $\boldsymbol{H}$ by (54) and Lemma 2, we have $\lim\limits_{m' \to \infty} \boldsymbol{B}[\varphi, \widetilde{Y}^{(m')}] = \boldsymbol{B}[\varphi, Y]$ and hence $\lim\limits_{m' \to \infty} \|\widetilde{Y}^{(m')}\|_b = \|Y\|_b$, or equivalently,

$$(61) \qquad \lim_{m'\to\infty} \|\tilde{Y}^{(m')} - Y\|_b = 0$$

by Lemma 5. So, by (57), we have

$$(62) \qquad \lim_{m'\to\infty} \|\tilde{Y}^{(m')} - Y\|_{\mathring{H}_1(G)} = 0 \quad \text{and} \quad Y \in \mathring{H}_1(G).$$

But since $\tilde{Y}^{(m)} = \tilde{\Phi}^{(m)} - \varphi$ converges uniformly in G to $\Phi - \varphi$ we have $Y = \Phi - \varphi$ and hence $\Phi - \varphi \in \mathring{H}_1(G)$, completing the proof of the main theorem.

10. Dirichlet's principle and its application.

We shall choose the Laplacian as the operator $-L$ and consider the original Dirichlet problem

$$(63) \qquad \begin{cases} \Delta u = 0 & \text{in } G, \\ u = \varphi & \text{on } \partial G, \end{cases}$$

where G is a bounded regular domain for the original Dirichlet problem. Then the bilinear form $B[u, \psi]$ is reduced to

$$(64) \qquad \boldsymbol{D}\{u, \psi\} = \sum_{i=1}^{n} \int_G \partial_i u(x) \partial_i \psi(x) dx$$

and $B[u, u]$ is nothing but the Dirichlet integral

$$(65) \qquad \boldsymbol{D}[u] = \sum_{i=1}^{n} \int_G (\partial_i u)^2 dx.$$

Then the solution $u(x)$ of the problem (63) has the property (*Dirichlet's principle*) that its Dirichlet integral is the minimum in the class $v \in W_1(G) \cap C(\bar{G})$: $v = \varphi$ on ∂G.

We can verify this property as follows: suppose v is a function in $W_1(G) \cap C(\bar{G})$ which is equal to φ on the boundary ∂G. Then the Dirichlet problem

$$\Delta U = 0 \text{ in } G \quad \text{and} \quad U = v \text{ on } \partial G,$$

has the same solution u as the problem (63). Therefore,

$$u = w + v, \quad w \in \mathring{H}_1(G) \cap C(\bar{G})$$

by Theorem 1. Since $\boldsymbol{D}\{u, w\} = 0$, we have

$$(66) \qquad \boldsymbol{D}[v] = \boldsymbol{D}[u] + \boldsymbol{D}[w] \geqq \boldsymbol{D}[u],$$

where the equality is valid only when $w =$ const. and hence $u = v$. This proves Dirichlet's principle.

In the proof given above we have verified that every $v \in W_1(G) \cap C(\bar{G})$ is decomposed into $v = u + (-w)$, where u is harmonic in G and continuous in the closure $\bar{G}$ and where $(-w) \in \mathring{H}_1(G) \cap C(\bar{G})$.

Theorem 2. *Let G be a bounded domain for which the original Dirichlet problem (for harmonic functions) is always soluble. Then we have the following statement:*

i) $W_1(G)\cap C(\bar{G})=H_1(G)\cap C(\bar{G})$. (*We have, in fact,* $W_1(G)=H_1(G)$).

ii) *Every* $v\in W_1(G)\cap C(\bar{G})$ *is decomposed in the form* $v=u_1+w_1$, *where* u_1 *is harmonic in* G *and continuous in the closure* $\bar{G}$ *and where* $w_1\in \overset{\circ}{H}_1(G)\cap C(\bar{G})$.

iii) *Every* $v\in W_1(G)\cap C(\bar{G})$ *is decomposed in the form* $v=u_2+w_2$, *where* u_2 *is real-analytic in* G *and continuous in the closure* $\bar{G}$ *and where* $w_2\in \overset{\circ}{H}_1(G)\cap C(\bar{G})$. *Moreover,* $(u_2, w)_{W_1(G)}=0$ *for all* $w\in \overset{\circ}{H}_1(G)$ *and hence, in particular,* $(u_2, w_2)_{W_1(G)}=0$.

Proof. We need only prove iii).

To see this we consider the Dirichlet problem

$$\begin{cases} -\Delta u+u=0 & \text{in } G \\ u=v & \text{on } \partial G. \end{cases}$$

Then the (unique) solution u_2 in $C^{\omega}(G)\cap C(\bar{G})$ of this problem satisfies

$$B[u_2, w]=(u_2, w)_{W_1(G)}=0 \quad \text{for all } w\in \overset{\circ}{H}_1(G).$$

Since it is evident that $w_2=v-u_2\in H_1(G)\cap C(\bar{G})$, we have iii).

11. Two remarks.

Remark 1. So far we have considered equations with real coefficients. But our method can be applied also to single equations with complex coefficients and also to systems of equations. But, for the sake of brevity we shall not enter into it here. We may also extend our results to the case where G is an unbounded domain. But, then, considerable complexity would creep into the theory, which surpasses the ability of the present author.

Remark 2. If we admit generalized classical solutions, then we have the following theorem.

Theorem 3. *Let Fundamental Assumptions I–III be satisfied with (II-a) replaced by the weaker*

II-a′): $a_{ij}, \partial_k a_{ij}, b_k$, *and* c *are bounded and continuous in the domain* G.

Consider the Dirichlet problem

$$(7)\qquad \begin{cases} Lu=f & \text{in } G, \\ u=\varphi & \text{on } \partial G, \end{cases}$$

where f *is bounded and continuous in* G *and where* $\varphi\in C(\bar{G})\cap W_1(G)$. *Then we have the following statement: Every generalized classical solution in* $C^1(G)\cap C(\bar{G})$ *of the problem (7) is a modern solution, and conversely, every modern solution* *of* (7) *is a generalized classical solution in* $C^1(G)\cap C(\bar{G})$.

As the proof is quite similar to that of Theorem 1, we may omit it.

Appendix
Regularity of solutions near the boundary

Let M be a second order elliptic differential operator of the form

$$(\sharp) \qquad Mu=a_{ij}(x)\partial_i\partial_j u+b_k(x)\partial_k u+c(x),$$

defined in a bounded subdomain G of E_n ($n\geqq 2$). Then we have the following theorem.

Theorem 4. *Let* G *be a domain of class* $C^{m+2+\sigma}$ ($0<\sigma<1$). *Suppose further that the coefficients* a_{ij}, b_k, c *are in* $C^{m+\sigma}(G)$. *Here* m *is a non-negative integer. Then for every generalized solution* u *in* $C^1(G)\cap C(\bar{G})$ *of the Dirichlet problem*

$$(\$) \qquad \begin{cases} Mu=f & in \ G,\\ u=\varphi & on \ \partial G,\end{cases}$$

where $f\in C^{m+\sigma}(\bar{G})$ *and* $\varphi\in C^{m+2+\sigma}(\bar{G})$, *we have* $u\in C^{m+2+\sigma}(\bar{G})$.

Proof. It is sufficient to prove the theorem in the case $m=0$.

We can reduce the problem ($) to the operational equation

$$(¥) \qquad (I-\lambda_0 T(u-\varphi))=T(f-M\varphi),$$

where Tv is given as the (unique) solution of

$$(£) \qquad \begin{cases} (M-\lambda_0)(Tv)=v & \text{in} \ G,\\ Tv=0 & \text{on} \ \partial G,\end{cases}$$

where $\lambda_0=\max|c(x)|$ and $v\in C(\bar{G})$. The operator T is completely continuous and its restriction to $C^\sigma(\bar{G})$ is also completely continuous by the Schauder-Caccioppoli boundary estimates (see J. Schauder [13] and C. Miranda [9]) and by E. Hopf's maximal principle ([8]). Since the natural embedding mapping $C^\sigma(\bar{G})\subset C(\bar{G})$ is continuous we have $C(\bar{G})^*\subset C^\sigma(\bar{G})^*$ for the dual spaces and hence the number of independent solutions of (¥) in the space $C^\sigma(\bar{G})$ (and hence $u\in C^{2+\sigma}(\bar{G})$) is equal to the number of independent solutions u of (¥) in the space $C(\bar{G})$ (and hence $u\in C^1(G)\cap C(\bar{G})$). Therefore we have proved our theorem.

The above Theorem 4 can be extended to the case of quasilinear elliptic differential equations.

Let us consider a quasilinear equation

$$(\sharp\sharp) \qquad Nu\equiv a_{ij}(x,u)\partial_i\partial_j u=g(x,u,\operatorname{grad} u),$$

where

i) the symmetric matrix $\|a_{ij}(x,u)\|$ is positive definite for $(x,u)\in\bar{G}\times\{u;\ |u|<\infty\}$, and

ii) $a_{ij}(x,u)$ and $g(x,u,p)$ are defined in $\mathfrak{D}$: $x\in\bar{G}$, $|u|<\infty$, $|p|=\{p_1^2+\cdots+p_n^2\}^{1/2}<\infty$ and are in $C^{m+\sigma}(\mathfrak{A})$ for every compact part $\mathfrak{A}$ of $\mathfrak{D}$, and finally

iii) $|g(x,u,p)|\leqq B(K)(|p|+1)$ for $x\in G$, $|u|\leqq K$, $|p|<\infty$, where $B(K)$ is a positive constant if K is.

Then we have

Theorem 5. *If the domain G is bounded and of class $C^{m+2+\sigma}$, every generalized solution in $C^1(G)\cap C(\bar{G})$ of the Dirichlet problem*

$$(¥¥)\qquad \begin{cases} Nu=g(x,u,\operatorname{grad}u) & \textit{in}\quad G,\\ u=\varphi & \textit{on}\quad \partial G,\end{cases}$$

where $\varphi\in C^{m+2+\sigma}(\bar{G})$, belongs to $C^{m+2+\sigma}(\bar{G})$.

Proof. We only prove Theorem 5 in the case $m=0$.

Let the least upper bound for the $u(x)$ be K and define

$$(££)\qquad \begin{cases} b_k(x)\equiv-\dfrac{g(x,u(x),\operatorname{grad}u(x))\partial_k u(x)}{|\operatorname{grad}u(x)|^2+1} & (k=1,\cdots,n)\\[2ex] f(x)\equiv\dfrac{g(x,u(x),\operatorname{grad}u(x))}{|\operatorname{grad}u(x)|^2+1} & \end{cases}$$

Then we can write the problem (##) as

$$M_0u=f \text{ in } G,\quad u=\varphi \text{ on } \partial G,$$

where

$$M_0u\equiv a_{ij}(x,u(x))\partial_i\partial_j u+b_k(x)\partial_k u.$$

By Cordes' estimates (see H. O. Cordes [4, 5]) $u\in C^{1+\sigma'}(\bar{G})$ for some σ', $0<\sigma'<1$. Hence b_k and f belong to $C^{\sigma'}(\bar{G})$. According to Theorem 4, $u\in C^{2+\sigma'}(\bar{G})$ and hence b_k and f belong to $C^{\sigma}(\bar{G})$. Thus, again by Theorem 4, we have the assertion: $u\in C^{2+\sigma}(\bar{G})$.

Remark. In this Appendix we have dealt only with single equations. But it is obvious that our method can be applied also to systems. For the sake of brevity, however, we restrict ourselves to the case of a single equation.

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(Ricevita la 2-an de aprilo, 1969)