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On Ura's Axioms and Local Dynamical Systems

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Introduction.

It appears that the formal introduction of a local dynamical system is due to Hajek [3] and Sell [7]. Recently, Hajek [5] gave a local characterization of a local semi-dynamical system, by introducing the notion of a germ, which can easily be generalized to the notion of a germ of a local dynamical system. On the other hand, several years before any of these contributions were made, Ura [8] had introduced two systems of axioms, referred to as an F-family and an $\boldsymbol{F}$-family. The main object of this paper is to show the essential equivalence of an F-family and a germ of a local dynamical system, and that of an $\boldsymbol{F}$-family and a local dynamical system.

In the definition of an F-family, given in [8], the phase space was assumed to be an open subset of a compact topological space. This is slightly modified here, in order to consider more general topological spaces as phase spaces. In [9], Ura added a somewhat stronger axiom to the system in [8], which is denoted by (F5) in this paper. In order to clarify the corresponding condition for a local dynamical system, we give two systems of axioms for a germ of a local dynamical system in § I. The equivalence of a germ and an F-family is proved in § I, for both, the case where (F5) is satisfied and the case where it is not satisfied.

In [5], Hajek introduced the notion of strong local equivalence of germs of local semi-dynamical systems and proved that strongly locally equivalent germs generate the same local-semi-dynamical system. Ura [10] has shown that by introducing isomorphisms of germs the same way as for local dynamical systems, isomorphisms of germs can be extended to isomorphism of the local dynamical systems generated by the given germs. In § II, we show that similar result holds for F-families, i. e., two strongly locally equivalent F-families, on the same phase space, generate the same $\boldsymbol{F}$-family. Similar modifications as for an F-families are given to $\boldsymbol{F}$-families and local dynamical systems. The proof of the uniqueness of an $\boldsymbol{F}$-family strongly equivalent to a given F-family remains the same as in [8]. Hence we only give the proof of the existence of such an $\boldsymbol{F}$-family, the technique of which clarifies Hajek's theorem for local dynamical systems.

§I. Basis of Solution-elements and Germs of Local Dynamical Systems

1. Basis of Solution-elements.

Definition 1. Let X be a topological space and F a family of mappings c such that c maps I_c into X, where I_c is an open interval of the set of real numbers R depending on c, and is called the interval of definition of c. If F satisfies the following conditions (F1) through (F4) or (F5), then F is called a *basis of solution-elements on the phase space* X (abbreviated as an *F-family* on X).

(F1) *Continuity with respect to t*: Every element c of F is continuous on I_c.

(F2) *Existence of solutions for the Cauchy problem and local uniformity of intervals of definition*: For each $x_0 \in X$ there exist $U \in \mathcal{V}(x_0)$[1] and $\delta > 0$ such that for every $x \in U$ and every $t_0 \in R$, there exists $c \in F$ with $(t_0-\delta, t_0+\delta) \subset I_c$ and $c(t_0)=x$[2].

(F3) *Homomorphism*: Let $c_1, c_2 \in F$. If $t_1 \in I_{c_1}$, $t_2 \in I_{c_2}$ and $c_1(t_1)=c_2(t_2)$, then $c_1(t_1+t)=c_2(t_2+t)$ for all $t \in (I_{c_1}-t_1) \cap (I_{c_2}-t_2)$[3].

(F4) *Continuity with respect to initial points*: Let $c \in F$ and $t_0, t_1 \in I_c$. For every $U \in \mathcal{V}(c(t_1))$, there exists $V \in \mathcal{V}(c(t_0))$ such that if $c_1 \in F, t_0, t_1 \in I_{c_1}$ and $c_1(t_0) \in V$, then $c_1(t_1) \in U$.

(F5) *Stronger uniformity*: For every $x_0 \in X$ and for every $U \in \mathcal{V}(x_0)$, there exists $V \in \mathcal{V}(x_0)$ and $\delta > 0$ satisfying the following condition: for every $x \in V$ and $t_0 \in R$, there exists $c \in F$ such that $(t_0-\delta, t_0+\delta) \subset I_c$, $c(t_0)=x$ and $c((t_0-\delta, t_0+\delta)) \subset U$ (see [11]).

It is obvious that (F5) implies (F2).

Proposition 1. *In the definition of an F-family,* (F3) *can be replaced by the following two conditions*:

(F3)′ *Homomorphism (restricted)*: *If $c, c' \in F$, $t_0 \in I_c$, $0 \in I_{c'}$ and $c(t_0)=c'(0)$, then $c(t_0+t)=c'(t)$ for all $t \in (I_c-t_0) \cap I_{c'}$.*

(F3)″ *Uniqueness for the Cauchy problem*: *If $c_1, c_2 \in F$ and $c_1(t_0)=c_2(t_0)$ for some $t_0 \in I_{c_1} \cap I_{c_2}$, then $c_1(t)=c_2(t)$ for all $t \in I_{c_1} \cap I_{c_2}$.*

Proof. See [8].

Proposition 2. *If X is a Hausdorff space, then* (F3) *can be replaced by the following weaker condition*:

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1) Throughout this paper, for a point or a subset P of X, $\mathcal{V}(P)$ denotes the neighborhood filter of P.

2) In order to treat a general topological space, (F2) of [8] is slightly modified.

3) If t is a number and I is an interval, then $I-t \equiv \{x-t : x \in I\}$.

(F3)* *Let* $c_1, c_2 \in F$. *If* $t_1 \in I_{c_1}, t_2 \in I_{c_2}$ *and* $c_1(t_1)=c_2(t_2)$, *then there exists an open interval* I *containing* 0 *such that* $c_1(t_1+t)=c_2(t_2+t)$ *for all* $t \in I$.

Proof. It is obvious that (F3) implies (F3)*. Assume (F3)* and let $(a,b)=(I_{c_1}-t_1)\cap(I_{c_2}-t_2)$, and $M=\{\beta \mid c_1(t_1+t)=c_2(t_2+t)$ for all $t\in[0,\beta),\ 0<\beta\leqq b\}$. Then by (F3)*, M is not empty. Let $\beta_0=\sup M$. Then $c_1(t_1+t)=c_2(t_2+t)$ for all $t\in[0,\beta_0)$. Assume $\beta_0<b$. Since X is Hausdorff and c_1 and c_2 are continuous on I_{c_1} and I_{c_2} respectively, we must have $c_1(t_1+t)=c_2(t_2+t)$ for all $t\in[0,\beta_0]$. Again by (F3)* there is an open interval I containing 0 such that $c_1(t_1+\beta_0+t)=c_2(t_2+\beta_0+t)$ for all $t\in I$, contradicting $\beta_0=\sup M$. Thus $\beta_0=b$, whence $c_1(t_1+t)=c_2(t_2+t)$ for all $t\in[0,b)$. Similarly, $c_1(t_1+t)=c_2(t_2+t)$ for all $t\in(a,0]$, and the proof is complete.

If X is Hausdorff (F3)′ and (F3)″ can be modified similarly.

Definition 2. Let F and G be two F-families on X. If for every pair (c,d) such that $c\in F$, $d\in G$ and $c(t_0)=d(t_0)$ for some $t_0\in I_c\cap J_d$[4], there exists $\delta>0$ satisfying $c(t)=d(t)$ for all $t\in(t_0-\delta,t_0+\delta)$ $(\subset I_c\cap J_d)$, then F is said to be *locally equivalent* to G; more simply, *equivalent* to G (δ may depend on c,d and t_0).

Definition 3. Let F and G be as in Definition 2. If for every $x_0\in X$ there exist $U\in\mathcal{V}(x_0)$ and $\delta>0$ such that $c\in F$, $d\in G$, $c(t_0)=d(t_0)\in U$ for some $t_0\in I_c\cap J_d$ implies $c(t)=d(t)$ for all $t\in I_c\cap J_d\cap(t_0-\delta,t_0+\delta)$, then F is said to be *strongly locally equivalent* to G; more simply, *strongly equivalent* to G.

The definitions of local equivalence and strong local equivalence for local semi-dynamical systems are given in [5].

The following proposition can be easily verified.

Proposition 3. *Each of the relations "equivalence" and "strong equivalence" is an equivalence relation.*

Proposition 4. *If the phase space X is Hausdorff, then equivalence implies strong equivalence.*

Proof. The proof of this proposition is similar to that of Proposition 2. In fact, using the same method of argument and replacing (F3)* by equivalence, one can prove even more. Namely, if $c\in F$, $d\in G$ and $c(t_0)=d(t_0)$ for some $t_0\in R$, then $c(t)=d(t)$ for all $t\in I_c\cap J_d$.

The assertion corresponding to Proposition 4 for local semi-dynamical systems is given in [5]. The proof suggested here is different.

2. Germs of Local Dynamical Systems.

Definition 4. Let X be a topological space, D a subset of $X\times R$ and μ a

4) Throughout this paper, J_d denotes the interval of definition of d, where d is an element of an F-family denoted by G.

mapping of D into X. If the following conditions ($\pi 0$) through ($\pi 3$) or ($\pi 4$) are satisfied, then μ is called a *germ of a local dynamical system on the phase space* X, or simply, a *germ* on X. μ is called *continuous* if it satisfies ($\pi 4$), otherwise it is called *separately continuous*.

($\pi 0$) D is a neighborhood of $X \times \{0\}$ and can be expressed as

$$D = \bigcup_{x \in X} \{x\} \times I_x,$$

where I_x is an open interval containing 0.

($\pi 1$) *Identity*: $\mu(x, 0) = x$ for all $x \in X$.

($\pi 2$) *Homomorphism*: If $(x, t) \in D$, then $\mu(\mu(x, t), s) = \mu(x, t+s)$ for sufficiently small s.

($\pi 3$) *Separate continuity*: $\mu(x, t)$ is separately continuous, i.e. for each $x \in X$ and $t \in R$, μ_x and μ^t are continuous in their domains of definition[5].

($\pi 4$) *Continuity*: μ is continuous in D.

Obviously, ($\pi 4$) implies ($\pi 3$).

Similar to (F3), ($\pi 2$) can be replaced by:

($\pi 2$)$'$ If $(x, t), (x, t+s), (\mu(x, t), s) \in D$, then $\mu(\mu(x, t), s) = \mu(x, t+s)$.

Definition 5. Let μ and ν be two germs on the same phase space X. If for every $x \in X$ there exists $\delta > 0$ such that μ_x and ν_x coincide in $(-\delta, \delta)$, then μ is said to be *locally equivalent* to ν, or simply, *equivalnet* to ν.

Definition 6. Let μ and ν be as in Definition 5. If there exists a neighborhood D' of $X \times \{0\}$ such that $D' \subset D \cap E$, where D and E are the domains of μ and ν respectively, and $\mu | D' = \nu | D'$, then μ is said to be *strongly locally equivalent* to ν, or simply, *strongly equivalent* to ν.

Theorem (Hajek). *Let μ be a germ on X. Then there exists a uniquely determined local system π on X which is strongly equivalent to μ.*

Proof. See [5].

From here on, if μ and ν are germs, then X and Y denote the phase spaces of μ and ν respectively and D and E denote the domains of μ and ν respectively. Further, for each $x \in X$ and $y \in Y$, I_x and J_y denote the intervals of definition of μ_x and ν_y respectively.

The following propositions are obvious.

Proposition 5. Each of the relations "equivalence" and "strong equiva-

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5) If μ is a mapping of a subset D of $X \times R$ into X, then for each $x_0 \in X$ and $t_0 \in R$, μ_{x_0} and μ^{t_0} denote the mappings defined by

$$\mu_{x_0}(t) = \mu(x_0, t) \quad \text{for } (x_0, t) \in D$$

and

$$\mu^{t_0}(x) = \mu(x, t_0) \quad \text{for } (x, t_0) \in D$$

respectively.

lence" of germs is an equivalence relation.

Proposition 6. *Let μ be a germ on X. Let D' be a neighborhood of $X\times\{0\}$ contained in the demain D of μ, and which can be expressed as*

$$D'=\bigcup_{x\in X}\{x\}\times I'_x,$$

where I'_x is an open interval containing 0 *(and contained in I_x). Then $\mu'=\mu|D'$ is a germ on X, strongly equivalent to μ (see* [10]*).*

3. Relations between a basis of solution-elements and a germ of a local dynamical system.

Process I. Let F be an F-family on X. Define a mapping μ as follows: Let $(x, t)\in X\times R$. If there exists $c\in F$ with $c(0)=x$ and $t\in I_c$, then $\mu(x, t)=c(t)$, otherwise (x, t) is not in the domain of μ.

We wish to show that μ is well-defined and is a germ on X. Further, if F satisfies (F5), then μ satisfies (π4).

It is easy to see that for every $x\in X$, the domain I_x of μ_x is

$$I_x=\bigcup_{c\in F, c(0)=x} I_c.$$

Therefore, I_x is an open interval containing 0. Let $c_1, c_2\in F$ such that 0, $t\in I_{c_1}\cap I_{c_2}$ and $c_1(0)=c_2(0)$. Then by (F3)$''$, $c_1(t)=c_2(t)$, and hence μ_x is well-defined.

Let $x_0\in X$. It follows from (F2) that there exist $U\in\mathcal{V}(x_0)$ and $\delta>0$ such that for every $x\in U$, there exists $c\in F$ with $c(0)=x$ and $(-\delta, \delta)\subset I_c$. Hence the domain D of μ contains $U\times(-\delta, \delta)$, and consequently (π0) holds.

It is obvious that μ satisfies (π1). Let $(x, t)\in D$. It follows from the definition of D that there exist $c\in F$ such that $0, t\in I_c$ and $\mu(x, t)=c(t)$. Also, there exists $c'\in F$ such that $0\in I_{c'}$ and $c'(0)=c(t)$, by (F2). Using (F3), we have $c'(s)=c(t+s)$ for all $s\in(I_c-t)\cap I_{c'}$. Hence, by definition of μ, we have $\mu(\mu(x, t), s)=c'(s)$ and $\mu(x, t+s)=c(t+s)$. This shows that (π2) holds. It is easy to verify that (π3) also holds.

Assume that F satisfies (F5), and let $(x_0, t_0)\in D$. Let $x_1=\mu(x_0, t_0)$ and $W\in\mathcal{V}(x_1)$. Since μ^{t_0} is continuous, there exists $U\in\mathcal{V}(x_0)$ such that $\mu^{t_0}(U)\subset W$. By (F5), there exist $V\in\mathcal{V}(x_0)$ and $\delta>0$ such that for every $x\in V$, there exists $c\in F$ with $(-\delta, \delta)\subset I_c$, $c(0)=x$ and $c((-\delta, \delta))\subset U$. Hence, $V\times(-\delta, \delta)\subset D$ and $\mu(V\times(-\delta, \delta))\subset U$. Thus if $(x, t)\in V\times(-\delta, \delta)$ and $(x, t+t_0)\in D$, then $\mu((x, t_0+t))=\mu(\mu(x, t_0), t)\in W$, i.e. if $(x, t)\in V\times(t_0-\delta, t_0+\delta)$ and $(x, t)\in D$, then $\mu(x, t)\in W$. This shows that (π4) holds.

Process II. Let μ be a germ on X. Define F to be the family of mappings given by

$$F=\{\mu_x(t-t_0)\,|\,t_0\in R,\ (x,t-t_0)\in D\},$$

i. e. $c\in F$ if and only if for some fixed $t_0\in R$ and $x_0\in X$, $c(t)=\mu_{x_0}(t-t_0)$ whenever $(x_0,t-t_0)\in D$. It is easy to verify that F is an F-family on X, and if μ satisfies (π4), then F satisfies (F5).

The proof of the following proposition is straight-forward.

Proposition 7. *Each of Process* I *and Process* II *keep equivalence and strong equivalence invariant.*

Theorem 1. *Let F be an F-family on X. Let μ be the germ obtained by applying Process* I *to F. Let G be the F-family obtained by applying Process* II *to μ. Then F and G are strongly equivalent.*

Proof. Let $x_0\in X$. Then, there exist $U\in\mathcal{V}(x_0)$ and $\delta>0$ satisfying (F2). Thus, $D\supset U\times(-\delta,\delta)$. Suppose $c(t_0)=d(t_0)=x\in U$ for some $t_0\in R$, $c\in F$ and $d\in G$. In view of Process II, $d(t)=\mu_x(t-t_0)$ in J_d. Since $x\in U$, there exists $c'\in F$ such that $c'(0)=x$ and $I_{c'}\subset(-\delta,\delta)$. Therefore, $c'(0)=c(t_0)$ and hence $c'(t)=c(t_0+t)$ in $(I_c-t_0)\cap(-\delta,\delta)$, by (F3). Thus, $c(t)=c'(t-t_0)$ in $I_c\cap(t_0-\delta, t_0+\delta)$. But, $c'(t-t_0)=\mu_x(t-t_0)=d(t)$ since $c'(0)=x$. Hence $c(t)=d(t)$ in $I_c\cap J_d\cap(t_0-\delta,t_0+\delta)$, and the proof is complete.

Theorem 2. *Let μ be a germ on X. Let F be the F-family obtained by applying Process* II *to μ. Let ν be the germ obtained by applying Process* I *to F. Then μ and ν are strongly equivalent; more precisely, $D\subset E$ and $\mu=\nu|D$.*

Proof. The first assertion follows from Proposition 7 and Theorem 2.

Let $x\in X$. Then the mapping c, defined by $c(t)=\mu_x(t)$ in I_x, belongs to F and $c(0)=x$. Therefore, $\nu_x(t)$ is defined at least in I_x and $\nu_x(t)=\mu_x(t)$ in I_x. This proves the last assertion.

Corollary. *If the phase space is Hausdorff, then equivalence of germs implies strong equivalence*[6].

Proof. Let μ and ν be two equivalent germs on the same phase space X. Let F and G be the F-families obtained by applying Process II to μ and ν respectively. By Proposition 7, F and G are equivalent. Hence F and G are strongly equivalent, by Proposition 4. Now, let μ' and ν' be the germs obtained by applying Process I to F and G respectively. By Proposition 7, μ' and ν' are strongly equivalent. On the other hand, μ' and ν' are also strongly equivalent to μ and ν respectivaly, by Theorem 2. Therefore, μ and ν are strongly equivalent.

6) See the remark made after the proof of Proposition 4 of § I.

§II. Complete Families of Non-extendable Solutions and Local Dynamical Systems

1. Complete Families of Non-extendable Solutions.

Definition 1. Let X be a topological space and $\boldsymbol{F}$ a family of mappings f such that f maps $\boldsymbol{I}_f$ into X, where $\boldsymbol{I}_f$ is an open interval in R and is called the interval of definition of f. If $\boldsymbol{F}$ satisfies the following conditions ($\boldsymbol{F}1$) through ($\boldsymbol{F}5$) or ($\boldsymbol{F}6$), then $\boldsymbol{F}$ is called a *complete family of non-extendable solutions on the phase space* X (abbreviated as an *$\boldsymbol{F}$-family* on X).

($\boldsymbol{F}1$) *Continuity with respect to t*: Every element f of $\boldsymbol{F}$ is continuous on $\boldsymbol{I}_f$.

($\boldsymbol{F}2$) *Exitence of solutions for the Cauchy problem*: For each $x \in X$ and $t_0 \in R$, there exists $f \in \boldsymbol{F}$ such that $f(t_0)=x$.

($\boldsymbol{F}3$) *Homomorphism*: If $f_1, f_2 \in \boldsymbol{F}$ and $f_1(t_1)=f_2(t_2)$, then $\boldsymbol{I}_{f_1}-t_1=\boldsymbol{I}_{f_2}-t_2$ and $f_1(t_1+t)=f_2(t_2+t)$ for all $t \in \boldsymbol{I}_{f_1}-t_1$.

($\boldsymbol{F}4$) *Continuity with respect to initial points*: Let $f_0 \in \boldsymbol{F}$ and $t_0, t_1 \in \boldsymbol{I}_{f_0}$. Then for every $U \in \mathcal{V}(f(t_1))$, there exists $V \in \mathcal{V}(f(t_0))$ such that if $f \in \boldsymbol{F}$, $t_0 \in \boldsymbol{I}_f$ and $f(t_0) \in V$, then $t_1 \in \boldsymbol{I}_{f_1}$ and $f(t_1) \in U$.

($\boldsymbol{F}5$) *Non-extendability*: Let $f \in \boldsymbol{F}$ and $\boldsymbol{I}_f=(a, b)$. If $b<\infty$, then the cluster set $L^+(f)$ of f as $t \uparrow b$ is empty. If $a>-\infty$, then the cluster set $L^-(f)$ of f as $t \downarrow a$ is empty.

($\boldsymbol{F}6$) *Continuity*: Let $f_0 \in \boldsymbol{F}$ and $t_0, t_1 \in \boldsymbol{I}_{f_0}$. For every $U \in \mathcal{V}(f_0(t_1))$, there exist $V \in \mathcal{V}(f_0(t_0))$ and $\delta>0$ such that if $f \in \boldsymbol{F}$ and $f(t_0) \in V$, then $(t_1-\delta, t_1+\delta) \subset \boldsymbol{I}_f$ and $f((t_1-\delta, t_1+\delta)) \subset U$.

Obviously, ($\boldsymbol{F}6$) implies ($\boldsymbol{F}1$) and ($\boldsymbol{F}5$).

If the phase space X is assumed to be an open subset of a topological space, then the statement that $L^+(f)=\phi$ is equivalent to saying that $L^+(f)$ is contained in the boundary of X (cf. [8], [9], [11]).

The following two propositions are easy to verify.

Proposition 1. *In the definition of an $\boldsymbol{F}$-family, ($\boldsymbol{F}3$) can be replaced by the following two condition*:

($\boldsymbol{F}3$)$'$ *Homomorphism (restricted)*: *Let $f \in \boldsymbol{F}$ and $t_0 \in R$. If g is a mapping of $\boldsymbol{I}_f-t_0$ into X given by $g(t)=f(t+t_0)$ for all $t \in \boldsymbol{I}_f-t_0$, then $g \in \boldsymbol{F}$.*

($\boldsymbol{F}3$)$''$ *Uniqueness for the Cauchy Problem*: *Let $f_1, f_2 \in \boldsymbol{F}$ and $t_0 \in \boldsymbol{I}_{f_1} \cap \boldsymbol{I}_{f_2}$. If $f_1(t_0)=f_2(t_0)$, then $\boldsymbol{I}_{f_1}=\boldsymbol{I}_{f_2}$ and $f_1=f_2$.*

Proposition 2. *An $\boldsymbol{F}$-family on X is an F-family on X.*

Before proving our next theorem, we state a few lemmas, proofs of which are easy (for Lemmas 1, 2 and 3 see [8]).

Lemma 1. *Let F be an F-family on X. Let $\boldsymbol{F}$ be a family of mappings*

f such that f maps an interval $\boldsymbol{I}_f$ of R into X satisfying the following two conditions.

(I) For every $c \in F$ there exists $f \in \boldsymbol{F}$ such that $I_c \subset \boldsymbol{I}_f$ and $c(t)=f(t)$ in I_c.

(II) Let $f \in \boldsymbol{F}$ and $t_0 \in \boldsymbol{I}_f$. If $c \in F$ and $c(t_0)=f(t_0)$, then $I_c \subset \boldsymbol{I}_f$ and $c(t)= f(t)$ in I_c.

Then, $\boldsymbol{F}$ is an $\boldsymbol{F}$-family on X, which is strongly equivalent to F as an F-family on X. Further, if F satisfies (F5), then $\boldsymbol{F}$ satisfies ($\boldsymbol{F}$6).

Lemma 2. *Let F and $\boldsymbol{F}$ be as in Lemma* 1. *Then, $\boldsymbol{F}$ is unique.*

Lemma 3. *Conditions* (I) *and* (II) *in Lemma* 1 *can be replaced by the following three conditions.*

(I)′ *For every $c \in F$ and $t_0 \in I_c$, there exist $f \in \boldsymbol{F}$ and $\delta > 0$ such that $(t_0-\delta, t_0+\delta) \subset I_c \cap \boldsymbol{I}_f$ and $c(t)=f(t)$ in $(t_0-\delta, t_0+\delta)$.*

(II)′ *For every $x_0 \in X$, there exist $U \in \mathcal{V}(x_0)$ and $\delta>0$ such that if $c \in F$, $f \in \boldsymbol{F}$ and $c(t_0)=f(t_0) \in U$, then $c(t)=f(t)$ in $I_c \cap \boldsymbol{I}_f \cap (t_0-\delta, t_0+\delta)$.*

(III)′ $\boldsymbol{F}$ satisfies ($\boldsymbol{F}$5).

Lemma 3*. *If X is Hausdorff, then* (II)′ *in Lemma* 3 *can be replaced by*

(II)′* *For every $f \in \boldsymbol{F}$ and $t_0 \in \boldsymbol{I}_f$, there exist $c \in F$ and $\delta>0$ such that $c(t)=f(t)$ for all $t \in (t_0-\delta, t_0+\delta) \subset I_c \cap \boldsymbol{I}_f$.*

Lemma 4. *Let F be an F-family on X and $\boldsymbol{F}$ an $\boldsymbol{F}$-family on X. Then, F and $\boldsymbol{F}$ are strongly equivalent as $\boldsymbol{F}$-families if and only if $\boldsymbol{F}$ satisfies conditions* (I) *and* (II) *of Lemma* 1 (*or* (I)′, (II)′ *and* (III)′ *of Lemma* 3).

Theorem 1. *Let F be an F-family on X. Then, there exists a unique $\boldsymbol{F}$-family $\boldsymbol{F}$ strongly equivalent to F. Further, if F satisfies* (F5), *then $\boldsymbol{F}$ satisfies* ($\boldsymbol{F}$6).

Proof. In view of the above lemmas, it is sufficient to show the existence of $\boldsymbol{F}$ satisfying condition (I)′, (II)′ and (III)′ of Lemma 3. Let $c_0 \in F$ and $t_0 \in I_{c_0}$. We shall show that there exists a mapping f^+ of an interval $[t_0, b)$ into X satisfying the following two conditions.

(II)* For every $\tau_0 \in [t_0, b)$, there exist $U \in \mathcal{V}(f^+(\tau_0))$ and $\delta>0$ such that if $\tau \in (\tau_0-\delta, \tau_0+\delta)$, $c \in F$ and $c(\tau)=f^+(\tau) \in U$, then $c(t)=f^+(t)$ in $(\tau-\delta, \tau+\delta)$.

(III)* $b=\infty$ or $L^+(f^+)=\phi$, where $L^+(f^+)$ denotes the cluster set of $f^+(t)$ as $t \uparrow b$.

One might remark that if a mapping f^+ of $[t_0, b)$ into X satisfies (II)*, then $\tau \in I_c \cap [t_0, b)$ and $c(\tau)=f(\tau)$ imply $c(t)=f^+(t)$ in $I_c \cap [t_0, b)$.

Without loss of generality, we can assume $t_0=0$. For each γ such that $0<\gamma<\infty$, let $P(\gamma)$ denote the statement that there is a mapping f_γ^+ of $[0, \gamma)$ into X satisfying (II)* in $[0, \gamma)$ and $L^+(f_\gamma^+) \neq \phi$, where $L^+(f_\gamma^+)$ denotes the cluster set of $f_\gamma^+(t)$ as $t \uparrow \gamma$.

Let $I_{c_0}=(\alpha, \beta)$. The interval (α, β) contains 0 by assumption. For each $\gamma \in (0, \beta)$, define f_γ^+ by $f_\gamma^+(t)=c_0(t)$ in $[0, \gamma)$. By (F2), it is clear that f_γ^+ satisfies (II)*, and since c_0 is continuous at γ, we have $c_0(\gamma) \in L^+(f_\gamma^+)$, whence (III)′ is satisfied. Thus, $P(\gamma)$ is true in $(0, \beta)$.

Assume that $P(\gamma)$ is true in $(0, \gamma_0)$, $\gamma_0>0$. Define $f_{\gamma_0}^+$ by $f_{\gamma_0}^+(t)=f_\gamma^+(t)$ for all $t \in [0, \gamma)$, $\gamma<\gamma_0$. It is easy to see that $f_{\gamma_0}^+$ is well-defined. Thus $f_{\gamma_0}^+$ is a mapping of $[0, \gamma_0)$ into X. If $L^+(f_{\gamma_0}^+)=\phi$, then $f^+ \equiv f_b^+$ satisfies (II)* and (III)* if $b=\gamma_0$. If $L^+(f_{\gamma_0}^+) \neq \phi$, then $P(\gamma)$ is true at γ_0, or in $(0, \gamma_0]$. Let $x_0 \in L^+(f_{\gamma_0}^+)$. Then there eixst $U \in \mathcal{V}(x_0)$ and $\delta>0$ satisfying (F2). Now, $x_0 \in L^+(f_{\gamma_0}^+)$ implies that there exists $\tau \in \left(\gamma_0-\frac{\delta}{2}, \gamma_0\right)$ with $f_{\gamma_0}^+(\tau) \in U$. By (F2), there exists $c \in F$ such that $c(\tau)=f_{\gamma_0}^+(\tau)$ and $(\tau-\delta, \tau+\delta) \subset I_c$, which implies $c(t)=f_{\gamma_0}^+(t)$ in $I_c \cap I_{f_{\gamma_0}^+}$. Let $0<\varepsilon<\frac{\delta}{2}$ and define $f_{\gamma_0+\varepsilon}^+$ by

$$f_{\gamma_0+\varepsilon}^+(t)=\begin{cases} f_{\gamma_0}^+(t) & \text{if } 0 \leqq t<\gamma_0, \\ c(t) & \text{if } \gamma_0 \leqq t<\gamma_0+\varepsilon. \end{cases}$$

Then, $f_{\gamma_0+\varepsilon}^+$ satisfies (II)* and (III)*, since the continuity of $c(t)$ at $\gamma_0+\varepsilon$ implies $c(\gamma_0+\varepsilon) \in L^+(f_{\gamma_0+\varepsilon}^+)$. This completes the proof of Theorem 1.

Definition 2. Let F be an F-family on X, and let $\boldsymbol{F}$ be the uniquely determined $\boldsymbol{F}$-family which is strongly equivalent to F according to Theorem 1. Then $\boldsymbol{F}$ is said to be *generated by* F.

Theorem 2. *Let F and G be two F-families on the same space X. Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be the $\boldsymbol{F}$-families generated by F and G respectively. If F and G are strongly equivalent, then $\boldsymbol{F}=\boldsymbol{G}$. Further, if X is Hausdorff, then equivalence of F and G implies $\boldsymbol{F}=\boldsymbol{G}$*[7].

Proof. Use Lemma 3 and Lemma 3*.

Proposition 3. *Let $\boldsymbol{F}$ be the $\boldsymbol{F}$-family on X, generated by an F-family F on X. For every $f \in \boldsymbol{F}$ and every compact interval $[\tau_0, \tau_1] \subset I_f$, there exists a finite subfamily $\{c_1, c_2, \cdots, c_n\}$ of F such that for every $t_0 \in [\tau_0, \tau_1]$ there is some c_k, $k=1, 2, \cdots, n$, with $f(t)=c_k(t)$ in* in I_{c_k}.

Proof. Let $\tau \in [\tau_0, \tau_1]$. Then by (II) of Lemma 1 and (F2), there exists $c_\tau \in F$ with $\tau \in I_{c_\tau}$ and $f(t)=c_\tau(t)$ in I_{c_τ}. Thus, $\{I_{c_\tau} \mid \tau \in [\tau_0, \tau_1]\}$ is an open covering of $[\tau_0, \tau_1]$ which is compact. Hence, there is a finite subset $\tau_1, \cdots, \tau_n$ of $[\tau_0, \tau_1]$ such that $\{I_{c_{\tau_k}} \mid k=1, \cdots, n\}$ covers $[\tau_0, \tau_1]$. Thus, $\{c_{\tau_1}, \cdots, c_{\tau_n}\}$ is the required subset of F.

7) See the remark following the proof of Proposition 4 of §I.

2. Local Dynamical Systems.

Definition 3. Let X be a topological space, $\mathcal{D}$ a subset of $X\times R$ and π a mapping of $\mathcal{D}$ into X. If the following conditions $(\pi 0)$ through $(\pi 3)$ or $(\pi 4)$ are satisfied, then π is called a *local dynamical system* (see [3], [7], [9]).

$(\pi 0)$ D is open in $X\times R$ and can be expressed as

$$\mathcal{D}=\bigcup_{x\in X}\{x\}\times\mathcal{J}_x,$$

where $\mathcal{J}_x$ denotes an open interval containing 0.

$(\pi 1)$ *Identity*: $\pi(x,0)=x$ for all $x\in X$.

$(\pi 2)$ *Homomorphism (stronger)*: If $(x,t)\in\mathcal{D}$, and either $(x,t+s)\in\mathcal{D}$ or $(\pi(x,t),s)\in\mathcal{D}$, then $(\pi(x,t),s)\in\mathcal{D}$ and $(x,t+s)\in\mathcal{D}$ and $\pi(\pi(x,t),s)=\pi(x,t+s)$.

$(\pi 3)$ *Separate Continuity*: For every $x\in X$, π_x is continuous, and for every $t\in R$, π^t is continuous (when it is defined).

$(\pi 4)$ *Continuity*: π is continuous on $\mathcal{D}$.

If π satisfies $(\pi 0)$ through $(\pi 3)$, then it is called a *local separately continuous dynamical system*. If π satisfies $(\pi 0)$ through $(\pi 4)$, then it is called a *local continuous dynamical system* (abbreviated as a *local system*).

If $\mathcal{J}_x=R$ for every $x\in X$, i.e. $\mathcal{D}=X\times R$, then π is said to be *global*.

We note that a local separately continuous dynamical system is a germ in which the domain $\mathcal{D}$ is an *open* neighborhood of $X\times\{0\}$. A local system is a germ in which $\mathcal{D}$ is open and $(\pi 4)$ holds.

Proposition 4. *Let X be a topological space and $\mathcal{D}\subset X\times R$. Then a mapping π of $\mathcal{D}$ into X is a local system if and only if π satisfies* $(\pi 0)$, $(\pi 1)$, $(\pi 4)$ *and the following two conditions*:

$(\pi 2)'$ *Homomorphism*: *If $(x,t)\in D$, $(x,t+s)\in D$ and $(\pi(x,t),s)\in D$, then $\pi(\pi(x,t),s)=\pi(x,t+s)$.*

$(\pi 3)'$ *Non-extendability*: *If $\mathcal{J}_x=(a,b)$ and a* (or b) *is finite, then the cluster set $L^-(x)$ (or $L^+(x)$), as $t\downarrow a$* (or $t\uparrow b$), *is empty.*

Proof. See [10].

The essential equivalence of (F5) and $(\pi 4)$ was demonstrated in Process I and Process II. Thus, we make the following conclusive remark:

Remark. By applying Process I to a given $\boldsymbol{F}$-family considered as an F-family, one obtains a germ μ which, in view of Hajek's Theorem, determines a unique local system strongly equivalent to μ. By applying Process II to a local system considered as a germ, one obtains an F-family F which in view of Theorem I, § II, determines a unique $\boldsymbol{F}$-family strongly equivalent to F.

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