On the Characteristic Boundary Value Problem for a Non-linear Hyperbolic Differential Equation

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Introduction

We consider a non-linear hyperbolic differential equation of the second order

$$(1)\qquad \frac{\partial^2 u}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right)+\sum_{k=1}^{n+1}b_k(x,t,u)\frac{\partial u}{\partial x_k}=f(x,t,u)$$

$$(x_{n+1}=t),$$

where the a_{ij} are defined and of class C^2 in the upper half-space $t\geqq 0$ of an Euclidean $(n+1)$-space E_{n+1}.

Let S be a direct characteristic conoid of the equation (1), say, the direct characteristic conoid with vertex at the origin. Denote by D the domain bounded by the characteristic conoid S and a hyperplane $\Omega:\ t=T(T>0)$.

The characteristic boundary value problem consists in finding a solution of (1) in D which satisfies the characteristic boundary condition

$$(2)\qquad u=\varphi \quad \text{on } S.$$

In a previous paper [1], we have already discussed this problem for a non-linear wave equation of the form (1). The purpose of the present paper is to treat the characteristic boundary value problem (1)-(2) in the same way as in [1]. Namely, we define in section 2 two kinds of weak solutions; one is a usual weak solution of hyperbolic differential equations and the other an interior weak solution. Their equivalence can, however, be proved by the well known fact that the transversal direction to the characteristic surface is tangential to that surface. Section 3 is concerned with the uniqueness of weak solutions. In section 4 we shall prove the existence of an interior weak solution by making use of the method of finite differences. In section 5 we consider the continuous dependence of weak solutions of the non-linear hyperbolic differential equation

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right)+\sum_{k=1}^{n+1}b_k(x,t)\frac{\partial u}{\partial x_k}=f(x,t,u)$$

on the function $f(x,t,0)$ and on the boundary data φ.

It should be remarked that, in general, the existence of the characteristic conoid is guaranteed only locally for the case of variable coefficients a_{ij}, However, our method proves the global existence of a weak solution, i. e., we can take the upper boundary $t=T$ of D arbitrarily large as long as the existence of the characteristic conoid is guaranteed.

1. Preliminaries

We assume throughout the paper that the coefficients a_{ij} of the second derivatives in (1) satisfy the following

Hypotheses on $a_{ij}(x,t)$:

(i) The a_{ij} are defined and of class C^2 in the upper half-space $t\geqq 0$ of E_{n+1}.

(ii) The matrix $||a_{ij}(x,t)||$ is symmetric and uniformly positive definite, i.e.,

$$(1.1)\qquad \alpha|\xi|^2\leqq\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\leqq\beta|\xi|^2\qquad(\alpha,\beta>0)$$

for any real vector $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$, where $|\xi|^2=\sum_{j=1}^{n}\xi_j^2$.

Notations: Let S: $t=\omega(x)$ be the direct characteristic conoid with vertex at the origin. As is well known, S is the locus of the bicharacteristics issuing from the origin which satisfy the system of ordinary differential equations

$$\frac{dx_i}{dt}=\sum_{j=1}^{n}a_{ij}\frac{\partial\omega}{\partial x_j}\qquad(i=1,2,\cdots,n).$$

We denote by $\lambda(x,t)$, or by briefly, by λ the bicharacteristic passing through a point (x,t) of S and define its positive direction as the direction from the origin towards infinity.

Let D be a bounded domain in E_{n+1} surrounded by the characteristic conoid S and a hyperplane Ω: $t=T(>0)$. Denote by $S(T)$ that (bounded) portion of S which is cut off by Ω, and by $\Omega(T)$ the upper boundary of D lying on the hyperplane $t=T$.

We shall now introduce the following function spaces :

$C^1(S(T))$: the set of all (real-valued) functions φ which are continuously differentiable in a neighbourhood of $S(T)$.

We define for $\varphi\in C^1(S(T))$

$$(1.2)\qquad ||\varphi||_{W_2^1(S(T))}=\left\{\int_{S(T)}\left[\varphi^2+\left(\frac{\partial\varphi}{\partial\lambda}\right)^2\right]dS\right\}^{\frac{1}{2}},$$

where dS denotes the surface element of $S(T)$.

$W_2^1(S(T))$: the completion of $C^1(S(T))$ with respect to the norm (1.2).

We define for $u\in C^1(D)$

$$(1.3)\qquad ||u||_{W_2^1(D)}=\left\{\int_D\left[u^2+\sum_{j=1}^{n+1}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx\,dt\right\}^{\frac{1}{2}}.$$

$W_2^1(D)$: the completion of $C^1(D)$ with respect to the norm (1.3).

$C_0^1(D)$: the set of all functions u in $C^1(D)$ whose supports are contained in the interior of D.

$W^1_{2,0}(D)$: the completion of $C^1_0(D)$ with respect to the norm (1.3).

$C^1_1(D)$: the set of all functions u in $C^1(D)$ such that $u \equiv 0$ in a neighbourhood of the upper boundary $\Omega(T)$.

$W^1_{2,1}(D)$: the completion of $C^1_1(D)$ with respect to the norm (1.3).

$W^1_2(D, \varphi)$: the set of all functions u in $W^1_2(D)$ such that

$$\lim_{\varepsilon \downarrow 0} \| u(x, \omega(x)+\varepsilon)-\varphi \|_{L_2(S(T))}=0,$$

where we have set $u=0$ in the exterior of D.

It should be observed that the functions in $W^1_2(D)$ have generalized first derivatives in the sense of distribution.

Preparations for the method of finite differences. Let us draw in E_{n+1} $(n+1)$ families of hyperplanes parallel to the coordinate axes

$$x_j = l_j h \qquad (j=1, \cdots, n), \quad t=mk.$$

Here $h=1/2^r$, r being a positive integer, and l_j, m take on successive integral values such that the slab $0 \leqq t \leqq T$ in E_{n+1} is covered with parallelepipeds thus formed. We denote by σ the mesh ratio k/h, where σ is a constant > 0.

Let $U_{l,m}$ be a function defined at all lattice points $P_{l,m}=P(lh, mk)$ of the net, where

$$(lh, mk)=(l_1 h, \cdots, l_n h, mk).$$

We use the following notations:

$(l, m)=(l_1, \cdots, l_n, m)$;

$(l_j+1)=(l_1, \cdots, l_{j-1}, l_j+1, l_{j+1}, \cdots, l_n, m)$;

$(l_j-1)=(l_1, \cdots, l_{j-1}, l_j-1, l_{j+1}, \cdots, l_n, m)$;

$(l_i+1, l_j-1, m)=(l_1, \cdots, l_i+1, \cdots, l_j-1, \cdots l_n, m)$;

$(l_i-1, l_j+1, m)=(l_1, \cdots, l_i-1, \cdots, l_j+1, \cdots, l_n, m)$;

$U_{l,m}=U(lh, mk)$;

$\Delta^+ U_{l,m}=U_{l,m+1}-U_{l,m}$; $\qquad \Delta^- U_{l,m}=U_{l,m}-U_{l,m-1}$;

$\delta^+_j U_{l,m}=U_{l_j+1,m}-U_{l,m}$; $\qquad \delta^-_j U_{l,m}=U_{l,m}-U_{l_j-1,m}$;

$\delta_j U_{l,m}=(\delta^+_j U_{l,m}+\delta^-_j U_{l,m})/2$;

$\delta^+_{ij} U_{l,m}=U_{l_i+1, l_j-1, m}-U_{l,m}$; $\qquad \delta^-_{ij} U_{l,m}=U_{l,m}-U_{l_i-1, l_j+1, m}$;

$\Delta^2 U_{l,m}=\Delta^+ U_{l,m}-\Delta^- U_{l,m}=\Delta^+(\Delta^- U_{l,m})=\Delta^-(\Delta^+ U_{l,m})$;

$\delta^2_j U_{l,m}=\delta^+_j U_{l,m}-\delta^-_j U_{l,m}=\delta^+_j(\delta^-_j U_{l,m})=\delta^-_j(\delta^+_j U_{l,m})$;

$\delta''_j U_{l,m}=U_{l_j-1, m+1}-U_{l,m}$; $\qquad \delta'_j U_{l,m}=U_{l_j+1, m+1}-U_{l,m}$;

$$\delta''_{ij}U_{l,m}=U_{l_i-1,l_j+1,m+1}-U_{l,m}\,; \qquad \delta'_{ij}U_{l,m}=U_{l_i+1,l_j-1,m+1}-U_{l,m}.$$

Let $\Omega^*=\{x;\ |x|<T^*\}$ be a domain in the x-space E_n such that $\Omega^*\supset\Omega(T)$ and consider the domain $\Omega_h\subset\Omega^*$ formed from those cubes which have no exterior points of Ω^*. Denote by Q and Q_h the domains $\Omega^*\times[0,T]$ and $\Omega_h\times[0,T]$ respectively. We say that a lattice point $P_{l,m}$ of the net is an interior point of Q_h if $P_{l,m}$, together with $2n$ lattice points $P_{l_j+1,m}$, $P_{l_j-1,m}(j=1,\cdots,n)$, belongs to Q_h. If $P_{l,m}$ is not a point of Q_h, we say that $P_{l,m}$ is an exterior point of Q_h. The remaining lattice points are the boundary points of Q_h.

Let us consider the domain $D_h\subset D$ formed from those parallelepipeds which have no exterior points of D. If a lattice point $P_{l,m}$, together with $2n$ lattice points $P_{l_j+1,m-1}$, $P_{l_j-1,m-1}(j=1,\cdots,n)$, belongs to D_h, we say that $P_{l,m}$ is an interior point of D_h. Concerning the exterior and the boundary points of D_h, we adopt the above definitions.

Let there be given a function $U_{l,m}$ defined at all lattice points of the net. Then, according to O. A. Ladyzhenskaia, we shall introduce two functions $\bar{U}_h$ and $\tilde{U}_h$ defined in the following way:

Take a (generating) parallelepipid of the net. Then $\bar{U}_h$ is defined there as the function which is separately linear in the $n+1$ variables x, t and which takes on the value $U_{l,m}$ at the vertices of the parallelepiped. As is readily verified, such a function always exists and is unique. If $U_{l,m}$ is defined at all lattice points of a domain Q_h^* formed from parallelepipeds, then $\bar{U}_h$ is defined at all points of Q_h^* and is Lipschitz-continuous. Hence $\bar{U}_h$ has generalized first derivatives.

Next, let us fix in each parallelepiped a vertex (for example, with minimum coordinates). Then $\tilde{U}_h$ is defined as the step function which, in the interior of each parallelepiped, takes on the value of $U_{l,m}$ at the assigned vertex.

Now we shall state some theorems, lemmas and formulas which we shall use in section 4. (For their proofs, see, e. g., O. A. Ladyzhenskaia [4].)

Let $\{G_{h,l,m}\}$ be a sequence of functions defined at all lattice points of the slab $0\leqq t\leqq T$ in E_{n+1} and vanishing at the exterior and the boundary points of Q_h.

Theorem 1.1. *Suppose that the inequality*

$$kh^n\sum_{Q_h}G^2_{h,l,m}<A_1$$

holds, where A_1 is a constant independent of h.

Then, if one of the sequences $\{\bar{G}_h\}$, $\{\tilde{G}_h\}$ converges weakly to a function G^* in $L_2(Q)$ as $h\to 0$, the other also converges weakly to G^*.

Theorem 1.2. *Suppose that the inequality*

$$kh^n \sum_{m=1}^{\left[\frac{T}{k}\right]-1} \sum_{\Omega_h} \left[\frac{1}{k^2} (\Delta^+ G_{h,l,m})^2 + \frac{1}{h^2} \sum_{j=1}^{n} (\delta_j G_{h,l,m})^2 \right] < A_2$$

holds, where A_2 is a constant independent of h.

Then, if one of the sequences $\{\bar{G}_h\}$, $\{\tilde{G}_h\}$ converges in the mean to a function G^* in $L_2(Q)$ as $h \to 0$, the other also converges in the mean to G^*.

Theorem 1.3. *Suppose that the inequality*

$$kh^n \sum_{m=0}^{\left[\frac{T}{k}\right]-1} \sum_{\Omega_h} \left[G^2_{h,l,m} + \frac{1}{k^2} (\Delta^+ G_{h,l,m})^2 + \frac{1}{h^2} \sum_{j=1}^{n} (\delta_j G_{h,l,m})^2 \right] < A_3$$

holds, where A_3 is a constant independent of h.

Then there exists a subsequence $\{G_{\alpha,l,m}\}$ such that the corresponding subsequence $\{\bar{G}_\alpha\}$ converges in the mean in $L_2(Q)$ to a function G^* in $W_2^1(Q)$ as $\alpha \to 0$. The corresponding sequences $\{\Delta^+ \bar{G}_\alpha / k\}$, $\{\delta_j \bar{G}_\alpha / h\}$ converge weakly in $L_2(Q)$ to $\partial G^* / \partial t$, $\partial G^* / \partial x_j$ respectively.

Lemma 1.1. *Let $U_{l,m}$ be a function defined at the lattice points of the net. Then, for $\sigma < 1/n$ and $0 \leqq m \leqq [T/k]-1$, the following inequality holds:*

$$\frac{1}{k^2} (\Delta^+ U_{l,m})^2 + \frac{1}{2h^2} \sum_{j=1}^{n} [(\delta_j'' U_{l_j+1,m})^2 + (\delta_j' U_{l_j-1,m})^2]$$

$$\geqq (1 - n\sigma) \left[\frac{1}{k^2} (\Delta^+ U_{l,m})^2 + \frac{1}{h^2} \sum_{j=1}^{n} (\delta_j U_{l,m})^2 \right].$$

Lemma 1.2. *Consider the lattice points of the net lying on the plane $t = pk$ with $0 \leqq p \leqq [T/k]$. If $U_{l,m}$ vanishes at the exterior and the boundary lattice points, then we have*

$$h^n \sum_{\Omega_h} U^2_{l,p} \leqq A_4 h^n \sum_{\Omega_h} \frac{1}{h^2} (\delta_1 U_{l,p})^2,$$

where A_4 is a constant (>0) independent of h.

Remark. A_4 is determined by the magnitude of Ω^*.

Formula 1.1.

$$(\Delta^+ U_{l,m} + \Delta^- U_{l,m}) \Delta^2 U_{l,m} = (\Delta^+ U_{l,m})^2 - (\Delta^+ U_{l,m-1})^2.$$

Formula 1.2.

$$(\Delta^+ U_{l,m} + \Delta^- U_{l,m}) \delta_j^2 U_{l,m} = (\Delta^+ U_{l,m})^2 - (\Delta^+ U_{l,m-1})^2$$

$$-\frac{1}{2}[(\delta_j''(U_{l_j+1,m})^2 + (\delta_j' U_{l_j-1,m})^2 - (\delta_j'' U_{l,m-1})^2 - (\delta_j' U_{l,m-1})^2].$$

Formula 1.3.

$$(\Delta^+ U_{l,m} + \Delta^- U_{l,m})(\delta_j^- (\delta_i^+ U_{l,m}) + \delta_i^- (\delta_j^+ U_{l,m})) = (\Delta^+ U_{l,m})^2 - (\Delta^+ U_{l,m-1})^2$$

$$-\frac{1}{2}[(\delta_i'' U_{l_i+1,m})^2 + (\delta_i' U_{l_i-1,m})^2 + (\delta_j'' U_{l_j+1,m})^2 + (\delta_j' U_{l_j-1,m})^2$$

$$-(\delta_i'' U_{l,m-1})^2-(\delta_i' U_{l,m-1})^2-(\delta_j'' U_{l,m-1})^2-(\delta_j' U_{l,m-1})^2]$$

$$+\frac{1}{2}[(\delta_{ij}'' U_{l_i+1,l_j-1,m})^2+(\delta_{ij}' U_{l_i-1,l_j+1,m})^2-(\delta_{ij}'' U_{l,m-1})^2-(\delta_{ij}' U_{l,m-1})^2].$$

Formula 1.4. (*Abel's transformation*)

$$\sum_{j=0}^{n-1} v_j(w_{j+1}-w_j)=v_n w_n-v_0 w_0-\sum_{j=1}^{n}(v_j-v_{j-1})w_j.$$

2. Definition of Weak Solutions

We begin by considering the solution $u(x,t)$ of the inhomogeneous equation

$$Lu\equiv\frac{\partial^2 u}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right)=f(x,t) \tag{2.1}$$

satisfying the boundary condition (2). For the sake of simplicity, assume that $u\in C^2(\bar{D})$[1] and take a function $\Phi\in C_1^1(\bar{D})$.

Then, integrating by parts, we have

$$\begin{aligned}\int_D f\Phi\,dxdt&=\int_D Lu\,\Phi\,dxdt\\&=\int_{S(T)}\left[\frac{\partial u}{\partial t}\cos(n,t)-\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\left(\sum_{j=1}^{n}a_{ij}\cos(n,x_j)\right)\right]\Phi\,dS\\&\quad-\int_D\left(\frac{\partial u}{\partial t}\frac{\partial\Phi}{\partial t}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\right)dx\,dt,\end{aligned} \tag{2.2}$$

where n denotes the outer normal to S. Since

$$\frac{\partial u}{\partial t}\cos(n,t)-\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\left(\sum_{j=1}^{n}a_{ij}\cos(n,x_j)\right)=-\frac{\partial u}{\partial\lambda},$$

(2.2) can be written as

$$\int_D\left(\frac{\partial u}{\partial t}\frac{\partial\Phi}{\partial t}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\Phi}{\partial x_j}+f\Phi\right)dx\,dt+\int_{S(T)}\frac{\partial\varphi}{\partial\lambda}\Phi\,dS=0, \tag{2.3}$$

where $\partial/\partial\lambda$ denotes the differentiation along the bicharacteristic λ.

We assume that the boundary data φ are in $W_2^1(S(T))$.

Using (2.3), we give

Definition 1. *Let u be a function in $W_2^1(D,\varphi)$. Then u is a weak solution of* (2.1)-(2) *if, for every $\Phi\in W_{2,1}^1(D)$, the integral relation* (2.3) *is satisfied.*

As for the non-linear equation (1), we give

Definition I. *A function u in $W_2^1(D,\varphi)$ is a weak solution of* (1)-(2) *if, for every $\Phi\in W_{2,1}^1(D)$, the integral relation* (2.3) *with f replaced by*

1) $\bar{D}$ denotes the closure of D.

$$F=-\sum_{k=1}^{n+1} b_k(x,t,u)\frac{\partial u}{\partial x_k}+f(x,t,u)$$

is satisfied.

We now give another definition of interior weak solutions.

Definition 2. *Let u be a function in $W_2^1(D,\varphi)$. Then u is an interior weak solution of* (2.1)-(2) *if for every $\Phi\in W_{2,0}^1(D)$, the following integral relation is satisfied*:

(2.4) $$\int_D\left(\frac{\partial u}{\partial t}\frac{\partial \Phi}{\partial t}-\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \Phi}{\partial x_j}+f\Phi\right)dx\,dt=0\,.$$

Definition II. *A function u in $W_2^1(D,\varphi)$ is an interior weak solution of* (1)-(2) *if for every $\Phi\in W_{2,0}^1(D)$, the integral relation* (2.4) *with f replaced by F is satisfied.*

In what follows throughout the paper, we shall understand by the term "weak solution" an interior weak solution in the sense of Definition II. A weak solution in this sense is, however, a weak solution in the sense of Definition I, as is seen by the following

Theorem 2.1 (*Equivalence of Definitions* 1 *and* 2). *If $f\in L_2(D)$, then Definitions* 1 *and* 2 *are equivalent.*

Let $C_2^1(D)$ denote the set of all functions Φ in $C_1^1(D)$ such that $\Phi\equiv 0$ in a neighbourhood of the origin.

The proof of Theorem 2.1 is based on the

Lemma 2.1. *If $\Phi\in C_1^1(\bar D)$ and $n\geqq 2$, then there exists a sequence $\{\Phi_m\}$ ($\Phi_m\in C_2^1(\bar D)$) such that $\|\Phi_m-\Phi\|_{W_2^1(D)}\to 0$ as $m\to\infty$.*

Proof. Set

$$\varphi(t)=\begin{cases}0 & t\leqq 0\\ 6\int_0^t \xi(1-\xi)d\xi & 0<t<1\\ 1 & t\geqq 1\end{cases}$$

and define $\varphi_\delta(t)=\varphi((2/\delta)t-1)$.

If we set $\Phi_\delta=\varphi_\delta\Phi$, we have

$$\partial\Phi_\delta/\partial t=\varphi_\delta\partial\Phi/\partial t+\varphi_\delta'\Phi$$

and

$$\int_D(\varphi_\delta'\Phi)^2dx\,dt\leqq C(\operatorname*{Max}_{x\in\bar D}\Phi)^2\delta^{n-1}\,,$$

since $|\varphi_\delta'|\leqq 3/\delta$ and since the characteristic conoid S is tangential at the origin to the cone $t=\sqrt{A_{ij}x_ix_j}$, where $\|A_{ij}\|$ is the inverse matrix of $\|a_{ij}(0,0)\|$.

Here C is a constant depending only on the dimension n and the positive definiteness of $||A_{ij}||$. Hence, obviously, $\Phi_m=\varphi_{\frac{1}{m}}\Phi$ is a required sequence provided that $n\geqq 2$.

Proof of Theorem 2.1. We shall prove the theorem only for the case $n\geqq 2$, since its validity can easily be proved for the case $n=1$.

That Definition 1 implies Definition 2 is evident. To prove the converse, take a function $\Phi\in C_2^1(\bar{D})$ and let u be an interior weak solution of (2.1)-(2) in the sense of Definition 2. It should be observed that u is a weak solution of (2.1) in the usual sense, i.e., $\int_D f\Phi\,dx\,dt=\int_D u\cdot L\,\Phi\,dx\,dt\;(\Phi\in C_0^\infty(D))$.

Let S_δ be the direct characteristic conoid with vertex at a point $(0,\delta)$ $(\delta>0)$ and consider, instead of D, the subdomain D_δ of D which is surrounded by the characteristic conoid S_δ and the hyperplane $t=T-\delta$. Here we take a positive number δ so small that Φ is also in $C_2^1(\bar{D}_\delta)$.

In the subdomain D_δ, we can find, as is well known, a sequence $\{u_m\}$ of functions in $C^2(\bar{D}_\delta)$ such that $||u_m-u||_{W_2^1(D_\delta)}\to 0$, $||Lu_m-f||_{L_2(D)}\to 0$ as $m\to\infty$. For $u_m\in C^2(\bar{D}_\delta)$, we have

$$\int_{D_\delta} Lu_m\cdot\Phi\,dx\,dt \tag{2.5}$$
$$=-\int_{S_\delta}\frac{\partial u_m}{\partial\lambda}\Phi\,dS-\int_{D_\delta}\left(\frac{\partial u_m}{\partial t}\frac{\partial\Phi}{\partial t}-\sum_{i,j=1}^n a_{ij}\frac{\partial u_m}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\right)dx\,dt\,.$$

Here $\partial/\partial\lambda$ denotes the differentiation along the bicharacteristics λ which generate the characteristic conoid S_δ, and dS is the surface element of S_δ. As is easily seen from the theory of characteristics, dS can be written as $dS=\lambda^{n-1}\rho_\delta(\lambda,\theta)d\lambda\, d\omega_n$, where the θ_i are $n-1$ angular coordinates, $d\omega_n$ is the surface element of the unit sphere in E_n, and $\rho_\delta(\lambda,\theta)$ is continuously differentiable in λ and is bounded away from zero. Hence, integration by parts with respect to λ yields

$$\int_{S_\delta}\frac{\partial u_m}{\partial\lambda}\Phi\,dS=-\int_{S_\delta}u_m\left(\frac{\partial\Phi}{\partial\lambda}+\left(\frac{n-1}{\lambda}+\frac{1}{\rho_\delta}\frac{\partial\rho_\delta}{\partial\lambda}\right)\Phi\right)dS\,,$$

so that (2.5) can be written as

$$\int_{D_\delta}Lu_m\cdot\Phi\,dx\,dt=\int_{S_\delta}u_m\left[\frac{\partial\Phi}{\partial\lambda}+\left(\frac{n-1}{\lambda}+\frac{1}{\rho_\delta}\frac{\partial\rho_\delta}{\partial\lambda}\right)\Phi\right]dS \tag{2.6}$$
$$-\int_{D_\delta}\left(\frac{\partial u_m}{\partial t}\frac{\partial\Phi}{\partial t}-\sum_{i,j=1}^n a_{ij}\frac{\partial u_m}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\right)dx\,dt\,.$$

Letting, first, $m\to\infty$ and then $\delta\to 0$ in (2.6), we have by virtue of the imbedding theorem of S. L. Sobolev

$$\int_D f\Phi\,dx\,dt=\int_{S(T)}\varphi\left[\frac{\partial\Phi}{\partial\lambda}+\left(\frac{n-1}{\lambda}+\frac{1}{\rho}\frac{\partial\rho}{\partial\lambda}\right)\Phi\right]dS$$

$$-\int_D\left(\frac{\partial u}{\partial t}\frac{\partial \Phi}{\partial t}-\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \Phi}{\partial x_j}\right)dx\,dt\,,$$

since $u\in W_2^1(D,\varphi)$.

Integrating by parts again, we get

$$\int_{S(T)}\varphi\left[\frac{\partial\Phi}{\partial\lambda}+\left(\frac{n-1}{\lambda}+\frac{1}{\rho}\frac{\partial\rho}{\partial\lambda}\right)\Phi\right]dS=-\int_{S(T)}\frac{\partial\varphi}{\partial\lambda}\Phi\,dS\,,$$

since $\varphi\in W_2^1(S(T))$. Thus we have shown that the integral relation (2.3) holds for every $\Phi\in C_2^1(\bar{D})$.

From Lemma 2.1 and the fact that the domain D is star-shaped, it follows that the function space $C_2^1(\bar{D})$ is dense in $W_{2,1}^1(D)$. Hence the integral relation (2.3) holds for every $\Phi\in W_{2,1}^1(D)$, which proves that u is a weak solution in the sense of Definition 1.

3. Uniqueness of Weak Solutions

Let $\mathfrak{D}$ denote the domain $\bar{D}\times(-\infty<u<+\infty)$ in the (x,t,u)-space.
Setting

$$B_k(x,t,u)=\int_0^u b_k(x,t,\xi)d\xi\,,$$

we begin with

Lemma 3.1. *Suppose that $b_k(x,t,u)$ satisfies the hypotheses:*

(i) $b_k(x,t,u)$, $\partial b_k(x,t,u)/\partial x_k\in C(\mathfrak{D})$.

(ii) $|b_k(x,t,u)|$, $|\partial b_k(x,t,u)/\partial x_k|\leqq M$.

Then

$$|B_k(x,t,u_1)-B_k(x,t,u_2)|\leqq M|u_1-u_2|.$$
$$|\partial B_k(x,t,u_1)/\partial x_k-\partial B_k(x,t,u_2)/\partial x_k|\leqq M|u_1-u_2|.$$

Lemma 3.2. *Let $b_k(x,t,u)$ satisfy the hypotheses of the preceding lemma. Then, for every pair of $u\in W_2^1(D,\varphi)$ and $\Phi\in W_{2,1}^1(D)$, we have*

$$\int_D b_k(x,t,u)\frac{\partial u}{\partial x_k}\Phi\,dx\,dt=\int_{S(T)}B_k(x,t,\varphi)\Phi\cos(n,x_k)dS$$
$$-\int_D\left(B_k(x,t,u)\frac{\partial\Phi}{\partial x_k}+\frac{\partial B_k}{\partial x_k}(x,t,u)\Phi\right)dx\,dt\,.$$

Proof. The proof is similar to that of Lemma 3.5 in [1].

Theorem 1 (*Uniqueness theorem*). *Let $b_k(x,t,u)$ $(k=1,2,\cdots,n+1)$ and $f(x,t,u)$ satisfy the hypotheses:*

(i) $b_k(x,t,u)$, $\partial b_k(x,t,u)/\partial x_k$ *are in* $C(\mathfrak{D})$ *and are bounded in* $\mathfrak{D}$:

$$|b_k(x,t,u)|,\quad|\partial b_k(x,t,u)/\partial x_k|\leqq M.$$

(ii) *$f(x,t,u)$ is Lipschitz continuous in u*:

$$|f(x,t,u_1)-f(x,t,u_2)|\leqq L|u_1-u_2|.$$

Then there exists at most one weak solution of the characteristic boundary value problem (1)-(2).

Proof. Suppose that u_1 and u_2 are two weak solutions of (1)-(2). If we set $v=u_1-u_2$, then $v\in W_2^1(D,0)$. We set

$$g=f(x,t,u_1)-f(x,t,u_2),$$

$$B_k(x,t,u)=\int_0^u b_k(x,t,\xi)d\xi \quad (k=1,2,\cdots,n+1).$$

Then Theorem 2.1 and Lemma 3.2 yield

$$\text{(3.1)}\quad \int_D\Big[\frac{\partial v}{\partial t}\frac{\partial \Phi}{\partial t}-\sum_{i,j=1}^n a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial \Phi}{\partial x_j}+\sum_{k=1}^{n+1}(B_k(x,t,u_1)-B(x,t,u_2))\frac{\partial \Phi}{\partial x_k}$$
$$+\sum_{k=1}^{n+1}\Big(\frac{\partial B_k}{\partial x_k}(x,t,u_1)-\frac{\partial B_k}{\partial x_k}(x,t,u_2))\Phi+g\,\Phi\Big]dx\,dt=0$$

for every $\Phi\in W^1_{2,1}(D)$.

Following O. A. Ladyzhenskaia, let us set

$$\Phi(x,t)=\begin{cases}-\int_t^{t_1}v(x,\xi)d\xi & \omega(x)\leqq t\leqq t_1(<T),\\ 0 & t_1\leqq t\leqq T.\end{cases}$$

Then $v=\partial\Phi/\partial t$ in $D(t_1)$, where $D(t_1)=\{(x,t);\ \omega(x)\leqq t\leqq t_1\}$.

Integration by parts yields

$$\text{(3.2)}\quad \int_{D(t_1)}\Big(\frac{\partial v}{\partial t}\frac{\partial \Phi}{\partial t}-\sum_{i,j=1}^n a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial \Phi}{\partial x_j}\Big)dx\,dt$$
$$=\int_{D(t_1)}\Big(\frac{\partial^2\Phi}{\partial t^2}\frac{\partial \Phi}{\partial t}-\sum_{i,j=1}^n a_{ij}\frac{\partial^2\Phi}{\partial x_i\partial t}\frac{\partial \Phi}{\partial x_j}\Big)dx\,dt=\frac{1}{2}\int_{D(t_1)}\frac{\partial}{\partial t}\Big[\Big(\frac{\partial\Phi}{\partial t}\Big)^2$$
$$-\sum_{i,j=1}^n a_{ij}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\Big]dx\,dt+\frac{1}{2}\int_{D(t_1)}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial t}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}dx\,dt$$
$$=\frac{1}{2}\int_{\Omega(t_1)}\Big[\Big(\frac{\partial\Phi}{\partial t}(x,t_1)\Big)^2+\sum_{i,j=1}^n a_{ij}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}(x,\omega(x))\Big)\Big]dx$$
$$+\frac{1}{2}\int_{D(t_1)}\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial t}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}dx\,dt,$$

since

$$\int_{S(t_1)}\Big(\frac{\partial\Phi}{\partial t}\Big)^2 dS=\int_{S(t_1)}v^2\,dS=0,\quad \int_{\Omega(t_1)}\Big(\frac{\partial\Phi}{\partial x_j}\Big)^2dx=0\quad (j=1,2,\cdots,n)$$

Here $\Omega(t_1)$ denotes the upper boundary of $D(t_1)$ lying on the hyperplane $t=t_1$.

We have in view of the hypotheses (i), (ii) on a_{ij}

$$\text{(3.3)}\quad \int_{\Omega(t_1)}\Big[\Big(\frac{\partial\Phi}{\partial t}(x,t_1)\Big)^2\sum_{i,j=1}^n a_{ij}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}(x,\omega(x))^2\Big]dx$$
$$\geqq\int_{\Omega(t_1)}\Big[\Big(\frac{\partial\Phi}{\partial t}(x,t_1)\Big)^2+\alpha\sum_{j=1}^n\Big(\frac{\partial\Phi}{\partial x_j}(x,\omega(x))\Big)^2\Big]dx$$

and

$$(3.4)\qquad \left|\int_{D(t_1)}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial t}\frac{\partial \Phi}{\partial x_i}\frac{\partial \Phi}{\partial x_j}dx\,dt\right|\leqq nN\int_{D(t_1)}\sum_{j=1}^{n}\left(\frac{\partial \Phi}{\partial x_j}\right)^2 dx\,dt\,,$$

where N is a constant such that $|\partial a_{ij}/\partial t|\leqq N$ in D.

The remaining integrals in (3.1) can be estimated as follows:

We define $v=0$ in $Q-D$ and extend v to a function in $W_2^1(Q)$, where Q is a cylindrical domain containing D. We set

$$\omega_j(x,t)=-\int_0^t\frac{\partial v}{\partial x_j}(x,\xi)d\xi \qquad (j=1,2,\cdots,n).$$

By virtue of Lemma 3.1 and Schwarz' inequality, we have for $k=1,2,\cdots,n$

$$(3.5)\qquad \left|\int_{D(t_1)}(B_k(x,t,u_1)-B_k(x,t,u_2))\frac{\partial \Phi}{\partial x_k}dx\,dt\right|$$

$$\leqq\frac{M}{2}\int_{D(t_1)}v^2\,dx\,dt+M\int_{D(t_1)}\omega_k^2\,dx\,dt+Mt_1\int_{\Omega(t_1)}\omega_k^2\,dx.$$

For $k=n+1$, we have

$$(3.6)\qquad \left|\int_{D(t_1)}(B_{n+1}(x,t,u_1)-B_{n+1}(x,t,u_2))\frac{\partial \Phi}{\partial t}dx\,dt\right|\leqq M\int_{D(t_1)}v^2\,dx\,dt\,.$$

For $k=1,2,\cdots,n+1$, we also have

$$(3.7)\qquad \left|\int_{D(t_1)}\left(\frac{\partial B_k}{\partial x_k}(x,t,u_1)-\frac{\partial B_k}{\partial x_k}(x,t,u_2)\right)\Phi\,dx\,dt\right|\leqq Mt_1\int_{D(t_1)}v^2\,dx\,dt.$$

Similarly,

$$(3.8)\qquad \left|\int_{D(t_1)}g\,\Phi\,dx\,dt\right|\leqq Lt_1\int_{D(t_1)}v^2\,dx\,dt.$$

Hence (3.1)-(3.8) yield

$$\int_{\Omega(t_1)}\left(v^2+\alpha\sum_{j=1}^{n}\omega_j^2\right)dx\leqq C\int_{D(t_1)}\left(v^2+\sum_{j=1}^{n}\omega_j^2\right)dx\,dt+2(M+nN)t_1\int_{\Omega(t_1)}\sum_{j=1}^{n}\omega_j^2 dx\,,$$

where $C=\mathrm{Max}(M(n+2+2(n+1)T)+2LT,\ 2(M+nN))$. If we choose $T_1<\alpha/2(M+nN)$, we therefore have for $0\leqq t_1\leqq T_1$

$$(3.9)\qquad \int_{\Omega(t_1)}\left(v^2+\sum_{j=1}^{n}\omega_j^2\right)dx\leqq C_1\int_{D(t_1)}\left(v^2+\sum_{j=1}^{n}\omega_j^2\right)dx\,dt\,,$$

where $C_1=C/[\alpha-2(M+nN)T_1]$.

If we set

$$\Gamma(t_1)=\int_{D(t_1)}\left(v^2+\sum_{j=1}^{n}\omega_j^2\right)dx\,dt\,,$$

then $\Gamma(t_1)$ satisfies the differential inequality

$$d\Gamma(t_1)/dt_1\leqq C_1\Gamma(t_1)$$

for $0\leqq t_1\leqq T_1$ and the condition $\Gamma(0)=0$. It thus follows readily that $\Gamma(t_1)=0$ for $0\leqq t_1\leqq T_1$. Hence we have by virtue of (3.9)

$$(3.10)\qquad ||u_1-u_2||_{L_2(\Omega(t_1))}=0$$

for $0 \leqq t_1 \leqq T_1$.

Repeating the above argument, we can easily prove that (3.10) holds for $0 \leqq t_1 \leqq T$. This completes the proof.

4. Existence of Weak Solutions

In this section we shall prove, by making use of the method of finite differences, the existence of an interior weak solution for the boundary data φ which is continuously differentiable in a neighbourhood of $S(T)$.

To the boundary data φ there corresponds a function $\varphi^*(x)=\varphi(x, \omega(x))$ of the variables x. Conversely, by this correspondence, every function $\varphi^*(x)$ becomes a boundary data φ. In what follows, we assume that the boundary data φ is given as a function of the variables x and that φ is in $C_0^1(\Omega^*)$.

Scheme of finite differences. Setting $\varphi=0$ for exterior points of Ω^*, we extend φ to a function defined on the whole of E_n.

Let us consider, instead of (1), the difference equations

$$(4.1) \qquad \Delta^2 U_{l,m}-\sigma^2 \sum_{i,j=1}^{n} \delta_j^-(a_{ij,h,l,m}(\delta_i^+ U_{l,m}))+\sigma k \sum_{k=1}^{n} b_{k,h,l,m}\, \delta_k U_{l,m} + kb_{n+1,h,l,m}\, \Delta^- U_{l,m}-k^2 f_{h,l,m}=0\,,$$

where we have set

$$a_{ij,h,l,m}=a_{ij}(lh, mk) \qquad (i, j=1, 2, \cdots, n),$$
$$b_{k,h,l,m}=b_k(lh, mk, U_{l,m}) \qquad (k=1, 2, \cdots, n+1),$$
$$f_{h,l,m}=f(lh, mk, U_{l,m}).$$

We set $U_{l,m}=\varphi(lh)$ at all the exterior and the boundary lattice points of D_h in the slab $0 \leqq t \leqq T$. Hence, obviously,

$$(4.2) \qquad U_{l,m}=\varphi(lh) \qquad \text{for} \quad m=0, 1.$$

If the values of $U_{l,j}$ have been determined for all lattice points with $j=1, 2, \cdots, m$ $(m \geqq 1)$, then the value of $U_{l,m+1}$ at the lattice point $P_{l,m+1}$ in the interior of D_h can be determined by the difference equation (4.1). Thus, by this procedure, the values of $U_{l,m}$ will be determined for all lattice points in the slab $0 \leqq t \leqq T$. Moreover, it follows from this procedure that $U_{l,m}$ satisfies the difference equations

$$(4.3) \qquad (\Delta^+ U_{l,m}+\Delta^- U_{l,m})(\Delta^2 U_{l,m}-\sigma^2 \sum_{i,j=1}^{n} \delta_j^-(a_{ij,h,l,m}(\delta_i^+ U_{l,m})) + \sigma k \sum_{k=1}^{n} b_{k,h,l,m}\, \delta_k U_{l,m}+kb_{n+1,h,l,m}\, \Delta^- U_{l,m}-k^2 f_{h,l,m})=0$$

for all lattice points with $1 \leqq m \leqq [T/k]-1$, where we set $b_k(x, t, u)=0$ $(k=1, 2, \cdots, n+1)$, $f(x, t, u)=0$ in the exterior of D.

It should be remarked that $U_{l,m}$, defined above, is a function defined at all lattice points of the net in the slab $0 \leqq t \leqq T$ and vanishing at the exterior and the boundary points of Q_h.

Let $g(x, t, u)$ be defined in $\mathfrak{D}$ and $U_{l,m}$ be a function defined at the lattice points of the net. We denote by $\tilde{g}_h$ the step function which is constructed from $g_{h,l,m} = g(lh, mk, U_{l,m})$ in the manner stated in section 1.

Let $\{U_{h,l,m}\}$, $\{V_{h,l,m}\}$, $\{W_{h,l,m}\}$ be the sequences of functions vanishing at the exterior and the boundary points of Q_h. We assume that $U_{h,l,m}$, $V_{h,l,m}$ satisfy the hypothesis of Theorem 1.2 and $W_{h,l,m}$ the hypothesis of Theorem 1.1.

The following lemmas are due to [1].

Lemma 4.1. *Suppose that $b(x, t, u) \in C(\mathfrak{D})$ satisfies the following hypotheses:*

(i) *$b(x, t, u)$ is bounded in $\mathfrak{D}$.*

(ii) *$b(x, t, u)$ is equi-continuous in D as a function of the variables x, t for all values of u, i.e., for every $\varepsilon > 0$ there exists a number $\delta\ (> 0)$ independent of u such that $|x_1 - x_2| < \delta$ and $|t_1 - t_2| < \delta((x_1, t_1), (x_2, t_2) \in D)$ imply $|b(x_1, t_1, u) - b(x_2, t_2, u)| < \varepsilon$ for all values of u.*

Then, if $\{\tilde{U}_h\}$, $\{\tilde{V}_h\}$ converge in the mean in $L_2(Q)$ to U, V respectively and $\{\tilde{W}_h\}$ weakly in $L_2(Q)$ to W as $h \to 0$, and if $V_{h,l,m}$ vanishes at the exterior and the boundary points of D_h, there exist the subsequences $\{\tilde{U}_\alpha\}$, $\{\tilde{V}_\alpha\}$, $\{\tilde{W}_\alpha\}$ such that

$$\lim_{\alpha \to 0} \int_D \tilde{b}_\alpha \tilde{V}_\alpha \tilde{W}_\alpha \, dx\, dt = \int_D b(x, t, U) V W \, dx\, dt, \tag{4.4}$$

where we have set $b_{h,l,m} = b(lh, mk, U_{l,m})$.

Lemma 4.2. *Suppose that $f(x, t, u) \in C(\mathfrak{D})$ satisfies the following hypotheses:*

(i) *$f(x, t, u)$ is Hölder-continuous in u with exponent $\alpha (0 < \alpha \leqq 1)$, i.e.,*

$$|f(x, t, u_1) - f(x, t, u_2)| \leqq L |u_1 - v_2|^\alpha$$

for $(x, t) \in D$, $-\infty < u_1, u_2 < +\infty$.

(ii) *$f(x, t, u)$ is equi-continuous in D as a function of the variables x, t for all values of u.*

Then, if $\{\tilde{U}_h\}$, $\{\tilde{V}_h\}$ converge in the mean in $L_2(Q)$ to U, V respectively as $h \to 0$, and if $V_{h,l,m}$ vanishes at the exterior and the boundary points of D_h, we have

$$\lim_{h \to 0} \int_D \tilde{f}_h \tilde{V}_h \, dx\, dt = \int_D f(x, t, U) V \, dx\, dt, \tag{4.5}$$

where we have set $f_{h,l,m} = f(lh, mk, U_{l,m})$.

Remark 4.1. *If $f(x, t, u) = c(x, t) g(u) + F(x, t)$ satisfies* i) *$c(x, t)$, $F(x, t) \in C(\bar{D})$,* ii) *$g(u)$ is Hölder-continuous with exponent $\alpha (0 < \alpha \leqq 1)$, then also (4.5) holds.*

Lemma 4.3. *If* $|g(x,t,u)| \leqq L|u|^{\alpha}$ $(0<\alpha \leqq 1)$, *then*

$$|g(x,t,u)|^2 \leqq L^2(1+|u|^2).$$

We now make the following hypotheses on $b_k(x,t,u)$ $(k=1,2,\cdots,n+1)$ and $f(x,t,u)$:

(i) $b_k(x,t,u)$, $f(x,t,u)$ are in $C(\mathfrak{D})$.

(ii) $b_k(x,t,u)$ are bounded in $\mathfrak{D}$: $|b_k(x,t,u)| \leqq M$.

(iii) $f(x,t,u)$ is Hölder-continuous in u with exponent $\alpha (0<\alpha \leqq 1)$:

$$|f(x,t,u_1)-f(x,t,u_2)| \leqq L|u_1-u_2|^{\alpha}$$

for $(x,t) \in D$, $-\infty < u_1, u_2 < +\infty$.

(iv) $b_k(x,t,u)$, $f(x,t,u)$ are equi-continuous in D as functions of the variables x, t for all values of u.

Theorem 2 (*Existence theorem*). *Under Hypotheses* (i)-(iv), *there exists at least one weak solution of the characteristic boundary value problem* (1)-(2).

Proof. Consider the difference equation (4.3) which is valid for all lattice points with $1 \leqq m \leqq [T/k]-1$ and rewrite

$$(4.6)\qquad (\Delta^+ U_{l,m}+\Delta^- U_{l,m})\left(\frac{\Delta^2 U_{l,m}}{k^2}-\frac{1}{h^2}\sum_{i,j=1}^{n}\delta_j^-(a_{ij,h,l,m}(\delta_i^+ U_{l,m}))\right.$$
$$\left.+\sum_{k=1}^{n} b_{k,h,l,m}\frac{\delta_k U_{l,m}}{h}+b_{n+1,h,l,m}\frac{\Delta^- U_{l,m}}{k}-f_{h,l,m}\right)=0.$$

By simple calculations we have

$$(4.7)\qquad \delta_j^-(a_{ij,h,l,m}(\delta_i^+ U_{l,m}))=a_{ij,h,l,m}\,\delta_j^-(\delta_i^+ U_{l,m})+(\delta_j^- a_{ij,h,l,m})(\delta_i^+ U_{l_j-1,m}),$$

$$(4.8)\qquad a_{ij,h,l,m}[(\Delta^+ U_{l,m})^2-(\Delta^+ U_{l,m-1})^2]=a_{ij,h,l,m}(\Delta^+ U_{l,m})^2$$
$$-a_{ij,h,l,m-1}(\Delta^+ U_{l,m-1})^2-(\Delta^- a_{ij,h,l,m})(\Delta^+ U_{l,m-1})^2,$$

$$(4.9)\qquad a_{ij,h,l,m}[(\delta_i'' U_{l_i+1,m})^2-(\delta_i'' U_{l,m-1})^2]=a_{ij,h,l,m}(\delta_i'' U_{l_i+1,m})^2$$
$$-a_{ij,h,l_{i-1},m-1}(\delta_i'' U_{l,m-1})^2-(\delta_i' a_{ij,h,l_i-1,m-1})(\delta_i'' U_{l,m-1})^2,$$

$$(4.10)\qquad a_{ij,h,l,m}[(\delta_i' U_{l_i-1,m})^2-(\delta_i' U_{l,m-1})^2]=a_{ij,h,l,m}(\delta_i' U_{l_i-1,m})^2$$
$$-a_{ij,h,l_{i+1},m-1}(\delta_i' U_{l,m-1})^2-(\delta_i'' a_{ij,h,l_i+1,m-1})(\delta_i' U_{l,m-1})^2,$$

$$(4.11)\qquad a_{ij,h,l,m}[(\delta_{ij}'' U_{l_i+1,l_j-1,m})^2-(\delta_{ij}'' U_{l,m-1})^2]=a_{ij,h,l_j-1,m}(\delta_{ij}'' U_{l_i+1,l_j-1,m})^2$$
$$-a_{ij,h,l_{i-1},m-1}(\delta_{ij}'' U_{l,m-1})^2+(\delta_j^- a_{ij,h,l,m})(\delta_{ij}'' U_{l_i+1,l_j-1,m})^2$$
$$-(\delta_i' a_{ij,h,l_i-1,m-1})(\delta_{ij}'' U_{l,m-1})^2,$$

$$(4.12)\qquad a_{ij,h,l,m}[(\delta_{ij}' U_{l_i-1,l_j+1,m})^2-(\delta_{ij}' U_{l,m-1})^2]=a_{ij,h,l_j+1,m}(\delta_{ij}' U_{l_i-1,l_j+1,m})^2$$
$$-a_{ij,h,l_i+1,m-1}(\delta_{ij}' U_{l,m-1})^2-(\delta_j^+ a_{ij,h,l,m})(\delta_{ij}' U_{l_i-1,l_j+1,m})$$
$$-(\delta_i'' a_{ij,h,l_i+1,m-1})(\delta_{ij}' U_{l,m-1})^2,$$

$$(4.13)\quad (\delta_i'' U_{l_i+1,m})^2+(\delta_j'' U_{l_j+1,m})^2-(\delta_{ij}'' U_{l_i+1,m})^2=2(\delta_i'' U_{l_i+1,m})(\delta_j'' U_{l_j+1,m})$$
$$-(\Delta^+ U_{l_j+1,m})^2+2(\Delta^+ U_{l_j+1,m})[(\delta_j'' U_{l_j+1,m})-(\delta_i'' U_{l_i+1,m})],$$

$$(4.14)\quad (\delta_i' U_{l_i-1,m})^2+(\delta_j' U_{l_j-1,m})^2-(\delta_{ij}' U_{l_i-1,m})^2=2(\delta_i' U_{l_i-1,m})(\delta_j' U_{l_j-1,m})$$
$$-(\Delta^+ U_{l_j-1,m})^2+2(\Delta^+ U_{l_j-1,m})[(\delta_j' U_{l_j-1,m})-(\delta_i' U_{l_i-1,m})].$$

Multiplying the difference equation (4.6) by h^n and summing over all integral values of l and $1\leqq m\leqq p$ $(p\leqq[T/k]-1)$, we get by virtue of Formulas 1.1-1.3 and (4.7)-(4.14)

$$(4.15)\quad h^n\sum_{\Omega_h}\Big\{\Big(1-\frac{1}{2}\sigma^2\sum_{i,j=1}^{n} a_{ij,h,l,p}-\frac{1}{2}\sigma^2\sum_{j=1}^{n} a_{jj,h,l,p}\Big)\frac{(\Delta^+ U_{l,p})^2}{k^2}$$
$$-\frac{1}{4}\sigma^2\sum_{i\neq j} a_{ij,h,l,p}\frac{[(\Delta^+ U_{l_j+1,p})^2+(\Delta^+ U_{l_j-1,p})^2]}{k^2}$$
$$+\frac{1}{2h^2}\sum_{i,j=1}^{n} a_{ij,h,l,p}[(\delta_i'' U_{l_i+1,p})(\delta_j'' U_{l_j+1,p})+(\delta_i' U_{l_i-1,p})(\delta_j' U_{l_j-1,p})$$
$$+\frac{\sigma}{2hk}\sum_{i\neq j} a_{ij,h,l,p}[(\Delta^+ U_{l_j+1,p})((\delta_j'' U_{l_j+1,p})-(\delta_i'' U_{l_i+1,p}))$$
$$+(\Delta^+ U_{l_j-1,p})((\delta_j' U_{l_j-1,p})-(\delta_i' U_{l_i-1,p}))]\Big\}$$
$$-h^n\sum_{\Omega_h}\Big\{\frac{1}{4h^2}\sum_{i,j=1}^{n} a_{ij,h,l,1}[(\delta_i^- U_{l,0})^2+(\delta_i^+ U_{l,0})^2+(\delta_j^- U_{l,0})^2$$
$$+(\delta_j^+ U_{l,0})^2]-\frac{1}{4h^2}\sum_{i\neq j} a_{ij,h,l,1}[(\delta_{ij}^+ U_{l,0})^2+(\delta_{ij}^- U_{l,0})^2]\Big\}$$
$$=(\mathrm{I})-(\mathrm{II})$$
$$=kh^n\sum_{m=1}^{p}\sum_{\Omega_h}\Big\{\frac{(\Delta^+ U_{l,m})+(\Delta^- U_{l,m})}{k}\Big(-\sum_{k=1}^{n} b_{k,h,l,m}\frac{\delta_k U_{l,m}}{h}$$
$$-b_{n+1,h,l,m}\frac{\Delta^+ U_{l,m-1}}{k}+f_{h,l,m}+\sum_{i,j=1}^{n}\frac{(\delta_j^- a_{ij,h,l,m})(\delta_i^+ U_{l_j-1,m})}{h^2}$$
$$-\frac{1}{2}\sigma^2\sum_{i,j=1}^{n}\frac{(\Delta^- a_{ij,h,l,m})(\Delta^+ U_{l,m-1})^2}{k^2}-\frac{1}{2}\sigma^2\sum_{j=1}^{n}\frac{(\Delta^- a_{jj,h,l,m})(\Delta^+ U_{l,m-1})^2}{k^2}$$
$$+\frac{1}{4h^2}\sum_{i,j=1}^{n}[(\delta_i' a_{ij,h,l_i-1,m-1})(\delta_i'' U_{l,m-1})^2+(\delta_i'' a_{ij,h,l_i+1,m-1})(\delta_i' U_{l,m-1})^2$$
$$+(\delta_j' a_{ij,h,l_j-1,m-1})(\delta_j'' U_{l,m-1})^2+(\delta_j'' a_{ij,h,l_j+1,m-1})(\delta_j' U_{l,m-1})^2]$$
$$-\frac{1}{4h^2}\sum_{i\neq j}[(\delta_j^- a_{ij,h,l,m})(\delta_{ij}'' U_{l_i+1,l_j-1,m})^2$$
$$-(\delta_i' a_{ij,h,l_i-1,m-1})(\delta_{ij}'' U_{l,m-1})^2-(\delta_j^+ a_{ij,h,l,m})(\delta_{ij}' U_{l_i-1,l_j+1,m})^2$$
$$-(\delta_i'' a_{ij,h,l_i+1,m-1})(\delta_{ij}' U_{l,m-1})^2]\Big\},$$

since, by (4.2), $U_{lm}=\varphi(lh)$ for $m=0,\ 1$.

We have by the hypotheses on a_{ij}

$$(4.16)\qquad (\mathrm{I}) \geqq h^n \sum_{\Omega_h} \Big\{ (1-\sigma^2\beta\, n - \frac{1}{2}\sigma(\sigma+1)n(n-1)N)\Big(\frac{\Delta^+ U_{l,p}}{k}\Big)^2 + (\alpha-\sigma(n-1)N)\frac{1}{2h^2}\sum_{j=1}^{n}[(\delta_j'' U_{l_j+1,p})^2+(\delta_j' U_{l_j-1,p})^2]\Big\}\,,$$

where N is a constant such that $|a_{ij}| \leqq N$, $|\partial a_{ij}/\partial x_k| \leqq N$ in $D(i,j=1,\cdots,n\,;$ $k=1,\cdots,n+1)$.

If we choose the mesh ratio σ so small that

$$(4.17)\qquad \begin{cases} 1-\sigma^2\beta\, n - \dfrac{1}{2}\sigma(\sigma+1)n(n-1)N > 0\,, \\ \alpha-\sigma(n-1)N > 0, \quad 1-n\sigma > 0\,, \end{cases}$$

we have

$$(4.18)\qquad (\mathrm{I}) \geqq \gamma\, h^n \sum_{\Omega_h}\Big[\Big(\frac{\Delta^+ U_{l,p}}{k}\Big)^2 + \frac{1}{2h^2}\sum_{j=1}^{n}[(\delta_j'' U_{l_j+1,p})^2+(\delta_j' U_{l_j-1,p})^2]\Big]\,,$$

where $\gamma = \mathrm{Min}(1-\sigma^2\beta\, n - \sigma(\sigma+1)n(n-1)N/2, \alpha-\sigma(n-1)N)$.

We also have

$$(4.19)\qquad (\mathrm{II}) \leqq 3\,nN \|\varphi_h\|^2_{W_2^1(\Omega^*)} \leqq C_0,$$

where C_0 is a constant independent of h.

Setting

$$R_p = kh^n \sum_{m=0}^{p}\sum_{\Omega_h}\Big\{\frac{(\Delta^+ U_{l,m})^2}{k^2} + \frac{1}{2h^2}\sum_{j=1}^{n}[(\delta_j'' U_{l_j+1,m})^2+(\delta_j' U_{l_j-1,m})^2]\Big\}$$

and estimating the right-hand side of (4.15), we easily have by virtue of Lemma 1.2 and Lemma 4.3

$$(4.20)\qquad \frac{R_p - R_{p-1}}{k} \leqq C_1 R + C_2\,,$$

where C_1 is a constant depending only on σ, L, M and N, and $C_2 = 2\gamma^{-1}(L^2 + N_1^2)m(D) + \gamma^{-1}C_0 (N_1 = \mathop{\mathrm{Max}}_{(x,t)\in D}|f(x,t,0)|\,;$ $m(D)$ is the measure of D).

For $p=0$, we have $R_0 \leqq kC_0$ by (4.19).

We shall show that R_p is bounded, independently of h, for $1 \leqq p \leqq [T/k]-1$. Rewriting (4.20) in the form

$$\frac{R_p - R_{p-1}}{k} \leqq \frac{C_1 R_{p-1} + C_2}{1-kC_1} \leqq C_3 R_{p-1} + C_4$$

where $C_3 = 2\,C_1$, $C_4 = 2\,C_2$ and we assume $k \leqq k_0 \leqq 1/2\,C_1$, and comparing R_p with the solution $R^*(t)$ of the differential equation $dR^*/dt = C_3 R^* + C_4$ with $R^*(0) = kC_0$, we can easily prove that $R_p \leqq R^*(pk)$. Hence Lemmas 1.1 and 1.2 yield

$$(4.21)\qquad kh^n \sum_{m=0}^{[\frac{T}{k}]-1}\sum_{\Omega_h}\Big[U_{l,m}^2 + \frac{1}{k^2}(\Delta^+ U_{l,m})^2 + \frac{1}{h^2}\sum_{j=1}^{n}(\delta_j U_{l,m})^2\Big] \leqq C\,,$$

where C is a constant independent of h.

If h, k take on the successive values $h = 1/2^r$, $k = \sigma h$ $(r = 1, 2, \cdots)$, then h, k

tend decreasingly to 0 as $r\to\infty$ and the sequence $\{U_{h,l,m}\}$, corresponding to $U_{l,m}$ defined above, satisfies, in view of (4.21), the hypotheses of Theorem 1.3. Hence we can find a subsequence $\{U_{\alpha,l,m}\}$ such that the corresponding sequence $\{\bar{U}_\alpha\}$ converges in the mean in $L_2(Q)$ to a function u in $W_2^1(Q)$ and $\{\Delta^+\bar{U}_\alpha/k\}$, $\{\delta_j\bar{U}_\alpha/h\}$ converge weakly in $L_2(Q)$ to $\partial u/\partial t$, $\partial u/\partial x_j$ respectively. From the imbedding theorem of S.L. Sobolev and the fact that $U_{l,m}=\varphi(lh)$ at the exterior and the boundary points of D_h, it is easily seen that the limit function u is in $W_2^1(D,\varphi)$.

We shall show that u is an interior weak solution in the sense of Definition II. To see this, take a function Φ in $C_0^1(D)$ and choose the mesh size h so small that Φ vanishes at all lattice points both on the boundary of D_h and on the plane $t=([T/k]-1)k$. Then $U_{l,m}$ satisfies the relation

$$(4.22)\qquad kh^n\sum_{m=1}^{\left[\frac{T}{k}\right]-1}\sum_{\Omega_h}\left\{\frac{\Delta^2U_{l,m}}{k^2}-\frac{1}{h^2}\sum_{i,j=1}^n\delta_j^-(a_{ij,h,l,m}(\delta_i^+U_{l,m}))\right.$$
$$\left.+\sum_{k=1}^n b_{k,h,l,m}\frac{\delta_kU_{l,m}}{h}+b_{n+1,h,l,m}\frac{\Delta^-U_{l,m}}{k}-f_{h,l,m}\right\}\Phi_{h,l,m}=0\,.$$

By means of Lemma 1.4, (4.22) becomes

$$kh^n\sum_{m=1}^{\left[\frac{T}{k}\right]-1}\sum_{\Omega_h}\left\{\frac{1}{k^2}(\Delta^-U_{l,m})(\Delta^-\Phi_{h,l,m})-\frac{1}{h^2}\sum_{i,j=1}^n a_{ij,h,l,m}(\delta_i^+U_{l,m})(\delta_j^+\Phi_{h,l,m})\right.$$
$$\left.+\left[-\sum_{k=1}^n b_{k,h,l,m}\frac{\delta_kU_{l,m}}{h}-b_{n+1,h,l,m}\frac{\Delta^-U_{l,m}}{k}+f_{h,l,m}\right]\right\}\Phi_{h,l,m}=0\,,$$

which can be written as

$$(4.23)\qquad \int_D\left\{\frac{1}{k^2}(\Delta^-\tilde{U}_h)(\Delta^-\tilde{\Phi}_h)-\frac{1}{h^2}\sum_{i,j=1}^n\tilde{a}_{ij,h}(\delta_i^+\tilde{U}_h)(\delta_j^+\tilde{\Phi}_h)\right.$$
$$\left.+\left[-\sum_{k=1}^n\tilde{b}_{k,h}\frac{\delta_k\tilde{U}_h}{h}-\tilde{b}_{n+1,h}\frac{\Delta^-\tilde{U}_k}{k}+\tilde{f}_h\right]\tilde{\Phi}_h\right\}dx\,dt=0\,,$$

since $\Phi\in C_0^1(D)$.

It thus follows from the integral relation (4.23), Lemmas 4.1 and 4.2 that we have for $\alpha\to 0$

$$(4.24)\qquad \int_D\left[\frac{\partial u}{\partial t}\frac{\partial\Phi}{\partial t}-\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\Phi}{\partial x_j}+\left(-\sum_{k=1}^{n+1}b_k\frac{\partial u}{\partial x_k}+f\right)\Phi\right]dx\,dt=0\,.$$

Here we have set $b_k=b_k(x,t,u)$, $f=f(x,t,u)$, u being the limit function obtained above.

From the validity of the integral relation (4.24) for every $\Phi\in C_0^1(D)$, it is obvious that (4.24) holds for every $\Phi\in W_{2,0}^1(D)$. Thus u is an interior weak solution of (1) in the sense of Definition II.

Remark 4.2. *If* $f(x,t,u)=c(x,t)g(u)+F(x,t)$ *satisfies* i) $c(x,t)$, $F(x,t)\in C(\bar{D})$, ii) $g(u)$ *is Hölder-continuous with exponent* $\alpha(0<\alpha\leqq 1)$, *then there exists at least one weak solution of* (1) *satisfying the boundary condition* (2).

5. Continuous Dependence of Weak Solutions on the Function $f(x,t,0)$ and on the Data φ

In this section we shall consider weak solutions of a non-linear hyperbolic differential equation of the form

$$\frac{\partial^2 u}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right)+\sum_{k=1}^{n+1}b_k(x,t)\frac{\partial u}{\partial x_k}=f(x,t,u) \tag{5.1}$$

satisfying the boundary condition (2).

We define for $\varphi\in C^1(\Omega^*)$

$$||\varphi||_{W_2^1(\Omega^*)}=\left\{\int_{\Omega^*}\left[\varphi^2+\sum_{j=1}^{n}\left(\frac{\partial\varphi}{\partial x_j}\right)^2\right]dx\right\}^{\frac{1}{2}}. \tag{5.2}$$

Setting

$$f(x,t,u)=f(x,t,0)+g(x,t,u)=F(x,t)+g(x,t,u),$$

we now make the following hypotheses :

(i) $b_k(x,t)$, $\partial b_k(x,t)/\partial x_k\in C(\bar{D})$ $(k=1,2,\cdots,n+1)$.

(ii) $F(x,t)\in C(\bar{D})$.

(iii) $g(x,t,u)$ is Lipschitz-continuous in u :

$$|g(x,t,u_1)-g(x,t,u_2)|\leqq L|u_1-u_2|$$

for $(x,t)\in D$, $-\infty<u_1,u_2<+\infty$.

(iv) $g(x,t,u)$ is equi-continuous in D as a function of the variables x, t for all values of u.

Under Hypotheses (i)-(iv) we can prove the

Theorem 3 (*Continuous dependence of weak solutions on* $f(x,t,0)$ *and on the data* φ). *Let* u_1 *and* u_2 *be two weak solutions of* (5.1)-(2), *corresponding respectively to* $f_1=F^{(1)}(x,t)+g(x,t,u)$, $\varphi^{(1)}$ *and* $f_2=F^{(2)}(x,t)+g(x,t,u)$, $\varphi^{(2)}$, *where* $\varphi^{(1)}$ *and* $\varphi^{(2)}$ *are in* $C_0^1(\Omega^*)$.

Then

$$||u_1-u_2||_{W_2^1(D)}\leqq C(||F^{(1)}-F^{(2)}||_{L_2(D)}+||\varphi^{(1)}-\varphi^{(2)}||_{W_2^1(\Omega^*)}), \tag{5.3}$$

where C is a constant independent of F and φ.

Proof. Under the hypotheses above, it follows from Theorems 1 and 2 that the unique solution u of (5.1)-(2) can be obtained as the limit of the approximate sequence $\{\bar{U}_\alpha\}$ constructed in the preceding section.

Let $U_{l,m}^{(1)}$ and $U_{l,m}^{(2)}$ be two functions defined at lattice points and belonging to the subsequences $\{U^{(1)}_{\alpha,l,m}\}$ $\{U^{(2)}_{\alpha,l,m}\}$ such that $\{\bar{U}^{(1)}_{\alpha}\}$, $\{\bar{U}^{(2)}_{\alpha}\}$ converge, for $\alpha\to 0$, in the mean in $L_2(D)$ to u_1, u_2 in $W^1_2(D)$ respectively. Then the values of $U^{(1)}_{l,m}$ and $U^{(2)}_{l,m}$ can be determined in the way indicated in the preceding section.

If we set $V_{l,m}=U^{(1)}_{l,m}-U^{(2)}_{l,m}$, then $V_{l,m}=\varphi^{(1)}(lh)-\varphi^{(2)}(lh)$ at the exterior and the boundary points of D_h, and $V_{l,m}$ satisfies the difference equations

$$(5.4)\qquad (\Delta^+V_{l,m}+\Delta^-V_{l,m})(\Delta^2V_{l,m}-\sigma^2\sum_{i,j=1}^{n}\delta_j^-(a_{ij,h,l,m}(\delta_i^+V_{l,m}))$$
$$+\sigma k\sum_{k=1}^{n}b_{k,h,l,m}\,\delta_k\,V_{l,m}+kb_{n+1,h,l,m}\,\Delta^-V_{l,m}-k^2G_{h,l,m})=0$$

for all lattice points with $1\leqq m\leqq[T/k]-1$, where

$$(5.5)\qquad G_{h,l,m}=F^{(1)}_{h,l,m}-F^{(2)}_{h,l,m}+g(lh,mk,U^{(1)}_{l,m})-g(lh,mk,U^{(2)}_{l,m}).$$

We set, as in the preceding section, $b_k=0$ $(k=1,2,\cdots,n+1)$, $F^{(1)}=F^{(2)}=g=0$ in the exterior of D.

We choose the mesh ratio σ in such a way that (4.17) hold.

If we set

$$R_p=kh^n\sum_{m=0}^{p}\sum_{\Omega_h}\left\{\frac{(\Delta^+V_{l,m})^2}{k^2}+\frac{1}{2h^2}\sum_{j=1}^{n}[(\delta_j''U_{l_j+1,m})^2+(\delta_j'V_{l_j-1,m})^2]\right.,$$

we have from (5.4), as in the proof of the preceding theorem,

$$(5.6)\qquad \frac{R_p-R_{p-1}}{k}\leqq C_1R_p+\gamma^{-1}\left[2\,kh^n\sum_{m=0}^{\left[\frac{T}{k}\right]}\sum_{\Omega_h}(F^{(1)}_{h,l,m}-F^{(2)}_{h,l,m})^2\right.$$
$$\left.+3\,nN\|\bar{\varphi}^{(1)}_h-\bar{\varphi}^{(2)}_h\|^2{}_{W_2^1(\Omega^*)}\right],$$

where C_1 is a constant depending only on σ, L, M and N, since we have from (5.5)

$$G^2_{h,l,m}\leqq 2(F^{(1)}_{h,l,m}-F^{(2)}_{h,l,m})^2+2[g(lh,mk,U^{(1)}_{l,m})-g(lh,mk,U^{(2)}_{l,m})]^2$$

and in view of (iii)

$$|g(lh,mk,U^{(1)}_{l,m})-g(lh,mk,U^{(2)}_{l,m})|\leqq L|V_{l,m}|\ .$$

The inequality (5.6) yields for $k\leqq 1/2\,C_1$

$$\frac{R_p-R_{p-1}}{k}\leqq 2\left\{C_1R_{p-1}+\gamma^{-1}\left[2\,kh^n\sum_{m=0}^{\left[\frac{T}{k}\right]}\sum_{\Omega_h}(F^{(1)}_{h,l,m}-F^{(2)}_{h,l,m})^2\right.\right.$$
$$\left.\left.+3\,nN\|\bar{\varphi}^{(1)}_h-\bar{\varphi}^{(2)}_h\|^2{}_{W_2^1(\Omega^*)}\right]\right\}.$$

Comparing again R_p with the solution of the differential equation

$$dR^*/dt=2\left\{C_1R^*+\gamma^{-1}\left[2\,kh^n\sum_{m=0}^{\left[\frac{T}{k}\right]}\sum_{\Omega_h}(F^{(1)}_{h,l,m}-F^{(2)}_{h,l,m})^2+3\,nN\,||\,\bar{\varphi}^{(1)}_h-\bar{\varphi}^{(2)}_h\,||^2{}_{W^1_2(\Omega^*)}\right]\right\}$$

with $R^*(0)=k\,||\,\bar{\varphi}^{(1)}_h-\bar{\varphi}^{(2)}_h\,||^2_{W^1_2(\Omega^*)}$, we get $R_p\leqq R^*(pk)$. Hence we have for $p=[T/k]-1$

$$\text{(5.7)}\qquad kh^n\sum_{m=0}^{\left[\frac{T}{k}\right]-1}\sum_{\Omega_h}\left[V^2_{l,m}+\frac{1}{k^2}(\varDelta^+V_{l,m})^2+\frac{1}{h^2}\sum_{j=1}^{n}(\delta_j\,V_{l,m})^2\right]$$
$$\leqq C\left(kh^n\sum_{m=0}^{\left[\frac{T}{k}\right]}\sum_{\Omega_h}(F^{(1)}_{h,l,m}-F^{(2)}_{h,l,m})^2+||\,\bar{\varphi}^{(1)}_h-\bar{\varphi}^{(2)}_h\,||^2{}_{W^1_2(\Omega^*)}\right),$$

where C is a constant independent of h, $F^{(1)}$, $F^{(2)}$, $\varphi^{(1)}$ and $\varphi^{(2)}$.

The inequality (5.7) can be written as

$$\text{(5.8)}\qquad ||\,\tilde{V}_h\,||^2_{L_2(Q)}+||\,\frac{1}{k}\varDelta^+\tilde{V}_h||^2_{L_2(Q)}+\sum_{j=1}^{n}\left\|\frac{1}{h}\delta_j\,\tilde{V}_h\right\|^2_{L_2(Q)}$$
$$\leqq C\left(\int_D(\tilde{F}^{(1)}_h-\tilde{F}^{(2)}_h)^2\,dx\,dt+||\,\bar{\varphi}^{(1)}_h-\bar{\varphi}^{(2)}_h\,||^2{}_{W^1_2(\Omega^*)}\right),$$

since $F^{(1)}=F^{(2)}=0$ in the exterior of D.

Using the well known fact that

$$||\,v\,||_{L_2(Q)}\leqq\varliminf_{h\to0}||\,\tilde{V}_h\,||_{L_2(Q)},\quad ||\,\partial\,v/\partial\,t\,||_{L_2(Q)}\leqq\varliminf_{h\to0}||\,\varDelta^+\tilde{V}_h/k\,||_{L_2(Q)},$$
$$||\,\partial\,v/\partial\,x_j\,||_{L_2(Q)}\leqq\varliminf_{h\to0}||\delta_j\,\tilde{V}_h/h\,||_{L_2(Q)},$$

where $v=u_1-u_2$, and letting $h\to0$, we get from (5.8)

$$||\,v\,||^2{}_{W^1_2(D)}=||\,u_1-u_2\,||^2{}_{W^1_2(D)}\leqq C(||\,F^{(1)}-F^{(2)}||^2{}_{L_2(D)}+||\,\varphi^{(1)}-\varphi^{(2)}\,||^2{}_{W^1_2(\Omega^*)})$$

An immediate consequence of Theorem 3 is the following theorem.

Let $W^1_{2,0}(\Omega^*)$ be the completion of $C^1_0(\Omega^*)$ with respect to the norm (5.2).

Theorem 4. *If $b_k(x,t)$, $f(x,t,u)$ satisfy the hypotheses* (i), (iii), (iv) *and* (ii′) $F(x,t)\in L_2(D)$, *then there exists a unique weak solution of* (5.1)-(2) *for any boundary data* $\varphi\in W^1_{2,0}(\Omega^*)$.

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