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Semi-decomposable Systems and Vector Subfunctions

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Introduction

In the previous notes [4, 5] the present author introduced the notion of subfunctions for single (nonlinear) elliptic or parabolic equations of the second order. Here, in this note, we introduce the notion of vector-valued subfunctions for especially simple second order systems of elliptic or parabolic type. In these systems dependent variables are only loosely connected with each other so that we may call them semi-decomposable systems[1].

For semi-decomposable systems there have been an increasing number of articles these several years[2]. So it appears sufficient, to the author, only to remark the fact that we can treat systems in just the same way as we can do single equations; namely, by the subfunction method.

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1. Semi-decomposable elliptic systems

In this section we are concerned with a system of the form

$$(1)\qquad F^k[\mathbf{u}]\equiv\sum_{i,j=1}^{N}a_{ij}^{k}(x,u^k,p^k)u_{x_ix_j}^{k}-f^k(x,\mathbf{u},p^k)=0\qquad(k=1,\cdots,N),$$

where x is a point in a subdomain G of the Euclidean n-space and $\mathbf{u}=(u^1,\cdots,u^N)$ is an unknown N-vector function, while $p^k=(u_{x_1}^k,\cdots,u_{x_N}^k)$ is the gradient vector derived from u^k $(k=1,\cdots,N)$. In the sequel we shall write (1) in the form

$$(2)\qquad \mathbf{F}[\mathbf{u}]=\mathbf{0}$$

and use this abbreviated form (2) when convenient.

DEFINITION 1: The system (1) is called a *semi-decomposable elliptic system in G* if the following conditions are satisfied:

1) For details see §§ 1 and 6.

2) We refer the reader to T. Kusano [7] and the papers cited there.

i) The matrices $\|a^k_{ij}(x, u^k, p^k)\|$ are continuous and positive-definitein $\mathfrak{D}_k$: $x\in G$, $|u^k|<\infty$, $|p^k|=\left\{\sum_{i=1}^N (p_i^k)^2\right\}^{\frac{1}{2}}<\infty$.

ii) $f^k(x, \mathbf{u}, p^k)$ are continuous in $\tilde{\mathfrak{D}}_k$: $x\in G$, $|u^j|<\infty$ $(j=1, \cdots, N)$, $|p^k|<\infty$.

iii) $f^k(x, \mathbf{u}, p^k)$ are non-increasing with respect to all u^j $(j\neq k)$ $(k=1, \cdots, N)$.

A semi-decomposable elliptic system is, in many points, similar to a single elliptic equation. This statement will be made clear in later sections.

2. F-subfunctions

In this section we shall generalize the notion of subfunctions so as to be suitable for the system (1).

DEFINITION 2: An N-vector function $\mathbf{v}(x)=(v^1(x), \cdots, v^N(x))$ [$\mathbf{w}(x)=(w^1(x), \cdots, w^N(x))$] is called an **F**-*subfunction in* G [an **F**-*subfunction in* G], if the following requirements are met:

i) $\mathbf{v}(x)\in\mathbf{C}(G)$[3], [$\mathbf{w}(x)\in\mathbf{C}(G)$].

ii) For each point x_0 of G there exist an open neighborhood U of x_0 and an N-vector function $\mathbf{V}(x)\in\mathbf{C}^2(U)$[4] [$\mathbf{W}(x)\in\mathbf{C}^2(\mathbf{U})$] such that

a) $\mathbf{V}(x)\leqq\mathbf{v}(x)$[5] [$\mathbf{W}(x)\geqq\mathbf{w}(x)$] in U,

b) $\mathbf{V}(x_0)=\mathbf{v}(x_0)$ [$\mathbf{W}(x_0)=\mathbf{w}(x_0)$], and

c) $\mathbf{F}[\mathbf{V}](x_0)\geqq\mathbf{0}$ [$\mathbf{F}[\mathbf{W}](x_0)\leqq\mathbf{0}$].

By strengthening the requirements iic) we obtain the following

DEFINITION 3: An **F**-subfunction $\mathbf{v}(x)$ in G [an **F**-superfunction $\mathbf{w}(x)$ in G] is said to be a *strong* **F**-*subfunction in* G [*a strong* **F**-*subfunction in* G] if the function $\mathbf{V}(x)$ [$\mathbf{W}(x)$], associated with $\mathbf{v}(x)$ and $x_0\in G$ [with $\mathbf{w}(x)$ and $x_0\in G$], satisfies the differential inequality

c′) $\mathbf{F}[\mathbf{V}]\geqq\mathbf{0}$ [$\mathbf{F}[\mathbf{W}]\leqq\mathbf{0}$] in U[6].

First we shall state and prove the following

THEOREM 1: *Let* $\mathbf{v}_1(x)$ *and* $\mathbf{v}_2(x)$ *be (strong)* **F**-*subfunctions in* G. *Then, so is the function* $\mathbf{v}(x)\equiv\max(\mathbf{v}_1(x), \mathbf{v}_2(x))$. *Here* $\max(\mathbf{v}_1(x), \mathbf{v}_2(x))$ *is the vector function* $(\max(v_1{}^1(x), v_2{}^1(x)), \cdots, \max(v_1{}^N(x), v_2{}^N(x)))$.

A similar statement for **F**-superfunctions is also valid if we replace "maxi-

3) This means $v^k(x)\in C(G)$ for every k. $1\leqq k\leqq N$.
4) I.e. $V^k(x)\in C^2(U)$ $(k=1, \cdots, N)$.
5) I.e. $V^k(x)\leqq v^k(x)$ for every k, $1\leqq k\leqq N$.
6) There is a variant in this definition. For details we refer the reader to K. AKÔ [4-II].

mum" by "minimum".

PROOF. Let x_0 be an arbitrary point of G. And let $\mathbf{V}_1(x)$ and $\mathbf{V}_2(x)$ be respectively N-vector functions in $\mathbf{C}^2(U_1)$ and $\mathbf{C}^2(U_2)$, U_i $(i=1, 2)$ being two neighborhood of x_0, which satisfy iia)–iic) when U is replaced by U_1 and U_2 respectively.

We define an N-vector function $\mathbf{V}(x)$ in $\mathbf{C}^2(U_1 \cap U_2)$ by

$$V^k(x) \equiv \begin{cases} \mathbf{V}_1{}^k(x) & \text{when} \quad v^k(x_0)=v_1{}^k(x_0), \\ \mathbf{V}_2{}^k(x) & \text{when} \quad v^k(x_0) \neq v_1{}^k(x_0). \end{cases} \tag{3}$$

Then it is clear that $\mathbf{V}(x) \leqq \mathbf{v}(x)$ in $U_1 \cap U_2$ and $\mathbf{V}(x_0)=\mathbf{v}(x_0)$. Moreover, by the definition of semi-decomposable elliptic systems

$$F^k[\mathbf{V}](x_0) \geqq F^k[\mathbf{V}_i](x_0) \geqq 0 \quad \text{for} \quad V^k(x)=V_i{}^k(x)$$

which concludes the inequality $\mathbf{F}[\mathbf{V}](x_0) \geqq \mathbf{0}$. Thus the fact that $\mathbf{v}(x)$ is an $\mathbf{F}$-subfunction in G has been established. Quite similarly, we can prove that if $\mathbf{v}_1(x)$ and $\mathbf{v}_2(x)$ are strong $\mathbf{F}$-subfunctions, so is $\mathbf{v}(x)$.

THEOREM 2: *Let $\mathbf{v}_1(x)$ be an (a strong) $\mathbf{F}$-subfunction in G and let $\mathbf{v}_2(x)$ be an (a strong) $\mathbf{F}$-subfunction in G', G' being a subdomain of G. We assume further that $\mathbf{v}_2(x) \in \mathbf{C}(\bar{G}' \cap G)$. If $\mathbf{v}_1(x) \leqq \mathbf{v}_2(x)$ in G' and if $\mathbf{v}_1(x)=\mathbf{v}_2(x)$ in the part of the boundary of G' contained in G, then the N-vector function $\mathbf{v}(x)$, defined by*

$$\mathbf{v}(x) \equiv \begin{cases} \mathbf{v}_1(x) & \text{for} \quad x \in G-G', \\ \mathbf{v}_2(x) & \text{for} \quad x \in G', \end{cases} \tag{4}$$

is also an (a strong) $\mathbf{F}$-subfunction in G.

A similar statement for $\mathbf{F}$-superfunctions is valid.

PPOOF. If $x_0 \in G'$, then the function $\mathbf{V}_2(x)$, associated with $\mathbf{v}_2(x)$ and x_0, satisfies the requirements i)–ii) of Definition 2 also for the function $\mathbf{v}(x)$. If $x_0 \in G-G'$, then the function $\mathbf{V}_1(x)$, associated with $\mathbf{v}_1(x)$ and x_0, satisfies the requirements i) and ii) of Definition 2 also for the function $\mathbf{v}(x)$. Hence we obtain the theorem.

3. Subfunction method

According to Theorems 1 and 2 we can discuss the Dirichlet-type boundary-value problem for a semi-decomposable elliptic system (1) in just the same way as in K. Akô [2, 4, 5]. The essential hypotheses are:

i) The presence of suitable a priori-estimates for solutions of (1).

ii) The existence of an $\mathbf{F}$-subfunction $\mathbf{v}(x)$ and an $\mathbf{F}$-superfunction $\mathbf{w}(x)$ in G such that $\mathbf{v}(x) \leqq \mathbf{w}(x)$ in G.

iii) For the prescribed boundary-values $\varphi(x)$

$$\overline{\lim_{y\to x}}\, \mathbf{v}(y) \leqq \varphi(x) \leqq \underline{\lim_{y\to x}}\, \mathbf{w}(y)^{7)} .$$

Then, we can choose a condidate for the Dirichlet-type boundary-value problem

$$(5) \qquad \begin{cases} \mathbf{F}[\mathbf{u}]=\mathbf{0} & \text{in } G, \\ \quad \mathbf{u}=\varphi & \text{on } \Gamma, \end{cases}$$

Γ being the boundary of G, in the following way.

Let $\mathfrak{F}_*$ be the family of $\mathbf{F}$-subfunctions $\mathbf{v}_1(x)$ in G, such that i) $\mathbf{v}_1(x) \leqq \mathbf{w}(x)$ in G and ii) $\overline{\lim}_{y\to x} \mathbf{v}_1(y) \leqq \varphi(x)$ on Γ. Then the N-vector function $\mathbf{u}_1(x) \equiv \sup \mathbf{v}_1(x)$ ($\mathbf{v}_1(x) \in \mathfrak{F}_*$) might be proved to be a solution of the system $\mathbf{F}[\mathbf{u}]=\mathbf{0}$ in G, if the hypotheses i)–iii) are satisfied. This vector function $\mathbf{u}_1(x)$ might be seen a solution of the problem (5) if we could construct "barrier" functions for our problem[8]. And if $\mathbf{u}_1(x)$ is known a solution of the problem (5), it is the "maximal" solution of it.

Quite similarly, we can construct a candidate $\mathbf{u}_2(x)$ of a solution of (5) by using a family of $\mathbf{F}$-superfunctions. And, if $\mathbf{u}_2(x)$ is seen a solution of (5), then it is the "minimal" solution. For details we refer the reader to K. Akô [2, 4, 5].

4. Semi-decomposable elliptic linear systems

Let us consider the following linear system

$$(6) \quad F^k[\mathbf{u}] \equiv \sum_{i,j=1}^{n} a_{ij}^k(x) u_{x_i x_j}^k + \sum_{i=1}^{n} b_i^k(x) u_{x_i}^k + \sum_{h=1}^{N} c^{kh}(x) u^h = f^k(x) \quad (k=1, \cdots, N)$$

in a domain G whose boundary Γ has no cusp. If all the functions a_{ij}^k, b_i^k, c^{kh}, and f^k are bounded and Hölder-continuous in G, and if the N matrices $\|a_{ij}^k(x)\|$ are strictly positive-definite in G[9], then we have the following alternative for the Dirichlet problem

$$(7) \qquad \begin{cases} \mathbf{F}[\mathbf{u}]=\mathbf{f} & \text{in } G, \\ \quad \mathbf{u}=\mathbf{0} & \text{on } \Gamma. \end{cases}$$

THEOREM 3: *Either the problem* (7) *with* $\mathbf{f}=0$ *has a non-trivial solution* $\mathbf{u}(x)$ ($\neq \mathbf{0}$) *in* $\mathbf{C}^2(G) \cap \mathbf{C}(\bar{G})$, *or the problem* (7) *has a unique solution* $\mathbf{u}(x)$ *in* $\mathbf{C}^2(G) \cap \mathbf{C}(\bar{G})$ *for every* $\mathbf{f}(x)$.

The method of proof is as follows.

First we choose λ_0 so large that $|c^{kh}(x)| \leqq \lambda_0$ ($k, h=1, \cdots, N$) in G and solve

7) Here $\overline{\lim}_{y\to x} \mathbf{v}(y) = (\overline{\lim}_{y\to x} v^1(y), \cdots, \overline{\lim}_{y\to x} v^N(y))$, etc.

8) As for the barrier functions for single equations see K. Akô [4-II, 5].

9) I.e. for any real n-tuple $\xi=(\xi_1, \cdots, \xi_n)$

$$\sum_{i,j=1}^{n} a_{ij}^k(x) \xi_i \xi_j \geqq A \sum_{i=1}^{n} (\xi_i)^2,\ A>0 \qquad (k=1, \cdots, N).$$

the problem

$$(8)\qquad \begin{cases} F_0^k[\mathbf{u}] \equiv F^k[\mathbf{u}] + \lambda_0(u^1 + \cdots + u^N - 2Nu^k) = f^k, & \text{in } G, \\ \mathbf{u} = \mathbf{0} \quad \text{on the boundary } \Gamma \text{ of } G. \end{cases}$$

The problem (8) may be solved easily and uniquely. We denote this solution $\mathbf{u}(x)$ by $(\mathfrak{M}\mathbf{f})(x)$. Then the problem (7) can be written as

$$(9)\qquad (\mathbf{I} + \lambda_0 \mathfrak{N})\mathbf{u} = \mathbf{g},$$

where $\mathbf{g} \equiv \mathfrak{M}\mathbf{f}$ and $\mathfrak{N}\mathbf{u} \equiv \mathfrak{M}\mathbf{u}^*$, $\mathbf{u}^*$ being an N-vector whose k^{th} component is $2Nu^k - \sum_{k=1}^N u^k$.

Thus we may restrict ourselves to the functional equation

$$(10)\qquad (\mathbf{I} + \lambda \mathfrak{N})\mathbf{u} = \mathbf{g}$$

in the Banach space $\mathrm{C}(\bar{G})$[10]. As in K. Akô [2], we can verify that (a suitable extension of) the operator $\mathfrak{N}$ is completely continuous in $\mathrm{C}(\bar{G})$ and hence we obtain Theorem 3 by virtue of the Riesz–Schauder theory in Banach spaces. For further details we refer the reader to K. Akô [3, 6].

5. A maximum principle

In this section we are concerned with the case where the functions $a_{ij}^k(x, u^k, p^k)$ and $f^k(x, \mathbf{u}, p^k)$ satisfy the Lipschitz condition. In this case we have a maximum principle formulated below[11].

THEOREM 4: *Let* $\mathbf{v}(x)$ *and* $\mathbf{w}(x)$ *be respecitively a strong* $\mathbf{F}$*-subfunction and a strong* $\mathbf{F}$*-superfunction in* G. *Moreover, we assume that the inequality* $\mathbf{v}(x) \leqq \mathbf{w}(x)$ *is valid in* G. *Then we have the following statements*:

i) *If* $v^{k_\lambda}(x_0) = w^{k_\lambda}(x_0)$ $(\lambda = 1, \cdots, N_0, N_0 < N)$ *and* $v^{k_\mu}(x_0) < w^{k_\mu}(x_0)$ $(\mu = N_0 + 1, \cdots, N)$, *then* $v^{k_\lambda}(x) \equiv w^{k_\lambda}(x)$ *and* $v^{k_\mu}(x) < w^{k_\mu}(x)$ *in* G.

ii) *If* $\mathbf{v}(x_0) = \mathbf{w}(x_0)$ *at some point* $x_0 \in G$, *then* $\mathbf{v}(x) \equiv \mathbf{w}(x)$ *in* G.

PROOF. Let $\mathbf{V}(x)$ and $\mathbf{W}(x)$ be two N-vector functions, associated respectively with $(\mathbf{v}(x), x_0)$ and with $(\mathbf{w}(x), x_0)$. Then, since $\mathbf{V} \leqq \mathbf{W}$ and since grad $V^{k_\lambda} =$ grad W^{k_λ} at x_0, $\sum_{i,j=1}^n a_{ij}^{k_\lambda}(x, V^{k_\lambda}, \text{grad } V^{k_\lambda}) V_{x_i x_j}^{k_\lambda} \geqq f^{k_\lambda}(x, \mathbf{V}, \text{grad } V^{k_\lambda})$ $\geqq f^{k_\lambda}(x, \mathbf{W}, \text{grad } W^{k_\lambda}) \geqq \sum_{i,j=1}^n a_{ij}^{k_\lambda}(x, W^{k_\lambda}, \text{grad } W^{k_\lambda})$ at the point x_0. Hence we have $V^{k_\lambda}(x) \equiv W^{k_\lambda}(x)$ in U by virtue of a maximum principle for single equations[12]. The validity of the identity $v^{k_\lambda}(x) \equiv w^{k_\lambda}(x)$ in U and hence in G is clear,

10) I.e. the space of all N-vector functions $\mathbf{g}(x)$, all of whose components are in $C(\bar{G})$ and whose norms $|\mathbf{g}|$ are $\max_{x \in \bar{G}}\{\sum_{k=1}^N |g^k(x)|\}$.

11) Probably, Osgood conditions imposed on a_{ij}^k and f^k may be sufficient for the following discussions. See R. Redheffer [8, 9].

12) See, e.g., K. Akô [1], Theorem 1.

since G is connected.

By virtue of the argument above we have $v^{k_\mu}(x) < w^{k_\mu}(x)$ in G. Thus we have proved the statement i). The proof of the statement ii) is similar and may be omitted.

At the end of this section we note that our maximum principle is often useful for a variety of problems.

6. Semi-decomposable parabolic systems

DEFINITION 4: A second order system

(11) $$\mathbf{G}[\mathbf{u}]=\mathbf{0},$$

where the differential operator $\mathbf{G}=(G^1, \cdots, G^N)$ is given by

(11′) $$G^k[\mathbf{u}] \equiv \sum_{i,j=1}^{n} a_{ij}^k(x, t, u^k, p^k) u_{x_i x_j}^k - u_t^k - f^k(x, t, \mathbf{u}, p^k) = 0 \qquad (k=1, \cdots, N)$$

is called a *semi-decomposable parabolic system* in an $(n+1)$-dimensional domain G, if it satisfies the following requirements:

i) For each k, $1 \leqq k \leqq N$, the matrix $\|a_{ij}^k(x, t, u^k, p^k)\|$ is continuous and positive-definite in D_k: $(x, t) \in G$, $|u^k| < \infty$, $|p^k| < \infty$, where G is an $(n+1)$-dimensional domain.

ii) For each k, $1 \leqq k \leqq N$, the function $f^k(x, t, \mathbf{u}, p^k)$ is continuous in $\widetilde{D}_k$: $(x, t) \in G, |\mathbf{u}| = \left\{\sum_{k=1}^{N} (u_i^k)^2\right\}^{\frac{1}{2}} < \infty$, $|p^k| < \infty$.

iii) $f^k(x, t, \mathbf{u}, p^k)$ is non-increasing with respect to all u^j $(j \neq k)$.

7. G-subfunctions

We can also define **G**-subfunctions for the system $\mathbf{G}[\mathbf{u}]=\mathbf{0}$ similarly as in § 2. For the sake of completeness we shall formulate these functions definitely.

DEFINITION 5: A **G**-*subfunction* $\mathbf{v}(x, t)$ [**G**-*superfunction* $\mathbf{w}(x, t)$] *in* G is an N-vector function in $\mathbf{C}(G)$, satisfying the following requirements:

i) For each point $(x_0, t_0) \in G$ there exists a finite circular cylinder U $(\subset G)$ with its lateral surface parallel to the t-axis and with its upper and lower bases perpendicular to the t-axis. Here we assume that the point (x_0, t_0) is situated at the center of the upperbasis of U.

ii) There exists an N-vector function $\mathbf{V}(x)[\mathbf{W}(x)]$ in $\mathbf{C}^2(U)$ such that

a) $\mathbf{V}(x, t) \leqq \mathbf{v}(x, t)$ $[\mathbf{W}(x, t) \geqq \mathbf{w}(x, t)]$ in U,

b) $\mathbf{V}(x_0, t_0) = \mathbf{v}(x_0, t_0)$ $\mathbf{W}(x_0, t_0) = \mathbf{w}(x_0, t_0)$, and

c) $\mathbf{G}[\mathbf{V}](x_0, t_0) \geqq \mathbf{0}$ $\mathbf{G}[\mathbf{W}](x_0, t_0) \leqq \mathbf{0}$.

DEFINITION 6: A **G**-subfunction $\mathbf{v}(x, t)$ [**G**-superfunction $\mathbf{w}(x, t)$] in G is said to be a *strong* **G**-*subfunction* [a *strong* **G**-*subfunction*] *in* G if the following condition is met: The function $\mathbf{V}(x,t)$ $[\mathbf{W}(x,t)]$, associated with $\mathbf{v}(x,t)$ and (x_0,t_0) [with $\mathbf{w}(x,t)$ and (x_0,t_0)], satisfies the differential inequality

c′) $\mathbf{G}[\mathbf{V}]\geqq\mathbf{0}$ $[\mathbf{G}[\mathbf{W}]\leqq\mathbf{0}]$ in U.

8. Remarks

As for the Dirichlet-type boundaryvalue problem for the semi-decomposable parabolic system (11) we can discuss the matter in a similar way as in semi-decomposable elliptic systems. For example, the subfunction method of §4 may be employed with some modifications. There are also some maximum principles similar to Theorem 4, but we shall not formulate them here. An interested reader may refer to T. Kusano [7] and the articles cited there.

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