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Fractional Integration and Inversion Formulae Associated with the Generalized Whittaker Transform*

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Abstract

In the present paper, we invoke the theories of the Mellin transform as well as fractional integration to investigate a solution of the integral equation

$$(1)\qquad \int_0^{\infty}(xt)^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k+\frac{1}{2},m}(xt)f(t)\,dt=W_{k,m}^{(\sigma)}\{f(t):x\},\ x>0,$$

which defines a generalized Whittaker transform of the unknown function $f(t)\in L_2(0,\infty)$ to be determined in terms of its image $W_{k,m}^{(\sigma)}\{f(t):x\}$.

It is shown that under certain constraints (1) can be reduced to the form of a Laplace integral which is readily solvable by familiar techniques.

1. Introduction.

Meijer [9] has given a generalization of the well-known Laplace transform (cf., e.g., [18])

$$(1.1)\qquad L[f(t):x]=\int_0^{\infty}e^{-xt}f(t)\,dt$$

by means of the integral equation

$$(1.2)\qquad \mathrm{M}_{k,m}\{f(t):x\}=\int_0^{\infty}(xt)^{-k-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k+\frac{1}{2},\,m}(xt)f(t)\,dt,$$

while a generalization of (1.1) due to Varma [15, p. 209] is in the form

$$(1.3)\qquad V_{k,m}\{f(t):x\}=\int_0^{\infty}(xt)^{m-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k,m}(xt)f(t)\,dt,$$

where $W_{k,m}(z)$ represents the Whittaker function [13, p. 14]

$$(1.4)\qquad W_{k,m}(z)=e^{-\frac{1}{2}z}\sum_{m,-m}\frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-k-m\right)}z^{m+\frac{1}{2}}{}_1F_1\left[\frac{1}{2}-k+m;\,1+2m;\,z\right]$$

in terms of Kummer's confluent hypergeometric function

$${}_1F_1[a;b;z]=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

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*) For a brief research announcement of the main results of the present paper see Pacific J. Math. **26** (1968) 357-377.

with, as usual,

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1).$$

When $k=\pm m$, (1.2) reduces to (1.1) and the reducibility of (1.3) to (1.1) in the special case $k+m=\frac{1}{2}$ is quite straightforward, since [3, p. 264]

(1.5) $$W_{\frac{1}{2}-m,\pm m}(z)=e^{-\frac{1}{2}z}z^{\frac{1}{2}-m}.$$

More than a decade ago, Erdélyi [1] pointed out that the kernel of (1.2) can be expressed as a fractional integral of e^{-xt} in terms of the operators of fractional integration introduced earlier by Kober (cf. [7]) and Erdélyi [2], while recently Fox [5] has made an appeal to Erdélyi-Kober's theory (cf. also [4]) to derive an inversion formula for the integral transform with nucleus $x^{\nu}K_{\nu}(x)$, where $K_{\nu}(x)$ is the modified Bessel function of the second kind [16, p. 78]. We investigate here a solution of the integral equation

(1.6) $$W_{k,m}^{(\sigma)}\{f(t):x\}=\int_0^{\infty}(xt)^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k+\frac{1}{2},m}(xt)f(t)dt, x>0,$$

which defines a generalized Whittaker transform [8, p. 23] of the unknown function $f(t)\in L_2(0,\infty)$ to be determined in terms of its image $W_{k,m}^{(\sigma)}\{f(t):x\}$, so that since

(1.7) $$W_{k,m}^{(-k)}\{f(t):x\}=M_{k,m}\{f(t):x\}$$

and

(1.8) $$W_{k-\frac{1}{2},m}^{(m)}\{f(t):x\}=V_{k,m}\{f(t):x\},$$

our results would readily enable us to invert the integral transforms given earlier by Meijer (cf. [1], [10]), and Varma (cf. [11], [12]).

In what follows we shall make a free use of the theories of fractional integration as well as the Mellin transform, and therefore we find it worthwhile to describe them briefly in the next section.

2. Fractional Integration and Mellin Transform.

In the familiar notation, the operators of fractional integration introduced by Kober [7] are defined as follows (cf. also [2]):

(2.1) $$I_{\eta,\alpha}^{(+)}f(x)=\frac{1}{\Gamma(\alpha)}x^{-\eta-\alpha}\int_0^x(x-u)^{\alpha-1}u^{\eta}f(u)du,$$

(2.2) $$K_{\zeta,\alpha}^{(-)}f(x)=\frac{1}{\Gamma(\alpha)}x^{\zeta}\int_x^{\infty}(u-x)^{\alpha-1}u^{-\zeta-\alpha}f(u)du,$$

where $f(x)\in L_p(0,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$, if $1<p<\infty$, and $\frac{1}{q}$ or $\frac{1}{p}=0$ if p or $q=1$,

$\alpha>0,\ \eta>-\dfrac{1}{q},\ \zeta>-\dfrac{1}{p}.$

Confining ourselves to the L_2-space theory, for simplicity of the conditions involved, so that l. i. m. indicates the limit in mean with index 2, it is readily seen that [14, p. 94] if

$$f(x)\in L_2(0,\infty),$$

then

$$(2.3)\qquad M\{f(x)\}=\underset{N\to\infty}{\text{l. i. m.}}\int_{1/N}^{N}f(x)x^{s-1}dx=F(s)\in L_2\left(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty\right),$$

and that if

$$F(s)\in L_2\left(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty\right),$$

then

$$(2.4)\qquad M^{-1}\{F(s)\}=\frac{1}{2\pi i}\underset{N\to\infty}{\text{l. i. m.}}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN}F(s)x^{-s}ds=f(x)\in L_2(0,\infty),$$

where M and M^{-1} represent Mellin and the inverse Mellin transforms respectively.

By investigating the asymptotic behaviour of the gamma functions occurring in the known formula [6, p. 114]

$$(2.5)\qquad M\left\{z^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}z}W_{k+\frac{1}{2},m}(z)\right\}=\frac{\Gamma(s+\sigma+m)\Gamma(s+\sigma-m)}{\Gamma(s+\sigma-k)},$$

where $s=\beta+i\delta$, β and δ being real, and $\beta+\sigma\pm m>0$, it can be shown that

(i) $M\left\{z^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}z}W_{k+\frac{1}{2},m}(z)\right\}$ is bounded on the line $s=\dfrac{1}{2}+i\delta$, where $-\infty<\delta<\infty$, and since, in view of (1.4),

$$e^{-\frac{1}{2}z}W_{k+\frac{1}{2},m}(z)$$

is an entire function, finite at the origin, such that [17, p. 343]

$$(2.6)\qquad W_{k+\frac{1}{2},m}(z)=e^{-\frac{1}{2}z}z^{k+\frac{1}{2}}\{1+0(z^{-1})\},$$

when $|z|$ is large and $|\arg(z)|<\pi$, it follows that

$$(\text{ii})\qquad z^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}z}W_{k+\frac{1}{2},m}(z)\in L_2(0,\infty).$$

When (i) and (ii) hold, from Fox's lemma (cf. [5], p. 458) we have

Lemma I. *If* $x>0$, $m\geqq 0$, *and* $f(x)\in L_2(0,\infty)$, *then*

$$(2.7)\qquad \int_0^\infty (xt)^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}xt}W_{k+\frac{1}{2},m}(xt)f(t)dt\in L_2(0,\infty)$$

and

(2.8)
$$M\left\{\int_0^\infty (xt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k+\frac{1}{2},m}(xt)f(t)dt\right\} = \frac{\Gamma(s+\sigma+m)\Gamma(s+\sigma-m)}{\Gamma(s+\sigma-k)} F(1-s)$$

wherein the integrals are regarded as functions of x.

Naxt we return to Kober's operator (2.2) and observe that under the sufficient conditions of its validity, viz.

(2.9) $\alpha > 0,\ S > -\frac{1}{2}$ assuming that $f(x) \in L_2(0, \infty)$,

we also have

(2.10) $$K^{(-)}_{\zeta,\alpha} f(x) \in L_2(0, \infty)$$

and [7, p. 203]

(2.11) $$M\{K^{(-)}_{\zeta,\alpha} f(x)\} = \frac{\Gamma(s+\zeta)}{\Gamma(s+\zeta+\alpha)} M\{f(x)\}.$$

From (2.7), (2.8) and (2.11) it is easy to see that

$$M\left\{K^{(-)}_{\sigma-k,\alpha}\int_0^\infty (xt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k+\frac{1}{2},m}(xt)f(t)dt\right\}$$
$$= \frac{\Gamma(s+\sigma+m)\Gamma(s+\sigma-m)}{\Gamma(s+\sigma-k+\alpha)} F(1-s)$$
$$= M\left\{\int_0^\infty (xt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k-\alpha+\frac{1}{2},m}(xt)f(t)dt\right\},$$

in view of (2.5), and therefore

Lemma II. *If* $x > 0,\ \alpha \geqq 0,\ \frac{1}{2}+\sigma-k > 0,$ *and* $f(x) \in L_2(0, \infty)$, *then*

(2.12)
$$K^{(-)}_{\sigma-k,\alpha}\int_0^\infty (xt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k+\frac{1}{2},m}(xt)f(t)dt = \int_0^\infty (xt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k-\alpha+\frac{1}{2},m}(xt)f(t)dt,$$

where the integrals are regarded as functions of x.

3. The Inversion Theorems.

We first establish the following theorem which exhibits the fact that under certain constraints the integral equation (1.6) can be reduced to (1.1) which is readily solvable by familiar techniques.

Theorem I. *Let f(x) be a solution of*

(3.1) $$\int_0^\infty (xt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k+\frac{1}{2},m}(xt)f(t)dt = W_{k,m}^{(\sigma)}\{f(t):x\}$$

which belongs to $L_2(0,\infty)$. *Then*

(3.2) $$f(x)=L^{-1}[K_{\sigma-k,\sigma+k}^{(-)}W_{k,m}^{(\sigma)}\{f(t):x\}],$$

provided (i) $x>0$, (ii) $\sigma+k\geqq 0$ *and* (iii) $\frac{1}{2}+\sigma-k>0$.

Proof. Since the conditions of Lemmas I and II are satisfied, we may operate upon (3.1) by means of

$$K_{\sigma-k,\sigma+k}^{(-)}$$

and by virtue of (2.12) and (1.5) we have

$$\int_0^\infty e^{-xt}f(t)dt = K_{\sigma-k,\sigma+k}^{(-)}W_{k,m}^{(\sigma)}\{f(t):x\},$$

whence (3.2) follows immediately.

Theorem II. *Let* $f(x)\in L_2(0,\infty)$ *and*

(3.3) $$L[f(t):x]=\int_0^\infty e^{-xt}f(t)dt.$$

Then

(3.4) $$K_{\sigma+m,\alpha}^{(-)}\, x^{\sigma-m}L[t^{\sigma-m}f(t):x]=W_{-m-\alpha,m}^{(\sigma)}\{f(t):x\},$$

provided (i) $x>0$, (ii) $\alpha\geqq 0$ *and* (iii) $\frac{1}{2}+\sigma+m>0$.

Proof. In view of (1.5) we have

$$x^{\sigma-m}L[t^{\sigma-m}f(t):x]=\int_0^\infty (xt)^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{\frac{1}{2}-m,m}(xt)f(t)dt,$$

and operating upon both sides by

$$K_{\sigma+m,\alpha}^{(-)},$$

we get (3.4) under the conditions stated earlier.

Finally we remark that when $\alpha=-k-m$, (3.4) is reduced to the interesting relationship

(3.5) $$W_{k,m}^{(\sigma)}\{f(t):x\}=K_{\sigma+m,-k-m}^{(-)}\, x^{\sigma-m}L[t^{\sigma-m}f(t):x],$$

which leads us to the construction of a table of generalized Whittaker transforms from that of the classical Laplace transforms, provided $k+m\leqq 0$, $\frac{1}{2}+\sigma+m>0$ and $x>0$.

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