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Linear System and its Perturbed System

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In studying the asymptotic behavior of solutions of the system

$$\frac{dx}{dt}=Ax+f(t, x) \tag{1}$$

with a constant matrix A, many authors (cf. [2], [5]) have used the following idea: The matrix A is assumed to be of form $\operatorname{diag}(A_1, A_2)$, where all of the characteristic roots of A_1 have negative real parts and those of A_2 have non-negative real parts. Let $\operatorname{diag}(X_1(t), X_2(t))$ be a fundamental matrix of the linear system

$$\frac{dx}{dt}=Ax$$

such that $X_i(t)$ corresponds to A_i $(i=1,2)$. Then, each solution $x(t)$ of (1) is represented by

$$x(t)=\begin{pmatrix} X_1(t) & 0 \\ 0 & X_2(t)\end{pmatrix}c+\int_0^t\begin{pmatrix} X_1(t-s) & 0 \\ 0 & 0\end{pmatrix}f(s, x(s))ds \\ -\int_t^\infty\begin{pmatrix} 0 & 0 \\ 0 & X_2(t-s)\end{pmatrix}f(s, x(s))ds \tag{2}$$

with a constant vector c, if the last integral exists.

Thus, there arises a question whether, for a given c, there exists a solution of (1) which satisfies (2) or the relation

$$\left\| x(t)-\begin{pmatrix} X_1(t) & 0 \\ 0 & X_2(t)\end{pmatrix}c\right\|\to 0 \quad \text{as} \quad t\to\infty.$$

This question is considered to be a generalization of the concept of the asymptotic integration constant discussed by Wintner [6] and is closely connected with the asymptotic equivalence (cf. [1]).

In this article, we shall give an answer to this question, by applying a special type of boundary value problem considered by Hukuhara [3]. This paper is based on Kato's paper titled "Asymptotic behaviors in functional differential equations" which will be published in Tohoku Math. J. (**18** (1966), 174–215).

Consider a linear system and its perturbed system

$$\frac{dx}{dt}=Ax+p(t) \tag{3}$$

and

$$\frac{dx}{dt}=Ax+p(t)+f(t, x), \tag{4}$$

where A is an $n\times n$ constant matrix and $p(t)$, $f(t,x)$ are continuous on $I\times R^n$, (I: $[0,\infty)$, R^n: a Euclidean n-space).

The following assumptions will be made;

(i) all solutions of (3) are bounded,

(ii) $\|f(t,x)\|\leqq\lambda(t)\varphi(\|x\|)$, where $\lambda(t)\geqq 0$ is continuous, $\int_0^\infty \lambda(t)dt<\infty$ and $\varphi(r)>0$ for $r\geqq 0$, $\varphi(r)$ is continuous, increasing, $\int^\infty \frac{dr}{\varphi(r)}=\infty$.

First of all, we shall transform (3) and (4) into

$$\frac{dy}{dt}=Ay \tag{5}$$

and

$$\frac{dy}{dt}=Ay+g(t,y) \tag{6}$$

by $x=y+x(t)$, where $x(t)$ is a solution of (3), which starts at $t=0$, and $g(t,y)=f(t,y+x(t))$. Since all solutions of (3) are bounded, there exists a $b>0$ such that $\|x(t)\|\leqq b$ for all $t\geqq 0$ and all characteristic roots of A have non-positive real parts and elementary divisors of characteristic roots which have zero real parts are linear. Therefore, there exists a non-singular constant matrix P such that

$$A^*=P^{-1}AP=\mathrm{diag}\,(B,0,C_1,\cdots,C_k),$$

where all characteristic roots of B have negative real parts, 0 is zero matrix and C_i is of form $\begin{pmatrix}0 & -\lambda_i\\ \lambda_i & 0\end{pmatrix}$, $\lambda_i>0$. Then, by $y=Pz$, the system (5) is transformed into

$$\frac{dz}{dt}=A^*z. \tag{7}$$

We set $Q_i(t)=\begin{pmatrix}\cos\lambda_i t & -\sin\lambda_i t\\ \sin\lambda_i t & \cos\lambda_i t\end{pmatrix}$ and set $Q(t)=\mathrm{diag}\,(E,Q_1,\cdots,Q_k)$, where E is the unit matrix which has the same degree as $\mathrm{diag}\,(B,0)$. Then, clearly $\det Q(t)\equiv 1$. We assume that B is an $l\times l$ matrix, 0 is an $m\times m$ matrix, and hence $k=\dfrac{n-(l+m)}{2}$. Let ξ be an l-vector and let η be an $(n-l)$-vector.

By $z=Q(t)\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, the system (7) is transformed into

$$\frac{d\xi}{dt}=B\xi,\quad \frac{d\eta}{dt}=0, \tag{8}$$

and by $y=PQ(t)\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, the system (6) is transformed into

$$\frac{d\xi}{dt}=B\xi+F_1(t,\xi,\eta),\quad \frac{d\eta}{dt}=F_2(t,\xi,\eta), \tag{9}$$

where $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = Q^{-1}(t)P^{-1}g\left(t, PQ(t)\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right)$. Clearly, there exists a $K>0$ such that $\|Q^{-1}(t)P^{-1}\| \leqq K$, $\|PQ(t)\| \leqq K$. If we consider a function $\psi(r)=\varphi(r+b)$, we have $\|g(t,y)\| \leqq \lambda(t)\psi(\|y\|)$, which implies that

$$(10) \qquad \|F_i(t,\xi,\eta)\| \leqq K\lambda(t)\psi(K\{\|\xi\|+\|\eta\|\}), \quad i=1,2,$$

and we have $\int^{\infty} \frac{dr}{\psi(Kr)} = \infty$. Therefore, for (8) and (9), the assumptions (i), (ii) hold good.

First of all, we shall prove the following lemmas.

Lemma 1. *If the assumptions above are satisfied, every solution* $y(t\,;y_0,t_0)$ *of* (6) *is continuable to the left.*

Proof. Since every solution of (5) is continuable to the left and (5) is a linear system, corrsponding to any $T>0$, there exists a continuous function $W(t,y)$ defined on $[0,T]\times R^n$ which satisfies the conditions

$$\|y\| \leqq W(t,y),$$
$$|W(t,y)-W(t,y')| \leqq M\|y-y'\|, \quad M>0,$$
$$W'(t,y) = \varliminf_{h\to+0} \frac{1}{h}\{W(t+h,\ y+hAy)-W(t,y)\} \geqq 0,$$

see Remark in Section 3 in [7]. Therefore, we have

$$W'(t,y(t\,;y_0,t_0)) \geqq -M\lambda(t)\psi(W(t,y(t\,;y_0,t_0))).$$

Comparing $W(t,y(t\,;y_0,t_0))$ with the maximal solution $u(t)$ of

$$\frac{du}{dt} = -M\lambda(t)\psi(u), \quad u(t_0)=W(t_0,y_0),$$

we have $W(t,y(t\,;y_0,t_0)) \leqq u(t)$ for any t, $0 \leqq t \leqq t_0$, and thus $y(t\,;y_0,t_0)$ is continuable to the left, because $u(t)$ is continuable.

Lemma 2. *Under the assumptions above, a solution* $(\xi(t),\ \eta(t))$ *of* (9) *is bounded. Moreover,* $\xi(t)\to 0$ *as* $t\to\infty$ *and* $\lim_{t\to\infty}\eta(t)$ *exists. Thus, for any given solution of* (9), *there exists a solution of* (8) *which tends to the given solution of* (9).

Proof. Since the solutions of (8) are uniform-bounded and the system is linear, by Theorem 19.1 with $c=0$ in [7], there exists a Liapunov function $V(t,\xi,\eta)$ defined on $I\times R^n$ and satisfying the conditions:

(iii) $\|\xi\|+\|\eta\| \leqq V(t,\xi,\eta) \leqq M\{\|\xi\|+\|\eta\|\}$, where $M>0$ is a constant,

(iv) $|V(t,\xi,\eta)-V(t,\xi',\eta')| \leqq M\{\|\xi-\xi'\|+\|\eta-\eta'\|\}$,

(v) $V'_{(8)}(t,\xi,\eta) \leqq 0$.

First of all, we shall see that a solution $(\xi(t),\eta(t))$ of (9) is bounded. By (iii), (iv), (v) and (10),

$$V'(t,\xi(t),\eta(t)) \leqq 2MK\lambda(t)\psi(K(\|\xi(t)\|+\|\eta(t)\|))$$

$$\leqq 2MK\lambda(t)\psi(KV).$$

By comparing $V(t, \xi(t), \eta(t))$ with the maximal solution $u(t)$ of

$$\frac{du}{dt} = 2MK\lambda(t)\psi(Ku), \quad u(t_0) = V(t_0, \xi(t_0), \eta(t_0)),$$

we can see that for any $r > 0$, there exists an $R > 0$ such that if $\|\xi(t_0)\| + \|\eta(t_0)\| \leqq r$, then $\|\xi(t)\| + \|\eta(t)\| \leqq R$ for $t \geqq t_0$, see Example 10.2 in [7].

Next, we shall see that $\xi(t) \to 0$ as $t \to \infty$ and $\lim_{t\to\infty} \eta(t)$ exists. As the solution $\xi = 0$ of $\dfrac{d\xi}{dt} = B\xi$ is exponential-asymptotically stable, there exist constants $M \geqq 1$, $c > 0$ and a Liapunov function $W(t, \xi)$ such that

$$\|\xi\| \leqq W(t, \xi) \leqq M\|\xi\|, \quad |W(t, \xi) - W(t, \xi')| \leqq M\|\xi - \xi'\|$$

and that $W'_{(*)}(t, \xi) \leqq -cW(t, \xi)$, where $(*)$ denotes the system $\dfrac{d\xi}{dt} = B\xi$. For a solution $\xi(t)$ of (9), consider a function $W(t, \xi(t))$. Assume that $\|\xi(t_0)\| + \|\eta(t_0)\| \leqq r$. From

$$W'(t, \xi(t)) \leqq -cW(t, \xi(t)) + MK\lambda(t)\psi(KR),$$

it follows that

$$\|\xi(t)\| \leqq e^{-c(t-t_0)} W(t_0, \xi(t_0)) + MK\psi(KR)e^{-ct} \int_{t_0}^{t} e^{cs}\lambda(s)ds,$$

which implies that $\|\xi(t)\| \to 0$ as $t \to \infty$. Since we have

$$\eta(t) = \eta(t_0) + \int_{t_0}^{t} F_2(s, \xi(s), \eta(s))ds$$

and

$$\int_{t_0}^{t} \|F_2(s, \xi(s), \eta(s))\| ds \leqq K\psi(KR) \int_{t_0}^{t} \lambda(s)ds < \infty,$$

we can see the existence of a constant $\eta(\infty)$ such that

$$\eta(\infty) = \eta(t_0) + \int_{t_0}^{\infty} F_2(s, \xi(s), \eta(s))ds,$$

or

$$\eta(t) = \eta(\infty) - \int_{t}^{\infty} F_2(s, \xi(s), \eta(s))ds.$$

Thus, for any given solution $(\xi(t; \xi_0, \eta_0, t_0), \eta(t; \xi_0, \eta_0, t_0))$ of (9), if we take a solution $(\xi^*(t; \xi_0^*, t_0), \eta^*(t; \eta(\infty), t_0))$ of (8), where ξ_0^* is arbitrary, $\xi^*(t; \xi_0^*, t_0) \to 0$ as $t \to \infty$ and $\eta^*(t; \eta(\infty), t_0) = \eta(\infty)$, and hence

$$\|\xi(t) - \xi^*(t)\| \to 0, \ \|\eta(t)\| \to 0 \quad \text{as} \quad t \to \infty,$$

that is, for any given solution of (9), there exists a solution of (8) which approaches the given solution of (9).

The following lemma is concerned with a kind of boundary value problem which Hukuhara applied to studying of behavior of solutions in a neighborhood of a singular point [3]. Here we shall state in a simple form, which we shall apply to the proof of Theorem. For the proof, see [4].

Lemma 3. *In a system*

$$\text{(11)} \qquad \begin{cases} \dfrac{dx}{dt} = f(t, x, y) \\ \dfrac{dy}{dt} = g(t, x, y), \end{cases}$$

where x, y are n, m-vectors, suppose that $f(t, x, y)$, $g(t, x, y)$ are continuous and bounded on $[a, b] \times R^n \times R^m$. Then, for any $(x_0, y_0) \in R^n \times R^m$, there exists at least one solution $(x(t), y(t))$ of (11) *which is defined on $[a, b]$ and satisfies $x(b) = x_0$, $y(a) = y_0$.*

Theorem. *Under the assumption* (i), (ii), *the systems* (3) *and* (4) *are asymptotically equivalent, that is, for any given solution of* (3) *there exists a solution of* (4) *which approaches the given solution of* (3) *as $t \to \infty$, and conversely.*

Proof. It is sufficient to see the asymptotic equivalence of the systems (8) and (9). Let $(\xi_1(t), \eta_1(t))$ be a given solution of (8), and there is a $B_0 > 0$ such that $B_0 \geqq \sup\{||\xi_1(t)||, ||\eta_1(t)||; t \geqq 0\}$. As we saw in the proof of Lemma 2, we have a Liapunov function $W(t, \xi)$. Let t_0 be chosen so that

$$\text{(12)} \qquad \int_{t_0}^{\infty} \lambda(t)\, dt < \frac{B_0}{MK\psi(4KB_0)}.$$

Now we define $F_1^*(t, \xi, \eta)$ and $F_2^*(t, \xi, \eta)$ by replacing (ξ, η) in F_1, F_2 by $\min\left\{1, \dfrac{4B_0}{||\xi|| + ||\eta||}\right\}(\xi, \eta)$. Moreover let $B^*(\xi)$ be such that

$$B^*(\xi) = \begin{cases} B\xi, & \text{if } ||\xi|| \leqq 2B_0 \\ \dfrac{2B_0}{||\xi||} B\xi, & \text{if } ||\xi|| > 2B_0. \end{cases}$$

Then, clearly these functions are continuous and bounded on $[t_0, T] \times R^l \times R^{n-l}$, where T is arbitrary, and we have

$$||F_1^*(t, \xi, \eta)|| \leqq K\lambda(t)\psi(4KB_0)$$

$$||F_2^*(t, \xi, \eta)|| \leqq K\lambda(t)\psi(4KB_0)$$

on $[t_0, \infty) \times R^l \times R^{n-l}$.

Consider a system

$$\text{(13)} \qquad \begin{cases} \dfrac{d\xi}{dt} = B^*(\xi) + F_1^*(t, \xi, \eta) \\ \dfrac{d\eta}{dt} = F_2^*(t, \xi, \eta). \end{cases}$$

By Lemma 3, for any $s > t_0$ and a fixed ξ_0 such that $||\xi_0|| \leqq \dfrac{M}{B_0}$, there exists a solution $(\xi^*(t; s), \eta^*(t; s))$ of (13) which satisfies $\xi^*(t_0; s) = \xi_0$ and $\eta^*(s; s) =$

$\eta_1(s)$. Since $\|\eta_1(s)\| \leqq B_0$ and we have

$$\|\eta^*(t\,;\,s)\| \leqq \|\eta_1(s)\| + \int_t^s K\lambda(u)\psi(4KB_0)du,$$

from (12), it follows that $\|\eta^*(t\,;\,s)\| \leqq 2B_0$ for all $t \in [t_0, s]$. As long as $\|\xi^*\| \leqq 2B_0$,

(14) $$W'(t, \xi^*(t\,;\,s)) \leqq -cW(t, \xi^*(t\,;\,s)) + MK\lambda(t)\psi(4KB_0),$$

and hence

$$W(t, \xi^*(t\,;\,s)) \leqq W(t_0, \xi_0) + MK\psi(4KB_0)\int_{t_0}^t \lambda(u)du$$
$$\leqq M\|\xi_0\| + B_0 \leqq 2B_0.$$

Thus, we have $\|\xi^*(t\,;\,s)\| \leqq 2B_0$ for all $t \in [t_0, s]$. Therefore, $(\xi^*(t\,;\,s), \eta^*(t\,;\,s))$ is a solution of (9) on $[t_0, s]$. By Lemma 2, there exists a solution $(\xi(t\,;\,s), \eta(t\,;\,s))$ of (9) defined on $[t_0, \infty)$ which satisfies $\xi(t_0\,;\,s) = \xi_0$ and $\eta(s\,;\,s) = \eta_1(s)$, and there exists a $R > 0$ such that $\|\xi(t\,;\,s)\| + \|\eta(t\,;\,s)\| \leqq R$ for all $t \geqq t_0$, because $\|\xi(t_0\,;\,s)\| = \|\xi_0\| \leqq B_0$, $\|\eta(t_0;\,s)\| \leqq 2B_0$. Thus, the family of functions $(\xi(t;s), \eta(t\,;\,s)\,;\,s \geqq t_0)$ is uniformly bounded and equicontinuous on any compact subinterval of $[t_0, \infty)$. Therefore, there exist a divergent sequence $\{s_k\}$ and a sequence $(\xi(t\,;\,s_k), \eta(t\,;\,s_k))$ which converges to a function $(\xi(t), \eta(t))$ defined on $[t_0, \infty)$ uniformly on any compact subinterval of $[t_0, \infty)$. Clearly, $(\xi(t), \eta(t))$ is a solution of (9) and $\|\xi(t)\| + \|\eta(t)\| \leqq R$ for all $t \geqq t_0$.

As we saw in Lemma 2, $\xi(t) \to 0$ as $t \to \infty$, and hence $\xi(t) \to \xi_1(t)$ as $t \to \infty$, because $\xi_1(t) \to 0$ as $t \to \infty$. Since we have

$$\|\eta(t\,;\,s) - \eta_1(t)\| \leqq K\psi(4KB_0)\int_t^s \lambda(u)du$$

for $t_0 \leqq t \leqq s$, if for any $\varepsilon > 0$, we chose a $\tau(\varepsilon) > 0$ such that

$$\int_{\tau(\varepsilon)}^\infty \lambda(u)du < \frac{\varepsilon}{K\psi(4KB_0)},$$

we have $\|\eta(t\,;\,s) - \eta_1(t)\| < \varepsilon$ for all $t \in [\tau(\varepsilon), s]$. This implies that

$$\|\eta(t) - \eta_1(t)\| \leqq \varepsilon \quad \text{for all} \quad t \geqq \tau(\varepsilon),$$

which shows $\eta(t) \to \eta_1(t)$ as $t \to \infty$.

Thus, the proof above and Lemma 2 show the asymptotic equivalence of the systems (3) and (4). Here it is noticed that, by Lemma 1, we can take the same initial time.

Remark. If $Y(t)$ denotes a fundemental matrix of the system (5), then for any constant c, $Y(t)c$ is a solution of (5). Therefore, this theorem shows the existence of a solution $y(t)$ of the system (6) , which satisfies

$$\|y(t) - Y(t)c\| \to 0 \quad \text{as} \quad t \to \infty$$

for any given constant c.

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