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Compact Sets of Nonlinear Operators*

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Abstract

Let $f(x,t)$ be a function defined on $R^n \times R$ with values in R^n, where R^n is Euclidean n-space, and define a nonlinear operator $\boldsymbol{f}$ by $\boldsymbol{f}: x(t) \rightarrow f(x(t),t)$. In this paper we shall consider collections F of such nonlinear operators $\boldsymbol{f}$, and we shall seek conditions under which F is conditionally compact in a suitable topological space. Our results have important applications in the theory of nonlinear integral equations. These applications are discussed in a joint paper of the author and R. K. Miller [4].

1. Notation and Basic Hypotheses.

In the following we shall let I denote an interval in R, the real line and R^m Euclidean m-space with norm $|\cdot|$. All functions referred to below will have range in R^m, the domain will be indicated in parentheses.

$\mathfrak{C}(I)$: Space of continuous functions.

$\mathfrak{PC}(I)$: Space of piece-wise continuous functions.

$\mathfrak{L}_p(I)$, $1 \leqq p \leqq \infty$: Lebesgue space with norm $\|\cdot\|_p$.

$\mathfrak{L}_p^{\mathrm{loc}}(R) = \mathfrak{L}_p^{\mathrm{loc}}$: Local $\mathfrak{L}_p$-space.

That is, $\mathfrak{L}_p^{\mathrm{loc}}$ consists of all functions $x(\cdot)$ defined on R such that $x(\cdot)$ is in $\mathfrak{L}_p(I)$ for every compact interval I. The spaces $\mathfrak{L}_p(I)$ will have the usual topology generated by the norm $\|\cdot\|_p$, $1 \leqq p \leqq \infty$. $\mathfrak{PC}(I)$ and $\mathfrak{C}(I)$ will have the topology of uniform convergence on compact subsets. (If I is compact, then $\mathfrak{C}(I)$ is a subspace of $\mathfrak{L}_\infty(I)$.) $\mathfrak{L}_p^{\mathrm{loc}}$ will have the topology of convergence in each $\mathfrak{L}_p(I)$, where I is compact.

Let $f: R^n \times R \rightarrow R^m$ be given. We shall say that f satisfies a *(weak) Carathéodory hypothesis for* p, $1 \leqq p < \infty$, if

(i) f is measurable, and

(ii) for every compact set $K \subset R^n$ there is a real-valued function m in $\mathfrak{L}_p^{\mathrm{loc}}$ such that $|f(x,t)| \leqq m(t)$ for $x \in K$ and almost all t in R, and

(iii) for every compact interval $I \subset R$ one has

$$\int_I |f(x_k(t),t) - f(x(t),t)|^p \, dt \rightarrow 0 \qquad (0)$$

whenever $x_k(\cdot) \rightarrow x(\cdot)$ in $\mathfrak{L}_\infty(I)$.

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If $m(\cdot)$ satisfies (ii) we shall say that $m(\cdot)$ is a *bounding function for f on K*. Condition (iii) is sometimes replaced by

(iii′) for almost all t, the function $f(x,t)$ is continuous in x.

We shall say that f satisfies a *(strong) Carathéodory hypothesis* if (i), (ii) and (iii′) are satisfied. It follows from the Lebesgue Dominated Convergence Theorem that the strong Carathéodory hypothesis implies the weak Carathéodory hypothesis.

We shall say that f satisfies an *α-Hölder condition* $(0<\alpha\leqq 1)$ if

(iv) for each compact set $K\subset R^n$ there is a real-valued function k in $\mathfrak{L}_p^{\mathrm{loc}}$ with

$$|f(x,t)-f(y,t)|\leqq k(t)|x-y|^{\alpha} \tag{1}$$

whenever $x, y\in K$ and for almost all t in R.

Any function $k(\cdot)$ satisfying (1) is said to be an *α-Hölder coefficient for f on K*. Since $k(t)$ is finite valued almost everywhere, we see that (iv) implies (iii).

The collection of all functions f that satisfy the weak Carathéodory hypothesis will be denoted by $\mathfrak{C}_p$, and the subcollection of functions that also satisfy an α-Hölder coefficient will be denoted by $\mathfrak{C}_p^{\alpha}$. We note that $\mathfrak{L}_p^{\mathrm{loc}}$ consists of precisely those functions from $\mathfrak{C}_p$ that are independent of x, that is of the form $f(t)$.

For f in $\mathfrak{C}_p$ we define the mapping $\boldsymbol{f}\colon \mathfrak{P}\mathfrak{C}(R)\to\mathfrak{L}_p^{\mathrm{loc}}$ by

$$\boldsymbol{f}(x)\colon\ t\to f(x(t),t). \tag{2}$$

This operator $\boldsymbol{f}$ is very important in the study of nonlinear integral equations, cf. [2], [3] and [4]. It follows from the Carathéodory hypothesis that for every compact interval $I\subset R$, $\boldsymbol{f}$ maps $\mathfrak{P}\mathfrak{C}(I)$ into $\mathfrak{L}_p(I)$ and that f is continuous with respect to the respective norms. In other words, the mapping $\boldsymbol{f}$ given by (2) is continuous.

Let X denote the collection of all continuous mappings of $\mathfrak{P}\mathfrak{C}(R)$ into $\mathfrak{L}_p^{\mathrm{loc}}$. We shall now define two topologies $\mathscr{T}_c$ and $\mathscr{T}_b$ on X. The first topology $\mathscr{T}_c$ is the topology of uniform convergence on compact sets. That is, we say that $\boldsymbol{f}_n\to\boldsymbol{f}$ in $(X,\mathscr{T}_c)$ if $\boldsymbol{f}_n(x)\to\boldsymbol{f}(x)$ uniformly for $x(\cdot)$ in compact sets K in $\mathfrak{P}\mathfrak{C}(R)$. Equivalently, this means that for every compact interval $I\subset R$ one has

$$\int_I|\boldsymbol{f}_n(x)-\boldsymbol{f}(x)|^p dt\to 0 \tag{3}$$

uniformly for $x(\cdot)$ in *compact* sets K in $\mathfrak{P}\mathfrak{C}(I)$. The second topology $\mathscr{T}_b$ is defined in exactly the same way but now the convergence in (3) is to be uniform for $x(\cdot)$ in *bounded* sets K in $\mathfrak{P}\mathfrak{C}(I)$.

These topologies represent nonlinear versions of two standard topologies for bounded linear operators from $\mathfrak{PC}(R)$ to $\mathfrak{L}_p^{\mathrm{loc}}$. In particular, $\mathcal{T}_c$ generalizes the strong topology and $\mathcal{T}_b$ generalizes the uniform topology, cf. [1].

Now let $\mathfrak{F}$ denote a collection of functions from $\mathfrak{C}_p$ and let $\hat{F}$ denote the corresponding collection of nonlinear mappings in X given by (2). In the next three sections we shall seek conditions (necessary and or sufficient) that $\hat{F}$ be conditionally compact in either $(X, \mathcal{T}_c)$ or $(X, \mathcal{T}_b)$.

2. Compact Sets in $\mathfrak{L}_p^{\mathrm{loc}}$.

We note that if $f \in \mathfrak{L}_p^{\mathrm{loc}}$, then the mapping $\boldsymbol{f}$ given by (2) maps every function $x(\cdot)$ into $f(\cdot)$, that is $\boldsymbol{f}$ is a constant mapping. Conversely every constant mapping $\boldsymbol{f}$ of $\mathfrak{PC}(R)$ in $\mathfrak{L}_p^{\mathrm{loc}}$ satisfies (2) for some f in $\mathfrak{L}_p^{\mathrm{loc}}$. In other words, (2) gives a one-to-one correspondence between $\mathfrak{L}_p^{\mathrm{loc}}$ and the constant functions in X. This correspondence is a homemorphism with respect to either $\mathcal{T}_c$ or $\mathcal{T}_b$, that is both topologies agree with the standard topology on $\mathfrak{L}_p^{\mathrm{loc}}$.

Now let $\mathfrak{F}$ be a collection in $\mathfrak{L}_p^{\mathrm{loc}}$. We now give a necessary and sufficient condition that $\mathfrak{F}$ be conditionally compact.

Theorem 1. *Let $\mathfrak{F} \subset \mathfrak{L}_p^{\mathrm{loc}}$, $1 \leqq p < \infty$. Then $\mathfrak{F}$ is conditionally compact if and only if*

(a) *for every compact interval $I \subset R$ there is a real number B such that $\int_I |f(t)|^p dt \leqq B$ for all f in $\mathfrak{F}$, and*

(b) *for every compact interval $I \subset R$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|\tau| \leqq \delta$ then*

$$\int_I |f(t+\tau) - f(t)|^p dt \leqq \varepsilon$$

for all f in $\mathfrak{F}$.

Proof. The proof of this is based on the fact that $\mathfrak{F}$ is conditionally compact in $\mathfrak{L}_p^{\mathrm{loc}}$ if and only if for each compact interval $I \subset R$, the restriction of $\mathfrak{F}$ to I is conditionally compact in $\mathfrak{L}_p(I)$. Now let $I = [a, b]$ be fixed. For each f in $\mathfrak{F}$ we define f^* by

$$\begin{aligned} f^*(t) &= f(t), \quad && t \in I \\ &= 0, && t \notin I. \end{aligned}$$

Let $\mathfrak{F}^*$ denote the collection of these f^*. Then $\mathfrak{F}^* \subset \mathfrak{L}_p(R)$ and we see that $\mathfrak{F}$ is conditionally compact in $\mathfrak{L}_p(I)$ if and only if $\mathfrak{F}^*$ is conditionally compact in $\mathfrak{L}_p(R)$.

Now assume that (a) and (b) hold. One then has

$$\int_R |f^*(t)|^p dt = \int_I |f(t)|^p dt \leqq B.$$

Also for $\tau \geqq 0$ one has

$$\int_R |f^*(t+\tau)-f^*(t)|^p dt \leqq \int_{a-\tau}^{a} |f(t+\tau)|^p dt$$

$$+\int_I |f(t+\tau)-f(t)|^p dt+\int_{b-\tau}^{b} |f(t)|^p dt$$

with a similar expression valid for $\tau<0$. By combining (a) and (b) we see that for every $\varepsilon>0$ there is an $\eta>0$ such that

$$\int_R |f^*(t+\tau)-f^*(t)|^p dt \leqq \varepsilon$$

whenever $|\tau| \leqq \eta$. By a well-known characterization of conditionally compact sets in $\mathfrak{L}_p(R)$, cf. [1; p. 298], we see that $\mathfrak{F}^*$ is conditionally compact in $\mathfrak{L}_p(R)$.

A simple verification establishes the converse statement. We shall omit the details. Q. E. D.

3. Compact sets in $\mathfrak{E}_p$.

In studying sets $\mathfrak{F}$ in $\mathfrak{E}_p$, it becomes necessary to distinguish between the two topologies $\mathscr{T}_c$ and $\mathscr{T}_b$. But before doing this we need another concept.

We have already observed that for every f in $\mathfrak{G}_p$, the bounding function $m(\cdot)$ lies in $\mathfrak{L}_p^{\mathrm{loc}}$. Also for f in $\mathfrak{G}_p^\alpha$, the α-Hölder coefficient $k(\cdot)$ lies in $\mathfrak{L}_p^{\mathrm{loc}}$. Both $m(\cdot)$ and $k(\cdot)$ depend not only on f but also on the compact set $K \subset R^n$. Moreover, even with f and K fixed it is evident that neither the bounding function nor the α-Hölder coefficient are uniquely determined.

Let $\mathfrak{F}$ be a collection of functions from $\mathfrak{E}_p$. We shall say that $\mathfrak{F}$ has *compact bounds* if for every compact set $K \subset R^n$ one can choose a bounding function $m(\cdot)$ for f on $K (f \in \mathfrak{F})$ such that the collection $\{m\}$ is conditionally compact in $\mathfrak{L}_p^{\mathrm{loc}}$. For $F \subset \mathfrak{E}_p^\alpha$ we shall say that $\mathfrak{F}$ has *compact α-Hölder coefficients* if for every compact set $K \subset R^n$ one can choose an α-Hölder coefficient $k(\cdot)$ for f on $K (f \in \mathfrak{F})$ such that the set $\{k\}$ is conditionally compact in $\mathfrak{L}_p^{\mathrm{loc}}$.

Let us now turn to the question of conditionally compact sets $\hat{F}$ in $(X, \mathscr{T}_c)$.

Theorem 2. *Let $\mathfrak{F} \subset \mathfrak{E}_p$ and assume that $\mathfrak{F}$ has compact bounds. Let $\hat{F}$ be the corresponding collection of mappings generated by (2). Then $\hat{F}$ is conditionally compact in $(X, \mathscr{T}_c)$ if and only if*

(a) *for every compact interval $I \subset R$, every $x(\cdot)$ in $\mathfrak{P}\mathfrak{C}(R)$ and every $\varepsilon>0$ there is a $\delta>0$ such that if $|\tau| \leqq \delta$ then*

$$\int_I |f(x(t+\tau), t+\tau)-f(x(t), t)|^p dt \leqq \varepsilon \tag{4}$$

for all f in $\mathfrak{F}$, and

(b) *the family of mappings $\boldsymbol{f}: \mathfrak{P}\mathfrak{C}(R) \rightarrow \mathfrak{L}_p^{\mathrm{loc}}$, $f \in \hat{F}$, is equicontinuous at* *each point of* $\mathfrak{P}\mathfrak{C}(R)$.

Proof. This is essentially a formulation of Ascoli's Theorem [6] to the family $\hat{F}$. Because of Theorem 1 and the assumption of compact bounds we see that condition (a) is equivalent to the following statement: "For each $x(\cdot)$ in $\mathfrak{P}\mathfrak{C}(R)$, the set $\{f(x): f\in\hat{F}\}$ is conditionally compact in $\mathfrak{L}_p^{\mathrm{loc}}$."
Statement (b) is the equicontinuity statement for $\hat{F}$, so this result is a direct application of Ascoli's Theorem as asserted. Q. E. D.

We have thus shown that $Cl\hat{F}$ (the closure of $\hat{F}$ in $(X, \mathscr{T}_c)$) is a compact set. This dose *not* necessarily mean that every mapping in $Cl\hat{F}$ is generated by a function from $\mathfrak{G}_p$. (The author thinks that this is true, but he is unable to verify it. In order to get the stronger conclusion one would have to show that the limit Ψ of any sequence $\{f_n\}$ from $\hat{F}$ is generated by a function f from $\mathfrak{G}_p$. A natural candidate for f is given by

$$f(x,t)=\Psi(x(\quad))$$

where $x(t)=x$ is a constant function.)

In the sequel we shall need a technically stronger version of Theorem 2. The problem arises when we study functions f that depend on a parameter h. Assume that we are given a function $f(x,h,t)$ where $f\colon R^n\times H\times R\rightarrow R^m$ and H is a locally compact metric space. The Carathéodory hypothesis is then augmented so that the bounding function m can be chosen independent of h in compact sets in H, the limit in (0) is uniform for h in compact sets and one assumes enough continuity in h so that the mapping $f\colon \mathfrak{P}\mathfrak{C}(R)\times H\rightarrow\mathfrak{L}_p^{\mathrm{loc}}$ given by

$$f(x(\cdot),h): t\rightarrow f(x(t),h,t) \tag{5}$$

is continuous. The α-Hölder condition is also changed so that the α-Hölder coefficient k can be chosen independent of h in compact sets.

Let $\mathfrak{G}_p$ now denote the collection of all $f(x,h,t)$ that satisfy this augmented Carathéodory hypothesis and let $\mathfrak{G}_p^{\alpha}$ now denote those functions that satisfy the augmented α-Hölder condition. Let X now denote the space of all continuous mappings of $\mathfrak{P}\mathfrak{C}(R)\times H$ into $\mathfrak{L}_p^{\mathrm{loc}}$. The analogue of Theorem 2 is now valid in this case. We will not include a formal statement, but only note that the δ in statement (a) may depend on h and that (4) becomes

$$\int_I |f(x(t+\tau),h,t+\tau)-f(x(t),h,t)|^p\,dt\leqq\varepsilon.$$

4. Compact sets in $\mathfrak{G}_p^{\alpha}$.

We return now to the original formulation given in terms of functions

$f(x,t)$. The next theorem, which is our main result, is a sufficient condition for conditional compactness in the $\mathscr{T}_b$-topology.

Theorem 3. *Let* $\mathfrak{F}\subset\mathfrak{S}_p^\alpha$ *and assume that* $\mathfrak{F}$ *has compact bounds and compact* α*-Hölder coefficients. The set* $\hat{F}$ *is conditionally compact in* $(X, \mathscr{T}_b)$ *if*

(a) *for every compact interval* $I\subset R$, *every* $x(\cdot)$ *in* $\mathfrak{PC}(R)$ *and every* $\varepsilon>0$ *there is a* $\delta>0$ *such that if* $|\tau|\leqq\delta$, *then*

$$\int_I |f(x(t+\tau), t+\tau)-f(x(t),t)|^p\,dt\leqq\varepsilon$$

for all f *in* $\mathfrak{F}$.

Furthermore, every mapping $\boldsymbol{g}$ *in* $Cl\hat{F}$ *satisfies* (2) *for some function* g *in* $\mathfrak{S}_p^\alpha$.

Proof. Part 1: Let us first show that $\hat{F}$ is conditionally compact in the weaker topology $\mathscr{T}_c$. That is, we want to show that the equicontinuity condition (b) of Theorem 2 is satisfied. Let I be a compact interval and $x(\cdot)$ in $\mathfrak{PC}(I)$. Let K be a compact set in R^n that contains the graph of $x(t)$, $t\in I$, in its interior. For each $f\in\mathfrak{F}$ choose the α-Hölder coefficient $k(\cdot)$ for f so that the family $\{k\}$ is conditionally compact in $\mathfrak{L}_p^{\mathrm{loc}}$. Since the mapping $k\to\int_I |k(t)|^p\,dt$ is a continuous mapping of $\mathfrak{L}_p^{\mathrm{loc}}$ into R, it follows that the set $\{k\}$ is mapped into a bounded set in R. This means that there is a b, $0\leqq b<\infty$, such that $\int_I |k(t)|^p\,dt\leqq b$ for all $k\in\{k\}$. If $y(t)$ is a piece-wise continuous function with graph in K one then has

$$\int_I |f(x(t),t), -f(y(t),t)|^p dt\leqq b \text{ ess.}\sup\{|x(t)-y(t)|^{p\alpha}:\ t\in I\}$$

for every f in $\mathfrak{F}$. Hence $\hat{F}$ is equicontinuous at $x(\cdot)$. This means that $Cl\hat{F}$ (the closure in $\mathscr{T}_c$) is a compact set in $(X,\mathscr{T}_c)$.

We will now show the following:

1. Every mapping $\boldsymbol{g}$ in $Cl\hat{F}$ satisfies (2) for some g in $\mathfrak{S}_p^\alpha$.
2. The set $Cl\hat{F}$ is compact in $(X,\mathscr{T}_b)$.

Part 2: To prove the first statement, we choose $\boldsymbol{g}$ in $Cl\hat{F}$. Then there is a sequence $\{f_n\}$ in $\mathfrak{F}\subset\mathfrak{S}_p^\alpha$ such that $\boldsymbol{f}_n\to\boldsymbol{g}$ in $\mathscr{T}_c$. Let K be a compact set in R^n. We can assume without any loss of generality that the sequence $\{f_n\}$ is chosen in such a way that the corresponding sequence of bounding functions $\{m_n\}$ and α-Hölder coefficients $\{k_n\}$ converge in $\mathfrak{L}_p^{\mathrm{loc}}$ to limits m and k respectively. Now define $g(x,t)$ by

$$g(x,t)=\boldsymbol{g}(x(t))$$

where $x(t)=x$ is a constant function. If $x\in K$, then

$$\int_I |g(x,t)|^p dt=\lim\int_I |f_n(x,t)|^p dt \leqq \lim\int_I m_n(t)^p dt=\int_I m(t)^p\, dt$$

for every compact interval I. Hence $|g(x,t)|\leqq m(t)$ almost everywhere, so m is a bounding function for g on K. Similarly if $x, y\in K$, then

$$\begin{aligned}\int_I |g(x,t)-g(y,t)|^p dt&=\lim\int_I |f_n(x,t)-f_n(y,t)|^p dt\\ &\leqq \lim\int_I k_n(t)^p |x-y|^{p\alpha} dt\\ &=\int_I k(t)^p |x-y|^{p\alpha} dt\end{aligned}$$

for every compact interval I. Hence

$$|g(x,t)-g(y,t)|\leqq k(t)|x-y|^{\alpha}$$

almost everywhere, so k is a α-Hölder coefficient for g on K. Hence $g\in\mathfrak{G}_p^{\alpha}$.

It now remains to show that for every $x(\cdot)$ in $\mathfrak{P}\mathfrak{C}(R)$ one has

$$\boldsymbol{g}(x(\ ))=g(x(\cdot),\cdot) \tag{6}$$

in $\mathfrak{L}_p^{\mathrm{loc}}$. It is clear, from the definition of g, that (6) holds for every piece-wise constant function $x(\cdot)$. By taking limits (in $\mathfrak{P}\mathfrak{C}(R)$) of piece-wise constant functions, we see that (6) holds for every $x(\cdot)$ in $\mathfrak{P}\mathfrak{C}(R)$.

Part 3: Let us now show that $Cl\hat{F}$ is compact in $(X, \mathscr{T}_b)$. We shall do this as follows: Let $\{f_n\}$ be a sequence in $\mathfrak{F}\subset\mathfrak{G}_p^{\alpha}$ and $f\in\mathfrak{G}_p^{\alpha}$ be given with $\boldsymbol{f}_n\to\boldsymbol{f}$ in $\mathscr{T}_c$. We will now show that there is a subsequence of $\{\boldsymbol{f}_n\}$ that converges to $\boldsymbol{f}$ in $\mathscr{T}_b$. To do this we shall use the augmented version of Theorem 2 discussed in the last section and the corresponding augmented version of Parts 1 and 2 of this proof.

Choose β so that $0<\beta<\alpha$, and let $\gamma=\alpha-\beta$. Now define $g_n(x,h,t)$ by

$$\begin{aligned}g_n(x,h,t)&=\frac{f_n(x+h,t)-f(x,t)}{h^{\beta}}, \quad & h\neq 0\\ &=0 \quad &, \quad h=0,\end{aligned}$$

for $x\in R^n$, $h\in R^n=H$ and $t\in R$.

It follows from a routine calculation that g_n satisfies a γ-Hölder condition in x. Also, if k_n is a α-Hölder coefficient for f_n, then $2k_n$ is a γ-Hölder coefficient for g_n. In other words the sequence $\{g_n\}$ lies in the augmented class $\mathfrak{G}_p^{\gamma}$. The equicontinuity of the corresponding family $\boldsymbol{g}_n$ of mappings from $\mathfrak{P}\mathfrak{C}(R)\times R^n$ into $\mathfrak{L}_p^{\mathrm{loc}}$ is established in a manner similar to that used in Part 1 of this proof. The only change is to note that the family $\{g_n(x,h,t)\}$ is equicontinuous in h for x and t fixed. It then follows from the augmented version of Theorem 2, that we can find a subsequence of $\{\boldsymbol{g}_n\}$ — call it $\{\boldsymbol{g}_n\}$ — that is convergent, say to $\boldsymbol{g}$. Using the argument of Part 2 it is easy to see that there is a g in the augmented $\mathfrak{G}_p^\gamma$ such that

$$\boldsymbol{g}(x(\cdot), h) = g(x(\cdot), h, \cdot)$$

for all h in R^n. It is easy to see that

$$g(x, h, t) = \frac{f(x+h, t) - f(x, t)}{h^\beta}.$$

If we let K be any compact set in R^n and let x be fixed, then

$$\sup_{h \in K} | g_n(x, h, t) - g(x, h, t) | = l_n(t), \tag{7}$$

and $l_n \to 0$ in $\mathfrak{L}_p^{\mathrm{loc}}$. Now choose K to be a compact set containing the origin and set $x=0$ in (7). One then has for $h \in K$

$$f_n(h, t) - f(h, t) = [g_n(0, h, t) - g(0, h, t)] h^\beta + f_n(0, t) - f(0, t).$$

Hence

$$| f_n(h, t) - f(h, t) | \leqq l_n(t) | h |^\beta + | f_n(0, t) - f(0, t) |. \tag{8}$$

If we replace h with $h(t)$ is a piece-wise continuous function with range in K, then (8) implies that $f_n \to f$ in $\mathscr{T}_b$, which completes the proof.

A useful corollary of this result arises when the functions $f(x, t)$ in $\mathfrak{F}$ are C^1 in the x variable. Before stating the result we note that the derivative $f_x(x, t)$, with components $\partial f_i / \partial x_j$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq n)$, can be viewed as a mapping of $R^n \times R$ into R^r where $r = nm$.

Corollary. *Let* $\mathfrak{E} \subset \mathfrak{G}_p$ *and assume that* $\mathfrak{F}$ *has compact bounds and that each* $f(x, t)$ *in* $\mathfrak{F}$ *is* C^1 *in the* x*-variable. Let* $\mathfrak{F}'$ *denote the collection of derivatives* $f_x(x, t)$, $f \in \mathfrak{F}$, *and assume that* $\mathfrak{F}'$ *has compact bounds. Assume further that for every compact interval* $I \subset R$, *every* $x(\cdot)$ *in* $\mathfrak{P}\mathfrak{C}(R)$ *and every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $|\tau| \leqq \delta$, *then*

$$\int_I | f(x(t+\tau), t+\tau) - f(x(t), t) |^p \, dt \leqq \varepsilon$$

for all f *in* $\mathfrak{F}$. *Then* $\hat{F}$ *is conditionally compact in* $(X, \mathscr{T}_b)$.

Proof. We note first that $\mathfrak{F} \subset \mathfrak{C}_p^1$ and next that $\mathfrak{F}$ has compact Lipschitz (l-Hölder) coefficients since $\mathfrak{F}'$ has compact bounds. The proof now follows from Theorem 3.

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