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On the Characteristic Initial Value Problem for Non-linear Wave Equations in Two Space Variables

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Introduction

In this paper we are concerned with the non-linear wave equation

$$\Box u \equiv \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = f(x_1, x_2, x_3, u) \tag{1}$$

in two space variables.

Let S be a characteristic half-cone of the equation (1), say, the direct characteristic cone with vertex at the origin. Denote by D the domain enclosed by S and a space-like surface Σ and denote also by S_Σ that (bounded) portion of S which is cut off by Σ.

Then the characteristic initial value problem, or the Darboux problem, consists in finding the solution of the equation (1) in D which satisfies the initial condition

$$u(x) = \varphi(x) \qquad \text{on} \quad S_\Sigma, \tag{2}$$

where $\varphi(x)$ is a function prescribed on S_Σ.

For the non-linear wave equation in one space variable

$$\frac{\partial^2 u}{\partial x \partial y} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \tag{3}$$

this problem has been the subject of many investigations and, under appropriate conditions on the function $f(x, y, u, p, q)$ figuring on the right-hand side of (3), the existence and the uniqueness of solutions of this problem have been established by various authors (see the bibliography at the end of the paper). Among them, T. Satō made his investigations on this problem by a systematic use of the comparison theorem and, under mild assumptions on the function $f(x, y, u, p, q)$, he obtained many interesting results analogous to those in the theory of ordinary differential equations, such as uniqueness conditions, maximal (minimal) solutions and Peano's property (see T. Satō [18][1]). The comparison theorem enables us to compare certain pairs of solutions of different equations having different initial conditions, thus in particular yielding an a priori estimate of solutions which

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

is most important in the existence proof of solutions of non-linear differential equations. Moreover, by making use of it, it will be seen in this paper that the above properties of solutions of (3) can be extended almost automatically to solutions of the equation (1). The comparison theorem may therefore be regarded as an effective tool for the study of the equation (1).

On the other hand, in his paper [15], M. Riesz treated this problem for the homogeneous wave equation

$$\Box u \equiv \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2} = 0$$

in n independent variables and, under the assumption that the initial data $\varphi(x)$ prescribed on the characteristic cone is sufficiently smooth, he gave an explicit solution formula.

The purpose of this paper is to deal with the characteristic initial value problem for the equation (1) and to extend to this case the results obtained for the equation (3) in [18].

To do this, we shall introduce the concept of generalized solutions of differential equations due to S. L. Sobolev. Based on the solution formula which follows readily from that obtained by M. Riesz, a precise definition of the generalized solution of the inhomogeneous wave equation

$$\Box u = f(x_1, x_2, x_3) \tag{4}$$

with initial condition (2) will be given in section 1 (see Definition 1.2). Then the problem is reduced to finding continuous solutions of the non-linear integral equation

$$u(x) = \frac{1}{2\pi} \int_{D_S^x} \frac{f(\xi, u(\xi))}{r} d\xi + \varphi(0) - \frac{1}{\pi} \int_{S^x} ds \int_0^{R_x} \frac{d\varphi}{dR} R^{-\frac{1}{2}} dR$$

of the Volterra type.

In section 2 we shall establish several comparison theorems under slightly different assumptions which, combined with Theorem 3.2 (the existence theorem), will play a fundamental role in sections 4, 5 and 6.

Section 3 is devoted to the proof of the existence of generalized solutions of the equation (1) with vanishing initial condition. We shall first prove that if $f(x)$ is continuous on the closure $\bar{D}$ of D, there exists a continuous generalized solution of the inhomogeneous equation (4) with vanishing initial condition. Then the existence of a continuous generalized solution of (1) with vanishing initial condition will be established under the assumption that $f(x, u)$ is continuous with respect to all its arguments. If $f(x, u)$ is continuously differentiable, then it will be shown that there exists a unique continuously differentiable generalized solution of (1) satisfying vanishing initial condition.

In sections 4 and 5, by making use of the comparison theorems obtained in section 2, we shall derive the existence of maximal (minimal) solutions, and uniqueness conditions respectively.

Section 6 deals with Peano's property for generalized solutions of (1) with vanishing initial condition.

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1. Generalized solutions

In this section we shall give a precise definition of the generalized solution of the inhomogeneous wave equation

$$\Box u = f(x_1, x_2, x_3) \tag{1.1}$$

with initial condition

$$u(x) = \varphi(x) \qquad \text{on} \quad S_\Sigma, \tag{1.2}$$

where $\varphi(x)$ is a once continuously differentiable function prescribed on S_Σ.

We begin with the explanation of the notation[2] which will be used throughout the paper. The letters x, y, ξ, η will always stand for the points $(x_1, x_2, x_3), \cdots, (\eta_1, \eta_2, \eta_3)$ in the space time of three dimensions, x_1 corresponding to the time variable and x_2, x_3 corresponding to the space variables. The Lorentz metric associated with the equation (1.1) is defined as the form

$$(x, y) = x_1y_1 - x_2y_2 - x_3y_3$$

for the scalar product of two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. *All metric notions, in particular arc length and surface element, are to be interpreted according to this Lorentz metric.* The (Lorentzian) distance between two points x and ξ will always be denoted by r, i.e.

$$r = \sqrt{(x_1-\xi_1)^2 - (x_2-\xi_2)^2 - (x_3-\xi_3)^2},$$

while the distance of a point x from the origin will be denoted by r_x. The volume element $d\xi_1 d\xi_2 d\xi_3$ will be abbreviated to $d\xi$.

Let S be the direct characteristic cone of the equation (1.1) with vertex at the origin, i.e.

$$S: \; x_1^2 - x_2^2 - x_3^2 = 0, \quad x_1 \geq 0$$

and let Σ[3] be a space-like surface which, together with the direct cone S, encloses a domain D. Denote by S_Σ that (bounded) portion of S which is cut off by Σ.

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[2] For the notation refer to M. Riesz [15].

[3] We assume that Σ is an open surface.

For any point x in D, denote by C^x the retrograde characteristic cone with vertex at x. Denote further by D_S^x the subdomain of D which is enclosed by two characteristic cones C^x and S, and denote also by S^x that (bounded) portion of S which is cut off by C^x. The one dimensional intersection of C^x and S will be denoted by s^x and its line element by ds.

The following Lemma 1.1 is essentially due to M. Riesz [15] pp. 107–115. However, since it is of fundamental importance to our definition of the generalized solution of the inhomogeneous equation (1.1) with initial condition (1.2), we shall give a brief sketch of its proof.

LEMMA 1.1. *Let $u(x)$ be a twice continuously differentiable function in $\bar{D}$*[4]*. Then $u(x)$ is expressible in $D \cup \Sigma$ in the form*

$$u(x)=\frac{1}{2\pi}\int_{D_S^x}\frac{\Box u(\xi)}{r}d\xi+u(0)-\frac{1}{\pi}\int_{s^x}ds\int_0^{R_x}\frac{du}{dR}R^{-\frac{1}{2}}dR\,,$$

where $R_x=r_x^2$ is the square of the distance of x from the origin, $R=r^2$ is the square of the distance between x and ξ, and the last integral is extended over the surface S^x.

Proof: Consider the domain D_S^x, assuming first that S is a space-like surface. Assuming further that $v(x)$ is a twice continuously differentiable function in $\bar{D}_S^x$ which vanishes together with its first derivatives on the retrograde cone C^x, we have from Green's formula

$$\int_{D_S^x}(u\Box v-v\Box u)d\xi=\int_{S^x}\left(u\frac{dv}{dn}-v\frac{du}{dn}\right)dS,^{5)} \tag{1.3}$$

where the differentiation d/dn is carried out in the direction of the outer normal. For a characteristic surface S, the formula (1.3) becomes invalid, since two geometrical elements dn and dS figuring on the right-hand side of (1.3) then vanish. However, with the aid of the solid angle from the point x and by letting S^x tend to the direct characteristic cone with vertex at the origin, the formula (1.3) is written as

$$\int_{D_S^x}(u\Box v-v\Box u)d\xi=\int_{s^x}ds\int_0^{R_x}\left(u\frac{dv}{dR}-v\frac{du}{dR}\right)\left(1-\frac{R}{R_x}\right)dR \tag{1.4}$$

for the characteristic surface S.

Assuming that α is sufficiently large, we set in (1.4)

$$v=\frac{r^{\alpha-1}}{H_3(\alpha+2)}=\frac{R^{\frac{\alpha-1}{2}}}{H_3(\alpha+2)}\,,$$

where

$$H_3(\alpha)=\pi^{\frac{1}{2}}2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-1}{2}\right).$$

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4) $\bar{D}$ denotes the closure of D.

5) dS is the surface element of S.

Then, setting

$$I^{\alpha}u(x)=\frac{1}{H_3(\alpha)}\int_{D_\xi^x} r^{\alpha-3}u(\xi)\,d\xi,$$

we obtain from (1.4)

$$\begin{aligned}I^{\alpha}u(x)&=I^{\alpha+2}\,\Box u(x)\\&\quad+\frac{1}{H_3(\alpha+2)}\int_{S^x}ds\int_0^{R_x}\left(u\frac{dR^{\frac{\alpha-1}{2}}}{dR}-\frac{du}{dR}R^{\frac{\alpha-1}{2}}\right)\left(1-\frac{R}{R_x}\right)dR\\&=I^{\alpha+2}\Box u(x)+U^{\alpha+2}(x).\end{aligned}\tag{1.5}$$

It is easily seen that the integrals figuring in (1.5) are all convergent for $\alpha>1$. Hence the formula (1.5) is valid for $\alpha>1$.

Since $u(x)$ is twice continuously differentiable in $\bar{D}$, the volume integrals $I^{\alpha}u(x)$ and $I^{\alpha+2}\Box u(x)$ in (1.5) can be analytically continued to $\alpha=0$, yielding

$$I^0u(x)=u(x)\quad\text{and}\quad I^2u(x)=\frac{1}{2\pi}\int_{D_\xi^x}\frac{\Box u(\xi)}{r}d\xi\tag{1.6}$$

respectively.

On the other hand, the analytic continuation of $U^{\alpha+2}(x)$ to $\alpha=0$ can be performed as follows:

For simplicity, setting $(\alpha-1)/2=\beta$, we first transform the integral with respect to R which figures in the expression of $U^{\alpha+2}(x)$ in (1.5). Since we have by simple calculations

$$u\frac{dR^{\beta}}{dR}-\frac{du}{dR}R^{\beta}=-R^{2\beta}\frac{d}{dR}(uR^{-\beta})$$

and

$$\frac{d}{dR}\left\{R^{2\beta}\left(1-\frac{R}{R_x}\right)\right\}=2\beta R^{2\beta-1}-\alpha\frac{R^{2\beta}}{R_x},$$

we obtain for $\beta>0$

$$\begin{aligned}\int_0^{R_x}\left(u\frac{dR^{\beta}}{dR}-\frac{du}{dR}R^{\beta}\right)\left(1-\frac{R}{R_x}\right)dR&=-\int_0^{R_x}R^{2\beta}\frac{d}{dR}(uR^{-\beta})\left(1-\frac{R}{R_x}\right)dR\\&=\int_0^{R_x}uR^{-\beta}\frac{d}{dR}\left\{R^{2\beta}\left(1-\frac{R}{R_x}\right)\right\}dR=2\int_0^{R_x}u\left(\beta R^{\beta-1}-\frac{\alpha}{2R_x}R^{\beta}\right)dR\\&=[2uR^{\beta}]_0^{R_x}-2\int_0^{R_x}\left(\frac{du}{dR}R^{\beta}+\frac{\alpha}{2R_x}R^{\beta}\right)dR\\&=2u(0)R_x^{\beta}-2\int_0^{R_x}\left(\frac{du}{dR}R^{\beta}+\frac{\alpha}{2R_x}R^{\beta}\right)dR\,.\end{aligned}$$

By the principle of analytic continuation, the last identity is valid for any β. Hence the analytic continuation of $U^{\alpha+2}(x)$ to $\alpha=0$ yields

$$U^2(x)=u(0)-\frac{1}{\pi}\int_{S^x}ds\int_0^{R_x}\frac{du}{dR}R^{-\frac{1}{2}}\,dR.\tag{1.7}$$

The desired expression for $u(x)$ is obtained from (1.6) and (1.7). q.e.d.

For a point x^0 on the surface S_Σ, denote by λ_{x^0} the generator of the direct cone S through x^0 and define its positive direction as the direction toward infinity.

The derivative of a function $u(x)$ in the positive direction of λ_{x^0} at a point x will be denoted by $\partial u(x)/\partial\lambda_{x^0}$ or, briefly, by $\partial u/\partial\lambda_{x^0}$.

A function $u(x)$ continuous in $\bar{D}$ and twice continuously differentiable in D is said to be of class $C^*[D]$[6] *if, when x approaches a point x^0 on the surface S_Σ along any path l which lies on the plane containing the point x^0 and the x_1-axis, its derivative $\partial u(x)/\partial\lambda_{x^0}$ approaches a (finite) limit and, moreover, the approach to the limit is uniform both for any choice of the path l and for any x^0 on S_Σ.*

A slight modification of Lemma 1.1 leads to the following

LEMMA 1.2. *Let $u(x)$ be a function in $C^*[D]$. Then $u(x)$ is expressible in $D \cup \Sigma$ in the form*

$$u(x)=\frac{1}{2\pi}\int_{D_S^x}\frac{\Box u(\xi)}{r}d\xi+u(0)-\frac{1}{\pi}\int_{s^x}ds\int_0^{R_x}\frac{du}{dR}R^{-\frac{1}{2}}\,dR. \tag{1.8}$$

Proof: Let S_ε be the direct characteristic cone with vertex at a point $0_\varepsilon=(\varepsilon, 0, 0)$ and let D_ε be the domain enclosed by S_ε and Σ. Then, for any point x in D, $u(x)$ is twice continuously differentiable in $\bar{D}^x_{S_\varepsilon}$, where ε is any sufficiently small positive number. Hence Lemma 1.1 applies, yielding the expression

$$u(x)=\frac{1}{2\pi}\int_{D^x_{S_\varepsilon}}\frac{\Box u(\xi)}{r}d\xi+u(0_\varepsilon)-\frac{1}{\pi}\int_{s^x_\varepsilon}ds\int_0^{R^\varepsilon_x}\frac{du}{dR}R^{-\frac{1}{2}}\,dR, \tag{1.9}$$

where s^x_ε is the intersection of C^x and S_ε, R^ε_x is the square of the distance between two points x and 0_ε, and the last integral is extended over the surface S^x_ε.

By letting ε tend to 0, the desired expression (1.8) follows immediately from (1.9), since $u(x)$ is in $C^*[D]$. q.e.d.

According to S. L. Sobolev[7], we give the following

DEFINITION 1.1. *A function $u(x)$ is said to be a generalized solution of the inhomogeneous equation* (1.1) *in a domain D if, for a suitable choice of an infinite sequence of functions $\{f_n(x)\}$ converging uniformly in D to the function $f(x)$, there exists an infinite sequence of twice continuously differentiable solutions $\{u_n(x)\}$ of the inhomogeneous equations*

$$\Box u_n=f_n(x) \qquad (n=1, 2, \cdots)$$

which converges uniformly in D to the function $u(x)$.

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[6] For second derivatives, it suffices to require the existence of continuous derivatives $\partial^2u/\partial x_1^2$, $\partial^2u/\partial x_2^2$ and $\partial^2u/\partial x_3^2$ figuring in (1.1).

[7] See I. G. Petrovsky [14] pp. 65-66.

As regards the meaning of the initial condition (1.2) for generalized solutions, a special definition is required. With the aid of Lemma 1.2, we shall give this only for once continuously differentiable functions $\varphi(x)$ defined on S_Σ.

DEFINITION 1.2. *Let* $\varphi(x)$ *be a once continuously differentiable function defined on* S_Σ. *Then a function* $u(x)$ *defined and continuous in the closure* $\bar{D}$ *is said to be a generalized solution in* D *of the inhomogeneous equation* (1.1) *with initial condition* (1.2) *if, for a suitable choice of two infinite sequences of functions* $\{f_n(x)\}$ *and* $\{\varphi_n(x)\}$ *satisfying the conditions :*

i) $\{f_n(x)\}$ *converges uniformly in* D *to the function* $f(x)$,

ii) *all the functions* $\varphi_n(x)$ *are once continuously differentiable on* S_Σ,

iii) $\{\varphi_n(x)\}$ *and all the sequences of its first derivatives converge uniformly on* S_Σ *to the function* $\varphi(x)$ *and its corresponding first derivatives respectively,*

there exists an infinite sequence of solutions $\{u_n(x)\}$ *in* $C^*[D]$ *of the inhomogeneous equations*

$$\Box u_n = f_n(x) \qquad (n=1, 2, \cdots)$$

with initial conditions

$$u_n(x) = \varphi_n(x) \quad \text{on} \quad S_\Sigma \qquad (n=1, 2, \cdots)$$

which converges uniformly in $\bar{D}$ *to the function* $u(x)$.

COROLLARY 1.1. *Let* $\varphi(x)$ *be a once continuously differentiable function defined on* S_Σ.

Assume that $u(x)$ *is a generalized solution in* D *of the inhomogeneous equation* (1.1) *with initial condition* (1.2). *Then* $u(x)$ *is expressible in* $D \cup \Sigma$ *in the form*

$$u(x) = \frac{1}{2\pi}\int_{D_S^x} \frac{f(\xi)}{r} d\xi + \varphi(0) - \frac{1}{\pi}\int_{S^x} ds \int_0^{R_x} \frac{d\varphi}{dR} R^{-\frac{1}{2}} dR$$

and hence the solution is unique.

Proof : This follows immediately from Definition 1.2 and Lemma 1.2.

COROLLARY 1.2. *Let* $\varphi(x)$ *be a once continuously differentiable function defined on* S_Σ *and let* $f(x, u)$ *be a function defined in* Δ, *where* Δ *is a set in the* (x, u) *space.*

Assume that $u(x)$ *is a generalized solution in* D *of the non-linear equation*

$$\Box u = f(x, u)$$

with initial condition (1.2). *Then* $u(x)$ *is a continuous solution in* $\bar{D}$ *of the non-linear integral equation*

$$u(x) = \frac{1}{2\pi}\int_{D_S^x} \frac{f(\xi, u(\xi))}{r} d\xi + \varphi(0) - \frac{1}{\pi}\int_{S^x} ds \int_0^{R_x} \frac{d\varphi}{dR} R^{-\frac{1}{2}} dR$$

of the Volterra type.

Proof : This is a consequence of the fact that $u(x)$ is a generalized solution in D of the inhomogeneous equation

$$\Box u = f(x, u(x))$$

with initial condition (1.2).

2. Comparison theorems

In this and succeeding sections the following notation will be used:

Let $\underline{\varphi}(x)$, $\varphi(x)$ and $\overline{\varphi}(x)$ be once continuously differentiable functions defined on S_Σ.

Let $\underline{\omega}(x)$ and $\overline{\omega}(x)$ be functions continuous in $\overline{D}$ and once continuously differentiable on S_Σ such that they are expressible in $D \cup \Sigma$ in the form

$$\left.\begin{aligned} \underline{\omega}(x) &= \frac{1}{2\pi}\int_{D^x_S} \frac{\Box\underline{\omega}(\xi)}{r} d\xi + \underline{\omega}(0) - \frac{1}{\pi}\int_{s^x} ds \int_0^{R_x} \frac{d\underline{\omega}}{dR} R^{-\frac{1}{2}} dR, \\ \overline{\omega}(x) &= \frac{1}{2\pi}\int_{D^x_S} \frac{\Box\overline{\omega}(\xi)}{r} d\xi + \overline{\omega}(0) - \frac{1}{\pi}\int_{s^x} ds \int_0^{R_x} \frac{d\overline{\omega}}{dR} R^{-\frac{1}{2}} dR \end{aligned}\right\} \tag{2.1}$$

respectively, where $\Box\underline{\omega}(x)$ and $\Box\overline{\omega}(x)$ are continuous functions in $\overline{D}$ which are in general independent of $\underline{\omega}(x)$ and $\overline{\omega}(x)$ respectively.

We assume that the following inequalities hold on S_Σ:

$$\underline{\omega}(0) \leq \underline{\varphi}(0) \leq \varphi(0) \leq \overline{\varphi}(0) \leq \overline{\omega}(0), \tag{2.2}$$

$$\frac{\partial\underline{\omega}(x)}{\partial\lambda_x} \leq \frac{\partial\underline{\varphi}(x)}{\partial\lambda_x} \leq \frac{\partial\varphi(x)}{\partial\lambda_x} \leq \frac{\partial\overline{\varphi}(x)}{\partial\lambda_x} \leq \frac{\partial\overline{\omega}(x)}{\partial\lambda_x}, \tag{2.3}$$

where λ_x denotes the generator of the direct cone S through a point x on S_Σ and the differentiation is carried out in the direction of λ_x toward infinity.

The following sets will be defined in the (x, u) space:

$$\mathcal{D} = \left\{(x, u);\ x \in \overline{D},\ \underline{\omega}(x) \leq u \leq \overline{\omega}(x)\right\}.$$

$$\mathcal{D}_0 = \left\{(x, u);\ x \in \overline{D},\ -\infty < u \leq \overline{\omega}(x)\right\}.$$

$$\mathcal{D}_1 = \left\{(x, u);\ x \in \overline{D},\ |u| \leq \omega(x)\right\},$$

where $\omega(x)$ is a function continuous in $\overline{D}$ and once continuously differentiable on S_Σ which is expressible in $D \cup \Sigma$ in the form (2.1) with $\Box\omega(x)$ continuous in $\overline{D}$ and which satisfies the inequalities

$$|\varphi(0)| \leq \omega(0), \quad \left|\frac{\partial\varphi(x)}{\partial\lambda_x}\right| \leq \frac{\partial\omega(x)}{\partial\lambda_x} \tag{2.4}$$

on S_Σ.

$$\mathcal{D}_2 = \left\{(x, u);\ x \in \overline{D},\ 0 \leq u < +\infty\right\}.$$

Δ denotes a set in the (x, u) space whose projection on the x-space contains the closed domain $\overline{D}$.

In what follows, when there is no danger of confusion, we shall simply speak of a solution instead of a generalized solution.

We shall now establish several comparison theorems for solutions of the non-linear wave equation

$$\Box u = f(x, u) \tag{2.5}$$

with initial condition

$$u(x) = \varphi(x) \quad \text{on} \quad S_\Sigma. \tag{2.6}$$

THEOREM 2.1. *Let $f(x, u)$ be a function defined and continuous in Δ and let the inequality*

$$f(x, u) < \Box\bar{\omega}(x) \tag{2.7}$$

hold in $\Delta \cap \mathcal{D}_0$.

Assume that $u(x)$ is a solution in D of the equation (2.5) *with initial condition* (2.6). *Then the inequality*

$$u(x) \leq \bar{\omega}(x) \tag{2.8}$$

holds in $\bar{D}$, where the equality sign can occur only on S_Σ.

Proof: We first observe from (2.2) and (2.3) that the inequality (2.8) holds on S_Σ. By the assumptions (2.2) and (2.7), we have

$$f(0, u(0)) < \Box\bar{\omega}(0)$$

and hence

$$f(x, u(x)) < \Box\bar{\omega}(x) \tag{2.9}$$

in the intersection of $\bar{D}$ and some neighbourhood U of the origin.

On the other hand, from Corollary 1.2, $u(x)$ is expressible in $D \cup \Sigma$ in the form

$$u(x) = \frac{1}{2\pi}\int_{D_S^x} \frac{f(\xi, u(\xi))}{r} d\xi + \varphi(0) - \frac{1}{\pi}\int_{s^x} ds \int_0^{R_x} \frac{d\varphi}{dR} R^{-\frac{1}{2}} dR.$$

Hence, comparing this expression with (2.1), we obtain from (2.2), (2.3) and (2.9)

$$u(x) < \bar{\omega}(x)$$

in $D \cap U$.

Assume now that the inequality (2.9) does not hold in $\bar{D}$ and denote by x_1^0 the least upper bound of the values of x_1 for which the inequality (2.9) holds in D_S^ξ. Then there must exist in D a point x^0 with coordinates (x_1^0, x_2^0, x_3^0) such that

$$f(x^0, u(x^0)) = \Box\bar{\omega}(x^0). \tag{2.10}$$

On the other hand, we have by the continuity of $u(x)$ and $\bar{\omega}(x)$

$$u(x^0) \leq \bar{\omega}(x^0)$$

and hence we have in virtue of (2.7)

$$f(x^0, u(x^0)) < \Box\bar{\omega}(x^0)$$

which is contrary to (2.10). Therefore x_1^0 does not exist and the proof is complete. q. e. d.

REMARK. It is evident that a similar theorem holds for $\underline{\omega}(x)$.

COROLLARY 2.1. *Let* $f(x,u)$ *be a function defined and continuous in* Δ *and let the inequality*

$$|f(x,u)| < \Box\omega(x)$$

hold in $\Delta \cap \mathcal{D}_1$.

Assume that $u(x)$ *is a solution in* D *of the equation* (2.5) *with initial condition* (2.6). *Then the inequality*

$$|u(x)| \leq \omega(x)$$

holds in $\bar{D}$, *where the equality sign can occur only on* S_Σ.

COROLLARY 2.2. *Let* $f(x,u)$ *and* $\bar{f}(x,u)$ *be functions defined and continuous in* Δ *and* $\bar{\Delta}$ *respectively, and let the inequality*

$$f(x,u) < \bar{f}(x,\bar{u})$$

hold for $(x,u)\in\Delta$, $(x,\bar{u})\in\bar{\Delta}$ *and* $u\leq\bar{u}$.

Assume that $u(x)$ *and* $\bar{u}(x)$ *are solutions in* D *of the equations*

$$\Box u = f(x,u), \quad \Box\bar{u} = \bar{f}(x,\bar{u})$$

with initial conditions

$$u(x) = \varphi(x), \quad \bar{u}(x) = \bar{\varphi}(x) \quad \text{on} \quad S_\Sigma$$

respectively. Then the inequality

$$u(x) \leq \bar{u}(x)$$

holds in $\bar{D}$, *where the equality sign can occur only on* S_Σ.

THEOREM 2.2. *Let* $f(x,u)$ *be a function defined and continuous in* Δ.

Assume that the inequality

$$f(x,u) \leq \Box\bar{\omega}(x)$$

holds in $\Delta\cap\mathcal{D}_0$ *and that* $\varphi(0) < \bar{\omega}(0)$.

Then, if $u(x)$ *is a solution in* D *of the equation* (2.5) *with initial condition* (2.6), *the inequality*

$$u(x) < \bar{\omega}(x)$$

holds in $\bar{D}$.

Proof: The proof is similar to that of Theorem 2.1.

Under the assumption that $f(x,u)$ is Lipschitz continuous with respect to u, we can prove the following

THEOREM 2.3. *Let* $f(x,u)$ *be a function defined and continuous in* Δ, *where* Δ *contains* $(x,\bar{\omega}(x))$ *for any* x *in* $\bar{D}$.

Assume that the inequality

$$f(x,u) \leq \Box\bar{\omega}(x) \tag{2.11}$$

holds in $\Delta\cap\mathcal{D}_0$ *and that* $f(x,u)$ *is Lipschitz continuous with respect to* u *in* Δ.

Then, if $u(x)$ *is a solution in* D *of the equation* (2.5) *with initial condition* (2.6), *the inequality*

$$u(x) \leq \bar{\omega}(x) \tag{2.12}$$

holds in $\bar{D}$.

Proof: Assume that the inequality (2.12) does not hold in $\bar{D}$ and denote by x_1^0 the greatest lower bound of the values of x_1 for which there exists a point x in D such that $u(x) > \bar{\omega}(x)$. Let $x^0 + \varepsilon = (x_1^0 + \varepsilon, x_2^0, x_3^0)$, with ε sufficiently small, be a point in D at which $u(x^0+\varepsilon) > \bar{\omega}(x^0+\varepsilon)$. Then we have

$$\begin{aligned}
&\bar{\omega}(x^0+\varepsilon) - u(x^0+\varepsilon) \\
&\quad = \frac{1}{2\pi}\int_{D_S^{x^0+\varepsilon}} \frac{\Box\bar{\omega}(\xi)}{r}\,d\xi + \bar{\omega}(0) - \frac{1}{\pi}\int_{S^{x^0+\varepsilon}} ds \int_0^{R_{x^0+\varepsilon}} \frac{d\bar{\omega}}{dR} R^{-\frac{1}{2}}\,dR \\
&\quad\quad - \left\{ \frac{1}{2\pi}\int_{D_S^{x^0+\varepsilon}} \frac{f(\xi, u(\xi))}{r}\,d\xi + \varphi(0) - \int_{S^{x^0+\varepsilon}} ds \int_0^{R_{x^0+\varepsilon}} \frac{d\varphi}{dR} R^{-\frac{1}{2}}\,dR \right\}
\end{aligned}$$

and hence in virtue of (2.2) and (2.3)

$$\bar{\omega}(x^0+\varepsilon) - u(x^0+\varepsilon) \geq \frac{1}{2\pi}\int_{D_S^{x^0+\varepsilon}} \frac{\Box\bar{\omega}(\xi) - f(\xi, u(\xi))}{r}\,d\xi. \tag{2.13}$$

The last integral may be divided into two parts, one from $\xi_1 = 0$ to $\xi_1 = x_1^0$, and the other from $\xi_1 = x_1^0$ to $\xi_1 = x_1^0 + \varepsilon$. The first integral is non-negative by the definition of x_1^0 and the inequality (2.11). To estimate the second integral from below, we note that

$$\begin{aligned}
&\Box\bar{\omega}(\xi) - f(\xi, u(\xi)) \\
&\quad = \Box\bar{\omega}(\xi) - f(\xi, \bar{\omega}(\xi)) + f(\xi, \bar{\omega}(\xi)) - f(\xi, u(\xi)) \\
&\quad \geq f(\xi, \bar{\omega}(\xi)) - f(\xi, u(\xi)).
\end{aligned}$$

Consequently, denoting by $D_{x_1 = x_1^0}^{x^0+\varepsilon}$ the subdomain of $D_S^{x^0+\varepsilon}$ with $x_1 \geq x_1^0$, we finally obtain from (2.13)

$$\bar{\omega}(x^0+\varepsilon) - u(x^0+\varepsilon) \geq \frac{1}{2\pi}\int_{D_{x_1=x_1^0}^{x^0+\varepsilon}} \frac{f(\xi, \bar{\omega}(\xi)) - f(\xi, u(\xi))}{r}\,d\xi. \tag{2.14}$$

Since $f(x, u)$ is Lipschitz continuous with respect to u, the integral (2.14) may be estimated by

$$\frac{1}{2\pi}\int_{D_{x_1=x_1^0}^{x^0+\varepsilon}} \frac{f(\xi, \bar{\omega}(\xi)) - f(\xi, u(\xi))}{r}\,d\xi \geq \frac{\varepsilon^2 L}{2} \operatorname*{Min}_{x \in D_{x_1=x_1^0}^{x^0+\varepsilon}} (\bar{\omega}(x) - u(x)), \tag{2.15}$$

where L is the Lipschitz constant of $f(x, u)$.

Therefore, choosing ε so small that $\varepsilon^2 L/2 < 1$ and then applying (2.15) to that point in $D_{x_1=x_1^0}^{x^0+\varepsilon}$ at which $\bar{\omega}(x) - u(x)$ attains its minimum, we obtain from (2.14)

$$\operatorname*{Min}_{x \in D_{x_1=x_1^0}^{x^0+\varepsilon}} (\bar{\omega}(x) - u(x)) \geq \frac{\varepsilon^2 L}{2} \operatorname*{Min}_{x \in D_{x_1=x_1^0}^{x^0+\varepsilon}} (\bar{\omega}(x) - u(x))$$

and hence

$$\operatorname*{Min}_{x \in D_{x_1=x_1^0}^{x^0+\varepsilon}} (\bar{\omega}(x) - u(x)) \geq 0,$$

which is contrary to the assumption that $\bar{\omega}(x)-u(x)$ is negative at $x^0+\mathcal{E}$.

q. e. d.

An immediate consequence of Theorem 2.3 is the following

COROLLARY 2.3. *Let $f(x,u)$ be a function defined and continuous in Δ. Further let $f(x,u)$ be Lipschitz continuous and non-decreasing with respect to u in Δ.*

Then there exists in D at most one solution of the equation (2.5) *with initial condition* (2.6).

3. Existence theorems

In this section we shall establish the existence of generalized solutions of the non-linear equation

$$\Box u=f(x,u) \tag{3.1}$$

with vanishing initial condition

$$u(x)=0 \quad \text{on} \quad S_\Sigma.$$

We assume throughout the section that $\underline{\omega}(x)$ and $\bar{\omega}(x)$ satisfy the inequalities (2.2) and (2.3) with $\varphi(x)\equiv 0$ on S_Σ.

We begin with the study of the inhomogeneous equation

$$\Box u=f(x) \tag{3.2}$$

with vanishing initial condition.

LEMMA 3.1. *Let $f(x)$ be a function continuous and bounded in D. Then the function $u(x)$ defined in $D\cup\Sigma$ by the expression*

$$u(x)=\frac{1}{2\pi}\int_{D_\xi^x}\frac{f(\xi)}{r}d\xi \tag{3.3}$$

is continuous in $\bar{D}$ and approaches 0 as x approaches a point x^0 on S_Σ.

Proof: It suffices to show that $u(x)$ approaches 0 as x approaches x^0. Assuming that $|f(x)|\leq M$ in D, we have[8]

$$\begin{aligned}|u(x)| &\leq \frac{M}{2\pi}\int_{D_\xi^x}\frac{1}{r}d\xi \\ &=\frac{M}{2\pi}\cdot\frac{1}{2}\int_{S^x}ds\int_0^{R_x}\left(1-\frac{R}{R_x}\right)R^{-\frac{1}{2}}dR\int_0^1\sigma\, d\sigma=\frac{MR_x}{6}\end{aligned}$$

and hence $u(x)\to 0$ as $x\to x^0$.

q. e. d.

LEMMA 3.2. *Let $f(x)$ be a continuous function in $\bar{D}$ having its first derivatives continuous and bounded in D.*

Then the function $u(x)$ defined by (3.3) *in $D\cup\Sigma$ is once continuously differentiable in D and, moreover, when x approaches a point x^0 on S_Σ along any path l which lies on the plane containing the point x^0 and the x_1-axis, its derivative*

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[8] See the proof of Lemma 3.4.

$\partial u(x)/\partial\lambda_{x^0}$ *approaches* 0 *and the approach to the limit is uniform both for any choice of the path* l *and for any* x^0 *on* S_Σ.

Proof : Setting

$$I^\alpha f(x)=\frac{1}{H_3(\alpha)}\int_{D_S^x} r^{\alpha-3}f(\xi)\,d\xi,$$

we have for sufficiently large α

$$\frac{\partial I^\alpha f(x)}{\partial x_1}=\frac{1}{H_3(\alpha)}\int_{D_S^x}\frac{\partial r^{\alpha-3}}{\partial x_1}f(\xi)\,d\xi=-\frac{1}{H_3(\alpha)}\int_{D_S^x}\frac{\partial r^{\alpha-3}}{\partial \xi_1}f(\xi)\,d\xi$$

and hence by integration by parts

$$\frac{\partial I^\alpha f(x)}{\partial x_1}=\frac{1}{H_3(\alpha)}\int_{S^x} r^{\alpha-3}f(\xi)\,d\xi_2\,d\xi_3+\frac{1}{H_3(\alpha)}\int_{D_S^x} r^{\alpha-3}\frac{\partial f(\xi)}{\partial\xi_1}d\xi, \tag{3.4}$$

since $r^{\alpha-3}=0$ on the cone C^x for $\alpha>3$.

By the assumptions on $f(x)$, the analytic continuation to $\alpha=2$ of the right-hand side of (3.4) can be performed, yielding

$$\frac{\partial I^2 f(x)}{\partial x_1}=\frac{1}{2\pi}\int_{S^x}\frac{f(\xi)}{r}d\xi_2\,d\xi_3+\frac{1}{2\pi}\int_{D_S^x}\frac{1}{r}\,\frac{\partial f(\xi)}{\partial\xi_1}d\xi\,.$$

Since $I^2 f(x)=u(x)$, we obtain

$$\frac{\partial u}{\partial x_1}=\frac{1}{2\pi}\int_{S^x}\frac{f(\xi)}{r}d\xi_2\,d\xi_3+\frac{1}{2\pi}\int_{D_S^x}\frac{1}{r}\,\frac{\partial f(\xi)}{\partial\xi_1}d\xi\,. \tag{3.5}$$

Similarly we have

$$\left.\begin{aligned}\frac{\partial u}{\partial x_2}&=\frac{1}{2\pi}\int_{S^x}\frac{f(\xi)}{r}(\pm d\xi_1\,d\xi_3)+\frac{1}{2\pi}\int_{D_S^x}\frac{1}{r}\,\frac{\partial f(\xi)}{\partial\xi_2}d\xi\,,\\ \frac{\partial u}{\partial x_3}&=\frac{1}{2\pi}\int_{S^x}\frac{f(\xi)}{r}(\pm d\xi_1\,d\xi_2)+\frac{1}{2\pi}\int_{D_S^x}\frac{1}{r}\,\frac{\partial f(\xi)}{\partial\xi_3}d\xi\,,\end{aligned}\right\} \tag{3.6}$$

where $(\pm d\xi_1 d\xi_3)$ is equal to $d\xi_1 d\xi_3$ or $-d\xi_1 d\xi_3$ according as $\xi_2<0$ or $\xi_2>0$, and $(\pm d\xi_1 d\xi_2)$ is equal to $d\xi_1 d\xi_2$ or $-d\xi_1 d\xi_2$ according as $\xi_3<0$ or $\xi_3>0$.

Hence $u(x)$ is once continuously differentiable in D.

To prove the second half of the lemma, we may assume without loss of generality that x^0 is a point on S_Σ with coordinates $(x_1^0, x_2^0, 0)$. Further, assuming that $x_2^0>0$, the derivative $\partial u(x)/\partial\lambda_{x^0}$ can be written as

$$\frac{\partial u(x)}{\partial\lambda_{x^0}}=\frac{1}{\sqrt{2}}\left(\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}\right). \tag{3.7}$$

Thus it suffices to prove that when x, with coordinates $(x_1, x_2, 0)$, approaches x^0 on S_Σ, the derivative (3.7) approaches 0 uniformly with respect to x^0 on S_Σ.

By means of the Lorentz transformation

$$\left.\begin{aligned}\eta_1&=\frac{x_1}{\sqrt{x_1^2-x_2^2}}\xi_1-\frac{x_2}{\sqrt{x_1^2-x_2^2}}\xi_2\,,\\ \eta_2&=\frac{x_2}{\sqrt{x_1^2-x_2^2}}\xi_1-\frac{x_1}{\sqrt{x_1^2-x_2^2}}\xi_2\,,\\ \eta_3&=\xi_3\,,\end{aligned}\right\} \tag{3.8}$$

the retrograde cone C^x is transformed into the retrograde cone C^y in the η-space, where $y=(r_x, 0, 0)$ corresponds to x by the transformation (3.8), while the direct cone S is transformed into itself. Observing that

$$\frac{\partial u}{\partial \xi_1}=\frac{x_1}{\sqrt{x_1^2-x_2^2}}\frac{\partial u}{\partial \eta_1}+\frac{x_2}{\sqrt{x_1^2-x_2^2}}\frac{\partial u}{\partial \eta_2},$$

$$\frac{\partial u}{\partial \xi_2}=-\frac{x_2}{\sqrt{x_1^2-x_2^2}}\frac{\partial u}{\partial \eta_1}-\frac{x_1}{\sqrt{x_1^2-x_2^2}}\frac{\partial u}{\partial \eta_2},$$

we have in virtue of (3.5) and (3.6)

$$\frac{\partial u}{\partial x_1}=\frac{x_1}{2\pi r_x}\left(\int_{S^y}\frac{f'(\eta)}{r}d\eta_2 d\eta_3+\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_1}d\eta\right)$$
$$+\frac{x_2}{2\pi r_x}\left(\int_{S^y}\frac{f'(\eta)}{r}(\pm d\eta_1 d\eta_3)+\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_2}d\eta\right)$$

and

$$\frac{\partial u}{\partial x_2}=-\frac{x_2}{2\pi r_x}\left(\int_{S^y}\frac{f'(\eta)}{r}d\eta_2 d\eta_3+\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_1}d\eta\right)$$
$$-\frac{x_1}{2\pi r_x}\left(\int_{S^y}\frac{f'(\eta)}{r}(\pm d\eta_1 d\eta_3)+\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_2}d\eta\right),$$

where we have set $f'(\eta)=f(\xi)$. Hence (3.7) yields

$$\frac{\partial u(x)}{\partial \lambda_{x^0}}=\frac{x_1-x_2}{2\sqrt{2}\,\pi r_x}\left(\int_{S^y}\frac{f'(\eta)}{r}d\eta_2 d\eta_3-\int_{S^y}\frac{f'(\eta)}{r}(\pm d\eta_1 d\eta_3)\right.$$
$$\left.+\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_1}d\eta-\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_2}d\eta\right). \tag{3.9}$$

By assumption we have the inequalities

$$|f'(\eta)|\leq M,\quad \left|\frac{\partial f'(\eta)}{\partial \eta_1}\right|\leq\frac{M}{r_x},\quad \left|\frac{\partial f'(\eta)}{\partial \eta_2}\right|\leq\frac{M}{r_x}$$

in D_S^y, where M is a constant depending only on $f(x)$ and D. Hence Lemma 3.1 applies, yielding

$$\left|\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_1}d\eta\right|\leq\frac{\pi M r_x}{3},\quad \left|\int_{D_S^y}\frac{1}{r}\frac{\partial f'(\eta)}{\partial \eta_2}d\eta\right|\leq\frac{\pi M r_x}{3}. \tag{3.10}$$

On the other hand, the surface element dS of S^y is given by

$$dS=\frac{1}{2\sqrt{2}}\left(1-\frac{R}{R_x}\right)dR\,d\theta\,,$$

where θ is an angular coordinate. Thus we obtain

$$\left|\int_{S^y}\frac{f'(\eta)}{r}d\eta_2\,d\eta_3\right|\leq M\int_{S^y}\frac{1}{r}d\eta_2 d\eta_3=\frac{M}{\sqrt{2}}\int_{S^y}\frac{1}{r}dS$$
$$=\frac{M}{4}\int_0^{2\pi}d\theta\int_0^{R_x}\left(1-\frac{R}{R_x}\right)R^{-\frac{1}{2}}dR=\frac{2\pi M r_x}{3} \tag{3.11}$$

and similarly

$$\left|\int_{S^y}\frac{f'(\eta)}{r}(\pm d\eta_1 d\eta_3)\right|\leq\frac{2\pi M r_x}{3}. \tag{3.12}$$

Therefore, observing that the difference $x_1 - x_2$ approaches 0 as x approaches x^0 on S_Σ, we conclude from (3.9), (3.10), (3.11) and (3.12) that $\partial u(x)/\partial \lambda_{x^0}$ approaches 0 as x approaches x^0 and that the approach to the limit is uniform for any x^0 on S_Σ. q.e.d.

LEMMA 3.3. *Let $f(x)$ be a twice continuously differentiable function in $\bar{D}$. Then the function $u(x)$ defined by* (3.3) *in $D\cup\Sigma$ is twice continuously differentiable in D and satisfies the relation*

$$\Box u(x) = f(x)$$

in D.

Proof : We have for sufficiently large α

$$\left.\begin{aligned} \frac{\partial^2 I^\alpha f(x)}{\partial x_1^2} &= \frac{1}{H_3(\alpha)} \int_{D_S^x} \frac{\partial^2 r^{\alpha-3}}{\partial x_1^2} f(\xi)\, d\xi = \frac{1}{H_3(\alpha)} \int_{D_S^x} \frac{\partial^2 r^{\alpha-3}}{\partial \xi_1^2} f(\xi)\, d\xi\,, \\ \frac{\partial^2 I^\alpha f(x)}{\partial x_2^2} &= \frac{1}{H_3(\alpha)} \int_{D_S^x} \frac{\partial^2 r^{\alpha-3}}{\partial \xi_2^2} f(\xi)\, d\xi\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{3.13}$$

and in particular

$$\Box I^\alpha f(x) = I^{\alpha-2} f(x). \tag{3.14}$$

By assumption the right-hand sides of (3.13) and (3.14) can be analytically continued to $\alpha=2$. Hence we see that $u(x)$ is twice continuously differentiable in D and that

$$\Box u(x) = \Box I^2 f(x) = I^0 f(x) = f(x). \qquad \text{q.e.d.}$$

We can now establish the existence of a unique generalized solution of the inhomogeneous equation (3.2) with vanishing initial condition for any continuous function $f(x)$ in $\bar{D}$.

THEOREM 3.1. *Let $f(x)$ be a function defined and continuous in $\bar{D}$. Then the function $u(x)$ defined by the expression* (3.3) *in $D\cup\Sigma$ is a unique generalized solution of the inhomogeneous equation* (3.2) *with vanishing initial condition.*

Proof : Since $f(x)$ is continuous in $\bar{D}$, there exists an infinite sequence of functions $\{f_n(x)\}$ twice continuously differentiable in $\bar{D}$ and converging uniformly in $\bar{D}$ to the function $f(x)$. Then it follows from Lemmas 3.1, 3.2 and 3.3 that the sequence of functions $\{u_n(x)\}$ defined in $D\cup\Sigma$ by

$$u_n(x) = \frac{1}{2\pi} \int_{D_S^x} \frac{f_n(\xi)}{r} d\xi, \quad (n=1, 2, \cdots) \tag{3.15}$$

is a sequence of solutions in $C^*[D]$ of the inhomogeneous equations

$$\Box u_n = f_n(x) \qquad (n=1, 2, \cdots)$$

with vanishing initial condition. From (3.15), it is obvious that the sequence $\{u_n(x)\}$ converges uniformly in $\bar{D}$ to the function $u(x)$. Hence, by definition, $u(x)$ is a generalized solution of (3.2) with vanishing initial condition.

The uniqueness follows from Corollary 1.1. q.e.d.

REMARK. The function $u(x)$ defined in $D \cup \Sigma$ by

$$u(x)=\varphi(0)-\frac{1}{\pi}\int_{S^x} ds\int_0^{R_x}\frac{d\varphi}{dR}R^{-\frac{1}{2}}dR \tag{3.16}$$

is continuous in $\bar{D}$ and takes on the prescribed value $\varphi(x)$ on S_Σ.[9] Moreover it is obvious that if $\varphi(x)$ is three times continuously differentiable on S_Σ, $u(x)$ is twice continuously differentiable in D and satisfies $\Box u=0$ in D. Hence the function $u(x)$ defined by (3.16) is a generalized solution of the homogeneous wave equation with initial condition $\varphi(x)$ on S_Σ if there exists an infinite sequence of functions $\{\varphi_n(x)\}$ three times continuously differentiable on S_Σ such that $\varphi_n(x)$ and its first derivatives converge uniformly on S_Σ to $\varphi(x)$ and its corresponding first derivatives respectively.

As a preparation for Theorem 3.2 which assures the existence of generalized solutions of (3.1) with vanishing initial condition, we shall prove the following

LEMMA 3.4. *Let* $\{f_\lambda(x);\ \lambda\in\Lambda\}$ *be a family of functions continuous and uniformly bounded in* D : $|f_\lambda(x)|\le M$.

Then the family of functions $\{u_\lambda(x)\}$ *defined in* $D\cup\Sigma$ *by*

$$u_\lambda(x)=\frac{1}{2\pi}\int_{D_S^x}\frac{f_\lambda(\xi)}{r}d\xi \qquad (\lambda\in\Lambda)$$

are uniformly bounded and equi-continuous in $\bar{D}$.

Proof : The uniform boundedness follows immediately from the inequality

$$|u_\lambda(x)|\le\frac{M}{2\pi}\int_{D_S^x}\frac{1}{r}d\xi=\frac{Mr_x^2}{6}. \tag{3.17}$$

To prove the equi-continuity of the family $\{u_\lambda(x)\}$, we first introduce a coordinate system into $\bar{D}_S^x$. Let ξ^0 be a point on S^x and ξ be a point on the straight segment joining ξ^0 and x. Denote by σ the ratio of $r_{x\xi}$ to $r_{x\xi^0}$: $r_{x\xi}=\sigma r_{x\xi^0}$, where $r_{x\xi}$ is the distance between x and ξ. Values of σ for points ξ on the retrograde cone C^x can be defined by continuity. Since $R_{x\xi^0}=r_{x\xi^0}^2$ varies in a linear manner along every generator of S^x, we shall choose $R_{x\xi^0}$ as a coordinate. Further, we shall choose an angular coordinate θ to specify directions on the cone S^x. Then $R_{x\xi^0}$, σ and θ form a coordinate system in $\bar{D}_S^x$. In terms of these variables, the volume element $d\xi$ is written, as is easily verified, in the form

$$d\xi=\frac{1}{2}ds\left(1-\frac{R}{R_x}\right)dR\,\sigma^2d\sigma,$$

where we have set $R=R_{x\xi^0}$.

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[9] See M. Riesz [15] pp. 115-125.

For any positive number ε, we choose a number δ_1 such that

$$\frac{M\delta_1^2}{6}<\frac{\varepsilon}{4}$$

and fix it. Then, from (3.17), we have for any x, x' with r_x, $r_{x'}\leq\delta_1$ in $\bar{D}$

$$|u_\lambda(x)-u_\lambda(x')|\leq|u_\lambda(x)|+|u_\lambda(x')|<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}. \tag{3.18}$$

Next we consider two arbitrary points x and x' with r_x, $r_{x'}\geq\delta_1$. We may assume without loss of generality that x has coordinates $(x_1, x_2, 0)$ and that $r_x\leq r_{x'}$. By means of the Lorentz transformation (3.8), the point x is transformed into a point y with coordinates $(r_x, 0, 0)$ in the η-space. Further we have

$$u_\lambda(x)=u_\lambda(y)=\frac{1}{2\pi}\int_{D_S^y}\frac{f_\lambda'(\eta)}{r_{y\eta}}d\eta$$

and

$$u_\lambda(x')=u_\lambda(y')=\frac{1}{2\pi}\int_{D_S^{y'}}\frac{f_\lambda'(\eta)}{r_{y'\eta}}d\eta,$$

where $f_\lambda'(\eta)=f_\lambda(\xi)$, y' corresponds to x' by (3.8), and $r_{y'\eta}$ denotes the distance between y' and η.

On the other hand, denoting by $D_{\delta_2}^y$ the subdomain of $\bar{D}_S^y$ which is written in terms of the coordinate system in $\bar{D}_S^y$

$$D_{\delta_2}^y=\{\eta;\ \delta_2^2\leq R\leq R_y,\ 0\leq\sigma\leq1,\ 0\leq\theta\leq2\pi\},$$

we have

$$\left|\int\limits_{D_S^y-D_{\delta_2}^y}\frac{f_\lambda'(\eta)}{r}d\eta\right|\leq M\int\limits_{D_S^y-D_{\delta_2}^y}\frac{1}{r}d\eta$$

$$=\frac{M}{2}\int\limits_{S^y}ds\int\limits_0^{\delta_2^2}\left(1-\frac{R}{R_y}\right)\frac{dR}{\sqrt{R}}\int\limits_0^1\sigma\,d\sigma<\frac{M\pi r_y\delta_2}{2}.$$

Hence we choose a number δ_2 such that

$$Mr_y\delta_2<\frac{\varepsilon}{4} \tag{3.19}$$

for any x in $\bar{D}$ and fix it.

Denoting further by $C^{y''}$ the smallest retrograde cone whose vertex y'' lies on the η_1-axis and which contains the cone $C^{y'}$ in its interior, we can rewrite the difference $u_\lambda(x)-u_\lambda(x')$ as follows:

$$u_\lambda(x)-u_\lambda(x')=(u_\lambda(y)-u_\lambda(y''))+(u_\lambda(y'')-u_\lambda(y')). \tag{3.20}$$

Assume now that $|y-y'|$[10] $<\delta_3$, where δ_3 is a positive number. Then it is obvious that we have

$$0<y_1''-y_1<\sqrt{2}\,\delta_3. \tag{3.21}$$

---

[10] $|y-y'|=\sqrt{\sum_{i=1}^{3}(y_i-y_i')^2}$.

We note from (3.8) that, for any number δ_3, there exists a number δ depending only on δ_3, δ_1 and D such that

$$|x-x'|<\delta \quad \text{implies} \quad |y-y'|<\delta_3. \tag{3.22}$$

The first difference on the right-hand side of (3.20) can be estimated in the following manner:

$$\begin{aligned} u_\lambda(y)-u_\lambda(y'') &= \frac{1}{2\pi}\int\limits_{D_S^y}\frac{f_\lambda'(\eta)}{r_{y\eta}}d\eta-\frac{1}{2\pi}\int\limits_{D_S^{y'}}\frac{f_\lambda'(\eta)}{r_{y''\eta}}d\eta \\ &= \frac{1}{2\pi}\int\limits_{D_S^y-D_{\delta_2}^y}\frac{f_\lambda'(\eta)}{r_{y\eta}}d\eta-\frac{1}{2\pi}\int\limits_{D_S^y-D_{\delta_2}^y}\frac{f_\lambda'(\eta)}{r_{y''\eta}}d\eta \\ &\quad +\frac{1}{2\pi}\int\limits_{D_{\delta_2}^y}\left(\frac{1}{r_{y\eta}}-\frac{1}{r_{y''\eta}}\right)f_\lambda'(\eta)\,d\eta. \end{aligned}$$

Then it follows from (3.19) that

$$\left|\frac{1}{2\pi}\int\limits_{D_S^y-D_{\delta_2}^y}\frac{f_\lambda'(\eta)}{r_{y\eta}}d\eta\right|<\frac{\varepsilon}{16}. \tag{3.23}$$

Further we have in virtue of (3.21)

$$\begin{aligned} \left|\int\limits_{D_S^{y'}-D_{\delta_2}^y}\frac{f_\lambda'(\eta)}{r_{y''\eta}}d\eta\right| &\leq M\int\limits_{D_S^{y'}-D_{\delta_2}^y}\frac{1}{r}d\eta \\ &\leq M\int\limits_{D_S^{y''}-D_{\delta_4}^y}\frac{1}{r}d\eta+M\cdot\frac{1}{2}\int\limits_{s^{y'}}ds\int\limits_{\delta_4^2}^{R_{y'}}\left(1-\frac{R}{R_{y''}}\right)\frac{dR}{\sqrt{R}}\int\limits_0^{\delta_4/\sqrt{R}}\sigma\,d\sigma \\ &< M\pi r_{y''}\delta_4, \end{aligned}$$

where

$$\delta_4=\sqrt{\delta_2^2\left(1+\frac{r_{y''}-r_y}{r_y}\right)+\sqrt{2}\,r_{y''}\delta_3}\,.$$

Hence it is evident from (3.21) that we can choose δ_3 so small that we have the inequality

$$Mr_{y''}\delta_4<\frac{\varepsilon}{4}$$

for any x with $r_x \geq \delta_1$ in $\overline{D}$, where δ_3 depends only on ε, δ_1, δ_2, M and D. Therefore we have

$$\left|\frac{1}{2\pi}\int\limits_{D_S^{y''}-D_{\delta_2}^y}\frac{f_\lambda'(\eta)}{r_{y''\eta}}d\eta\right|<\frac{\varepsilon}{8}. \tag{3.24}$$

Since $r_{y\eta}$ is continuous with respect to all its arguments, we can choose δ_3 so small that $|y-y'|<\delta_3$ implies $|r_{y\eta}-r_{y''\eta}|<\delta_5$ for any x, x' in $\overline{D}$,where δ_5 is an arbitrary positive number. Hence we have

$$\left|\frac{1}{2\pi}\int_{D^y_{\delta_2}}\left(\frac{1}{r_{y\eta}}-\frac{1}{r_{y''\eta}}\right)f'_\lambda(\eta)d\eta\right| \leq \frac{M}{2\pi}\int_{D^y_{\delta_2}}\frac{|r_{y\eta}-r_{y''\eta}|}{r_{y\eta}r_{y''\eta}}d\eta$$
$$< \frac{M\delta_5}{2\pi\delta_2^2}\int_{D^y_S} d\eta = \frac{Mr_y^3\delta_5}{24\delta_2^2}.$$

Therefore, by taking δ_3 sufficiently small, we have

$$\left|\frac{1}{2\pi}\int_{D^y_{\delta_2}}\left(\frac{1}{r_{y\eta}}-\frac{1}{r_{y''\eta}}\right)f'_\lambda(\eta)d\eta\right| < \frac{\varepsilon}{16}, \tag{3.25}$$

where δ_3 depends only on ε, δ_1, δ_2, M and D.

Combining (3.23), (3.24) and (3.25), we obtain

$$|u_\lambda(y)-u_\lambda(y'')| < \frac{\varepsilon}{4} \tag{3.26}$$

for any y, y' with $|y-y'|<\delta_3$, where δ_3 depends only on ε, δ_1, δ_2, M and D.

Similarly, we obtain the inequality

$$|u_\lambda(y'')-u_\lambda(y')| < \frac{\varepsilon}{4} \tag{3.27}$$

for any y, y' with $|y-y'|<\delta_3$.

Thus the inequalities (3.26) and (3.27) yield

$$|u_\lambda(y)-u_\lambda(y')| < \frac{\varepsilon}{2} \tag{3.28}$$

for any y, y' with $|y-y'|<\delta_3$, where δ_3 depends only on ε, δ_1, δ_2, M and D.

Therefore, observing that δ_1 and δ_2 depend only on ε, M and D, we can conclude from (3.18), (3.22) and (3.28) that we have the inequality

$$|u_\lambda(x)-u_\lambda(x')| < \varepsilon$$

for any x, x' with $|x-x'|<\delta$ in $\bar{D}$, where δ depends only on ε, M and D. Hence the proof is complete. q. e. d.

Now we can prove the following existence theorem.

THEOREM 3.2. *Let $f(x,u)$ be a continuous function defined in $\mathcal{D}$*[11] *and let the inequality*

$$\Box\underline{\omega}(x) \leq f(x,u) \leq \Box\bar{\omega}(x) \tag{3.29}$$

hold in $\mathcal{D}$.

Then there exists a generalized solution $u(x)$ of the equation (3.1) *with vanishing initial condition which satisfies the inequality*

$$\underline{\omega}(x) \leq u(x) \leq \bar{\omega}(x) \tag{3.30}$$

in $\bar{D}$.

Proof: Let $\mathcal{F}$ be the family of all continuous functions in $\bar{D}$ satisfying the inequality (3.30). Then $\mathcal{F}$ is not empty, since $v(x)\equiv 0$ belongs to $\mathcal{F}$.

---

[11] Note that $\underline{\omega}(x)$ and $\bar{\omega}(x)$ satisfy the inequalities (2.2) and (2.3) with $\varphi(x)\equiv 0$ on S_Σ.

It is also evident that $\mathscr{F}$ is a closed and convex set in the Banach space $C[\bar{D}]$[12] with the norm $\|v\|=\underset{x\in\bar{D}}{\mathrm{Max}}|v(x)|$ $(v\in C[\bar{D}])$.

Since the function $f_{[v]}=f(x, v(x))$ is continuous in $\bar{D}$ for any $v(x)\in\mathscr{F}$, it follows from Theorem 3.1 that there exists a unique generalized solution $u(x)$ in D of the inhomogeneous equation

$$\Box u = f_{[v]}(x)$$

with vanishing initial condition. Moreover, we see from Corollary 1.1, (2.1), (2.2) and (2.3) that $u(x)$ belongs also to $\mathscr{F}$. Hence we can define a mapping

$$T: \quad v(x)\in\mathscr{F} \rightarrow u(x)=T(v(x))\in\mathscr{F}.$$

It is obvious from Corollary 1.1 and Theorem 3.1 that T is continuous. Compactness of $T(\mathscr{F})$ in the Banach space $C[\bar{D}]$ follows from Lemma 3.4.

Therefore it follows from the fixed point theorem of Schauder-Tychonoff that there exists a function $u(x)\in\mathscr{F}$ such that

$$u(x) = T(u(x)).$$

Then $u(x)$ is a generalized solution of (3.1) with vanishing initial condition satisfying (3.30). q.e.d.

Immediate consequences of Theorem 3.2 and Lemma 3.4 are the following

COROLLARY 3.1. *Let* $f(x, u)$ *be a continuous function defined in* $\mathscr{D}_1$ *and let the inequality*

$$|f(x, u)| \leq \Box\omega(x)$$

hold in $\mathscr{D}_1$, *where we assume that* $\omega(x)$ *satisfies the inequalities* (2.4) *with* $\varphi(x)\equiv 0$ *on* S_Σ.

Then there exists a generalized solution $u(x)$ *of* (3.1) *with vanishing initial condition satisfying the inequality*

$$|u(x)| \leq \omega(x)$$

in $\bar{D}$.

COROLLARY 3.2. *Let* $f(x, u)$, $\underline{f}(x, u)$ *and* $\bar{f}(x, u)$ *be continuous functions defined in* Δ, $\underline{\Delta}$ *and* $\bar{\Delta}$ *respectively, and let the inequality*

$$\underline{f}(x, \underline{u}) \leq f(x, u) \leq \bar{f}(x, \bar{u})$$

hold for any $(x, u)\in\Delta$, $(x, \underline{u})\in\underline{\Delta}$, *and* $(x, \bar{u})\in\bar{\Delta}$ *such that* $\underline{u}\leq u\leq\bar{u}$.

Assume that there exist generalized solutions $\underline{u}(x)$ *and* $\bar{u}(x)$ *in* D *of the equations*

$$\Box\underline{u} = \underline{f}(x, \underline{u}), \quad \Box\bar{u} = \bar{f}(x, \bar{u})$$

having the initial conditions

$$\underline{u}(x) = \underline{\varphi}(x), \quad \bar{u}(x) = \bar{\varphi}(x) \qquad \text{on} \quad S_\Sigma$$

---

[12] $C[\bar{D}]$ denotes the set of all continuous functions in $\bar{D}$.

respectively, where $\underline{\varphi}(x)$ *and* $\overline{\varphi}(x)$ *satisfy the inequalities* (2.2) *and* (2.3) *with* $\varphi(x)\equiv 0$ *on* S_Σ. *Assume further that the inequality*

$$\underline{u}(x) \leq \overline{u}(x)$$

holds in $\overline{D}$.

Then, if $\varDelta$ *contains the domain defined by*

$$x \in \overline{D},\ \underline{u}(x) \leq u \leq \overline{u}(x),$$

there exists a generalized solution $u(x)$ *of* (3.1) *with vanishing initial condition satisfying the inequality*

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x)$$

in $\overline{D}$.

Under additional assumptions on the function $f(x, u)$, we can prove the existence of a unique once continuously differentiable generalized solution of (3.1) with vanishing initial condition.

LEMMA 3.5. *Let* $f(x, u)$ *be a function once continuously differentiable in* $\mathcal{D}$ *and non-decreasing with respect to* u.

Assume that $u(x)$ *is a generalized solution of* (3.1) *with vanishing initial condition, having its first derivatives continuous and bounded in* D. *Then the inequality*

$$\left|\frac{\partial u}{\partial x_1}\right|,\ \left|\frac{\partial u}{\partial x_2}\right|,\ \left|\frac{\partial u}{\partial x_3}\right| \leq e^{\alpha x_1} + \frac{M x_1}{3}$$

holds in D, *where* α *is a constant depending only on* $f(x, u)$ *and* D, *and* M *is a constant such that* $|f(x, 0)| \leq M$ *in* $\mathcal{D}$.

Proof: Since $u(x)$ is written in $D \cup \Sigma$ in the form

$$u(x) = \frac{1}{2\pi}\int_{D_S^x} \frac{f(\xi, u(\xi))}{r}\, d\xi,$$

we have in virtue of (3.5)

$$\frac{\partial u}{\partial x_1} = \frac{1}{2\pi}\int_{S^x} \frac{f(\xi, 0)}{r}\, d\xi_2\, d\xi_3 + \frac{1}{2\pi}\int_{D_S^x}\left(\frac{\partial f}{\partial \xi_1} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial \xi_1}\right)\frac{1}{r} d\xi. \tag{3.31}$$

If we set

$$v(x) = \frac{\partial u}{\partial x_1} - \frac{1}{2\pi}\int_{S^x} \frac{f(\xi, 0)}{r}\, d\xi_2\, d\xi_3\,, \tag{3.32}$$

(3.31) is written as

$$v(x) = \frac{1}{2\pi}\int_{D_S^x}\left(\frac{\partial f}{\partial \xi_1} + \frac{\partial f}{\partial u}\cdot\frac{1}{2\pi}\int_{S^\xi}\frac{f(\eta, 0)}{r_{\xi\eta}} d\eta_2\, d\eta_3 + \frac{\partial f}{\partial u} v(\xi)\right)\frac{1}{r} d\xi. \tag{3.33}$$

Hence $v(x)$ is a continuous solution of the integral equation (3.33) in D.

On the other hand, observing that $u(x) = (x_1^2 - x_2^2 - x_3^2)/6$ is the unique solution of the equation $\Box u = 1$ having vanishing condition, we get

$$\frac{1}{2\pi}\int_{S^x} \frac{1}{r} d\xi_2\, d\xi_3 = \frac{1}{3} x_1,$$

which yields

$$\left|\frac{1}{2\pi}\int_{S^x}\frac{f(\xi,0)}{r}d\xi_2\,d\xi_3\right| \le \frac{M}{2\pi}\int_{S^x}\frac{1}{r}d\xi_2\,d\xi_3=\frac{Mx_1}{3}. \tag{3.34}$$

Hence $v(x)$ is bounded in D. Therefore we see from (3.33), (3.34) and the assumptions on $f(x,u)$ that $v(x)$ approaches 0 as x approaches a point x^0 on S_Σ.

We now choose α so large that the inequality

$$\alpha^2 \ge \left|\frac{\partial f}{\partial x_1}\right| + \left(1+\frac{Mx_1}{3}\right)\left|\frac{\partial f}{\partial u}\right|$$

holds in $\mathcal{D}$. Then we can take as $\bar{\omega}(x)$ in Theorem 2.2 the function $e^{\alpha x_1}$ and we obtain the inequality

$$v(x) \le e^{\alpha x_1}$$

in D, where α is a constant depending only on $f(x,u)$ and D.

Similarly, we have the inequality

$$v(x) \ge - e^{\alpha x_1}$$

in D and hence

$$|v(x)| \le e^{\alpha x_1}$$

in D. Therefore, from (3.32) and (3.34), we obtain the inequality

$$\left|\frac{\partial u}{\partial x_1}\right| \le e^{\alpha x_1} + \frac{Mx_1}{3}$$

in D, where α is a constant depending only on $f(x,u)$ and D.

$\partial u/\partial x_2$ and $\partial u/\partial x_3$ can be estimated in a similar manner. q.e.d.

THEOREM 3.3. *Let $f(x,u)$ be a function once continuously differentiable in $\mathcal{D}$ and non-decreasing with respect to u, and let the inequality* (3.29) *hold in $\mathcal{D}$. Then there exists a unique generalized solution $u(x)$ of the equation* (3.1) *with vanishing initial condition which is once continuously differentiable in D.*

Proof: Let $\mathcal{F}$ be the family of all continuous functions $v(x)$ in $\bar{D}$ satisfying the inequality (3.30) such that $v(x)$ is Lipschitz continuous in $\bar{D}$:

$$|v(x)-v(x')| \le \left(e^{\alpha x_1}+\frac{Mx_1}{3}\right)(|x_1-x_1'|+|x_2-x_2'|+|x_3-x_3'|),$$

where $x_1 \ge x_1'$ and $|f(x,u)| \le M$ in $\mathcal{D}$.

Then, using the same notation as in Theorem 3.2, it is easy to see that $u(x)=T(v(x))$ is once continuously differentiable in D. Moreover, in virtue of Lemma 3.5, we have $T(\mathcal{F})\subset\mathcal{F}$ for sufficiently large α.

Therefore, applying the fixed point theorem, we see that there exists a generalized solution of (3.1) with vanishing condition which is once continuously differentiable in D.

The uniqueness follows from Corollary 2.3. q.e.d.

4. Maximal and minimal solutions

We begin with the following

DEFINITION 4.1. *Let $\bar{u}(x)$ be a solution*[13] *in D of the equation*

$$\Box u = f(x, u) \tag{4.1}$$

with initial condition

$$u(x) = \varphi(x) \quad \text{on } S_{\Sigma}. \tag{4.2}$$

Then $\bar{u}(x)$ is said to be the maximal solution of the equation (4.1) *with initial condition* (4.2) *if, for any solution $u(x)$ of* (4.1) *with initial condition* (4.2), *$\bar{u}(x)$ satisfies the inequality*

$$\bar{u}(x) \geq u(x)$$

in $\bar{D}$.

The minimal solution is defined in a similar manner.

In what follows, we assume that $\underline{\omega}(x)$ and $\bar{\omega}(x)$ satisfy the inequalities (2.2) and (2.3) with $\varphi(x) \equiv 0$ on S_{Σ}. Combining Theorem 2.1 and Theorem 3.2, we can prove the

THEOREM 4.1. *Let $f(x, u)$ be a function continuous in $\mathscr{D}$ and non-decreasing with respect to u.*

Assume that the inequality

$$\Box\underline{\omega}(x) \leq f(x, u) < \Box\bar{\omega}(x) \tag{4.3}$$

holds in $\mathscr{D}$. Then there exists in D the maximal solution of the equation (4.1) *with vanishing initial condition.*

Proof: Since $\bar{D}$ and $\mathscr{D}$ are closed, we can choose a positive number ε such that

$$f(x, u) + \varepsilon < \Box\bar{\omega}(x).$$

Then we choose a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$, $\varepsilon_n \leq \varepsilon$. In virtue of Theorem 3.2, there exists a solution $u_n(x)$ of the equation

$$\Box u = f(x, u) + \varepsilon_n$$

with vanishing condition. Then we have in virtue of Theorem 2.1

$$u_n(x) \geq u_{n+1}(x) \geq u(x)$$

in $\bar{D}$, where $u(x)$ is an arbitrary solution in D of (4.1) with vanishing condition. Further it follows from Lemma 3.4 that the sequence $\{u_n(x)\}$ converges uniformly in $\bar{D}$ to a limit function $\bar{u}(x)$.

Then $\bar{u}(x)$ is obviously a solution of (4.1) with vanishing condition and satisfies the inequality

$$\bar{u}(x) \geq u(x)$$

in $\bar{D}$. q.e.d.

COROLLARY. *Let $f(x, u)$ satisfy the hypotheses in Theorem* 4.1.

---

[13] By a solution, we mean a generalized solution.

Assume that the inequality

$$\Box \underline{\omega}(x) < f(x, u) \leq \Box \bar{\omega}(x)$$

holds in $\mathcal{D}$. *Then there exists in* D *the minimal solution of* (4.1) *with vanishing condition.*

EXAMPLE. Let $f(u)$ be a function defined by

$$f(u) = \begin{cases} \lambda(\lambda+1) u^{1-2/\lambda} & 0 < u < +\infty, \\ 0 & -\infty < u \leq 0, \end{cases}$$

where $\lambda > 2$.

Then $u \equiv 0$ and $u = (x_1^2 - x_2^2 - x_3^2)^{\lambda/2}$ are obviously the solutions of the equation

$$\Box u = f(u) \tag{4.4}$$

with vanishing condition.

It is evident that $u \equiv 0$ is the minimal solution.

We shall show that $u = (x_1^2 - x_2^2 - x_3^2)^{\lambda/2}$ is the maximal solution. Setting

$$\bar{\omega}(x) = ((x_1 + h)^2 - x_2^2 - x_3^2)^{\lambda/2},$$

where $h > 0$, it is easy to see that $\bar{\omega}(0) > 0$ and $\partial \bar{\omega}(x)/\partial \lambda_x > 0$ for any x on S_Σ. Hence Theorem 2.2 applies, yielding

$$\bar{\omega}(x) > u(x)$$

in $\bar{D}$, where $u(x)$ is an arbitrary solution of (4.4) with vanishing condition. Since h is arbitrary, we get

$$(x_1^2 - x_2^2 - x_3^2)^{\lambda/2} \geq u(x)$$

in $\bar{D}$.

5. Uniqueness conditions

In this section, by making use of the comparison theorem obtained in section 2, we shall derive some uniqueness conditions for solutions of the equation

$$\Box u = f(x, u) \tag{5.1}$$

with vanishing initial condition.

We begin with the following

LEMMA 5.1. *Let* $\omega_n(x)$ $(n = 1, 2, \cdots)$ *be a sequence of functions continuous in* $\bar{D}$ *and once continuously differentiable on* S_Σ *such that they are expressible in* $D \cup \Sigma$ *in the form*

$$\omega_n(x) = \frac{1}{2\pi} \int_{D_S^x} \frac{\Box \omega_n(\xi)}{r} d\xi + \omega_n(0) - \frac{1}{\pi} \int_{S^x} ds \int_0^{R_x} \frac{d\omega_n}{dR} R^{-\frac{1}{2}} dR \qquad (n = 1, 2, \cdots),$$

where $\Box \omega_n(x)$ *are continuous in* $\bar{D}$ *and* $\omega_n(0) \geq 0$, $\partial \omega_n(x)/\partial \lambda_x \geq 0$ *on* S_Σ.

Assume that $F(x, u)$ *is a continuous function in* $\mathcal{D}_2$ *with*

$$F(x, 0) = 0$$

and that the inequality

$$0 \leq F(x, u) < \Box\omega_n(x)$$

holds for $0 \leq u \leq \omega_n(x)$.

Then, if the sequence $\{\omega_n(x)\}$ *converges uniformly to* 0 *in* $\bar{D}$, $u(x) \equiv 0$ *is the unique solution in* D *of the equation*

$$\Box u = F(x, u) \tag{5.2}$$

with vanishing initial condition.

Proof: That $u(x) \equiv 0$ is a solution is evident. We define $F_0(x, u)$ as follows:

$$F_0(x, u) = F(x, [u]_0), \tag{5.3}$$

where

$$[u]_0 = \begin{cases} u & 0 \leq u < +\infty, \\ 0 & -\infty < u < 0. \end{cases}$$

Then $F_0(x, u)$ is continuous in the domain

$$\Delta = \{(x, u);\ x \in \bar{D},\ -\infty < u < \infty\}.$$

It is obvious that a solution of (5.2) is a solution of the equation

$$\Box u = F_0(x, u). \tag{5.4}$$

Conversely, if a solution $u(x)$ of (5.4) satisfies the inequality $u(x) \geq 0$, $u(x)$ is a solution of (5.2). Assume now that $u(x)$ is a solution in D of (5.4) satisfying vanishing initial condition.

Setting

$$\underline{\omega}(x) = -\varepsilon(x_1^2 - x_2^2 - x_3^2),$$

where ε is an arbitrary positive number, we have in virtue of Theorem 2.1

$$-\varepsilon(x_1^2 - x_2^2 - x_3^2) \leq u(x) \leq \omega_n(x).$$

Since ε is arbitrary, we have

$$u(x) \equiv 0. \qquad \text{q.e.d.}$$

THEOREM 5.1. *Let* $F(x, u)$ *be a function continuous in* $\mathcal{D}_2$ *and non-decreasing with respect to* u.

Assume that, for $F(x, u)$, *there exists a function* $\omega(x)$ *satisfying the inequalities* (2.4) *with* $\varphi(x) \equiv 0$ *on* S_Σ *and that the inequality*

$$0 \leq F(x, u) < \Box\omega(x)$$

holds for $0 \leq u \leq \omega(x)$.

Then $u(x) \equiv 0$ *is the unique solution in* D *of the equation* (5.2) *with vanishing initial condition if and only if there exists a sequence of functions* $\{\omega_n(x)\}$ *converging uniformly to* 0 *in* $\bar{D}$ *which satisfy the conditions*: i) $\omega_n(x)$ *fulfills the hypotheses in Lemma* 5.1 *and* $\omega_n(x) \leq \omega(x)$, ii) *the inequality*

$$0 \leq F(x, u) < \Box\omega_n(x)$$

holds for $0 \leq u \leq \omega_n(x)$.

Proof: The sufficiency follows from Lemma 5.1.

Assume that $u(x) \equiv 0$ is a unique solution in D of (5.2) with vanishing initial condition. From Theorem 4.1, there exist maximal solutions $\omega_n(x)$ in D of the equations

$$\Box u = F(x, u)+\varepsilon_n \qquad (n=1, 2, \cdots) \tag{5.5}$$

with vanishing initial condition, where $\varepsilon_n \downarrow 0$ and ε_1 is sufficiently small. Thus the inequality

$$\omega_n(x) \geq \omega_{n+1}(x)$$

holds in $\bar{D}$ and, since $u(x)\equiv 0$ is a unique solution of (5.2), the sequence $\{\omega_n(x)\}$ converges uniformly to 0 in $\bar{D}$.

Moreover, from the property of $F(x, u)$, (5.2) and (5.5), we have the inequality

$$0 \leq F(x, u) < \Box\omega_n(x)$$

for $0 \leq u \leq \omega_n(x)$. Hence $\{\omega_n(x)\}$ is a desired sequence. q.e.d.

A function $F(x, u)$ is said to satisfy *Condition* (*U*) in $\mathcal{D}_2$ if $F(x, u)$ satisfies the hypotheses in Theorem 5.1 and $u(x)\equiv 0$ is a unique solution in D of (5.2) with vanishing initial condition.

THEOREM 5.2. *Let $f(x, u)$ be a continuous function defined in Δ and let the inequality*

$$|f(x, u)-f(x, u')| \leq F(x, |u-u'|) \tag{5.6}$$

hold in $\Delta \cap \mathcal{D}_2$, where $F(x, u)$ satisfies Condition (*U*) *in $\mathcal{D}_2$.*

Then the equation (5.1) *admits at most one solution in D having vanishing initial condition.*

Proof: Assume that $u_1(x)$ and $u_2(x)$ are solutions in D of (5.1) with vanishing condition. Then the difference $u(x)=u_1(x)-u_2(x)$ is a solution in D of the equation

$$\Box u = f(x, u+u_2(x))-f(x, u_2(x))$$

with vanishing condition. By assumption, we have

$$|f(x, u+u_2(x))-f(x, u_2(x))| \leq F(x, |u|). \tag{5.7}$$

Further, by the assumptions on $F(x, u)$, there exists a sequence $\{\omega_n(x)\}$ converging uniformly to 0 in $\bar{D}$ such that the inequality

$$0 \leq F(x, |u|) < \Box\omega_n(x)$$

holds for $|u| \leq \omega_n(x)$. Hence we obtain from (5.7)

$$|f(x, u+u_2(x))-f(x, u_2(x))| < \Box\omega_n(x)$$

and from Corollary 2.1

$$|u(x)| \leq \omega_n(x)$$

in $\bar{D}$, which yields

$$u(x) \equiv 0.$$

q.e.d.

REMARK. We can take as $F(x, u)$ in Theorem 5.2 the following functions:

$$\text{i) } Au, \quad \text{ii) } Au\left|\log\frac{1}{u}\right|, \quad \text{iii) } Au\left(\log\frac{1}{u}\right)^2$$

for $0 \leq u < +\infty$, where we set $u \log \frac{1}{u} = 0$ and A is a constant ≥ 0.

Proof : This follows immediately from Theorem 2.1.

THEOREM 5.3. *Let $f(x, u)$ be a continuous function defined in $\mathcal{D}_2$ and satisfying the inequalities*

$$0 \leq f(x, u), \quad f(x, 0) = 0,$$

$$f(x, u) \leq \frac{A + (x_1^2 - x_2^2 - x_3^2)\delta(x_1)}{x_1^2 - x_2^2 - x_3^2} u,$$

$$\left(0 \leq \delta(x_1), \quad \int_0^{x_1} \delta(\xi)\, d\xi < +\infty\right)$$

where A is a non-negative constant < 6.

Then $u(x) \equiv 0$ is a unique solution in D of the equation (5.1) *with vanishing initial condition.*

Proof : This is a direct consequence of Lemma 5.1, if we set

$$\omega_n(x) = \varepsilon_n (x_1^2 - x_2^2 - x_3^2) \exp\left\{\int_0^{x_1} \left(\int_0^{\eta} \delta(\xi)\, d\xi\right) d\eta\right\},$$

where $\varepsilon_n \downarrow 0$.

Setting $\delta(x_1) \equiv 0$, we obtain the following

COROLLARY 5.1. *If the inequality*

$$f(x, u) \leq \frac{A}{x_1^2 - x_2^2 - x_3^2} u \qquad (A < 6)$$

holds, $u(x) \equiv 0$ is a unique solution in D of (5.1) *with vanishing initial condition.*

6. Peano's property

We have showed in section 4 that if $f(x, u)$ satisfies the hypotheses in Theorem 4.1, there exists a maximal solution of the equation

$$\Box u = f(x, u) \tag{6.1}$$

with vanishing condition. Moreover, Peano's property concerning maximal and minimal solutions in the theory of ordinary differential equations can be extended to solutions of (6.1).

To do this, we need the following

LEMMA 6.1. *Let $\underline{\omega}(x)$ and $\bar{\omega}(x)$ satisfy the inequality*

$$\underline{\omega}(x) < \bar{\omega}(x)$$

in $\bar{D}$. Let $f(x, u)$ be a function continuous in $\mathcal{D}$ and non-decreasing with respect to u such that the inequality

$$\Box \underline{\omega}(x) < f(x, u) < \Box \bar{\omega}(x) \tag{6.2}$$

holds in $\mathcal{D}$.

Then there exists a function $F(x, u)$ continuous in $\mathcal{D}$, non-decreasing and Lipschitz continuous with respect to u such that the inequalities

$$\Box \underline{\omega}(x) < F(x, u) < \Box \bar{\omega}(x), \tag{6.3}$$

$$|f(x,u)-F(x,u)|<\varepsilon \tag{6.4}$$

hold in $\mathfrak{D}$, *where* ε *is a given positive number.*

Proof: Setting

$$d=\operatorname*{Min}_{x\in\bar{D}}(\bar{\omega}(x)-\underline{\omega}(x)),$$

we have $d>0$. Since $f(x,u)$ is continuous in the closed domain $\mathfrak{D}$, there exists a number δ (>0) such that $0<|u-u'|<\delta$ implies the inequality

$$|f(x,u)-f(x,u')|<\varepsilon$$

for any (x,u), (x,u') in $\mathfrak{D}$. Further, since $\Box\underline{\omega}(x)$ and $\Box\bar{\omega}(x)$ are also continuous in $\bar{D}$, we have the inequalities

$$\Box\underline{\omega}(x)<f(x,u)-\varepsilon,$$
$$\Box\bar{\omega}(x)>f(x,u)+\varepsilon$$

for ε sufficiently small.

Let m be an integer such that the inequality

$$0<\frac{\bar{\omega}(x)-\underline{\omega}(x)}{m}<\delta$$

holds in $\bar{D}$. Then, setting

$$\omega_j(x)=\underline{\omega}(x)+\frac{\bar{\omega}(x)-\underline{\omega}(x)}{m}j, \qquad (j=0,1,2,\cdots,m)$$

we define $F(x,u)$ as follows:

$$F(x,u)=f(x,\omega_{j-1})\frac{u-\omega_j}{\omega_{j-1}-\omega_j}+f(x,\omega_j)\frac{u-\omega_{j-1}}{\omega_j-\omega_{j-1}}$$

for $\omega_{j-1}(x)\leq u\leq\omega_j(x)$. Then we have

$$|f(x,\omega_j)-f(x,\omega_{j-1})|<\varepsilon,$$
$$|f(x,u)-F(x,u)|<\varepsilon,$$

whence

$$\Box\underline{\omega}(x)<F(x,u)<\Box\bar{\omega}(x).$$

It is obvious that $F(x,u)$ is non-decreasing with respect to u. Observing that

$$|D_u^{\pm}F(x,u)|\leq\frac{m\varepsilon}{d}$$

where D_u^+F and D_u^-F are defined in $\underline{\omega}(x)\leq u<\bar{\omega}(x)$ and $\underline{\omega}(x)<u\leq\bar{\omega}(x)$ respectively, we see that $F(x,u)$ is Lipschitz continuous with respect to u.

q.e.d.

LEMMA 6.2. *Let* $f(x,u,\lambda)$ *be continuous in* $\mathfrak{D}\times\Lambda$ *and let the inequalities*

$$\Box\underline{\omega}(x)\leq f(x,u,\lambda)\leq\Box\bar{\omega}(x), \tag{6.5}$$
$$|f(x,u,\lambda)-f(x,u',\lambda)|\leq F(x,|u-u'|) \tag{6.6}$$

hold in $\mathfrak{D}\times\Lambda$, *where* Λ *is an interval* $l_1\leq\lambda\leq l_2$ *and* $F(x,u)$ *satisfies Condition (U) in* $\mathfrak{D}_2$.

Then a solution $u(x,\lambda)$ *in* D *of the equation*

$$\Box u = f(x, u, \lambda) \tag{6.7}$$

with vanishing initial condition is continuous in $\bar{D}\times\Lambda$.

Proof: From (6.5) and (6.6), we see that there exists a unique solution $u(x,\lambda)$ of (6.7) with vanishing condition.

We shall first show that $u(x,\lambda)$ is equi-continuous with respect to λ in Λ for $x\in\bar{D}$. Since $f(x,u,\lambda)$ is continuous in the closed domain $\mathscr{D}\times\Lambda$, we can choose a positive number δ such that $|\lambda_1-\lambda_2|<\delta$ $(\lambda_1,\lambda_2\in\Lambda)$ implies the inequality

$$|f(x,u,\lambda_1)-f(x,u,\lambda_2)|<\varepsilon,$$

where ε is an arbitrary number.

Then we have

$$\begin{aligned}&|f(x,u(x,\lambda_1),\lambda_1)-f(x,u(x,\lambda_2),\lambda_2)|\\ &\quad\leq|f(x,u(x,\lambda_1),\lambda_1)-f(x,u(x,\lambda_1),\lambda_2)|\\ &\quad+|f(x,u(x,\lambda_1),\lambda_2)-f(x,u(x,\lambda_2),\lambda_2)|\\ &\quad<F(x,|u(x,\lambda_1)-u(x,\lambda_2)|)+\varepsilon.\end{aligned}$$

By Theorem 4.1, there exists in D a maximal solution $\omega(x,\varepsilon)$ of the equation

$$\Box u = F(x,u)+\varepsilon$$

with vanishing condition. It is evident from the proof of Theorem 4.1 that $\omega(x,\varepsilon)$ converges uniformly to 0 in $\bar{D}$ as $\varepsilon\to 0$. It is also obvious that we have the inequality

$$|u(x,\lambda_1)-u(x,\lambda_2)|\leq\omega(x,\varepsilon).$$

Hence $u(x,\lambda)$ is equi-continuous with respect to λ in Λ for $x\in\bar{D}$.

The continuity of $u(x,\lambda)$ in $\bar{D}\times\Lambda$ follows from the inequality

$$\begin{aligned}&|u(x_1,\lambda_1)-u(x_2,\lambda_2)|\\ &\quad\leq|u(x_1,\lambda_1)-u(x_2,\lambda_1)|+|u(x_2,\lambda_1)-u(x_2,\lambda_2)|.\end{aligned}$$

q.e.d.

Now we can prove the following

THEOREM 6.1. *Let* $\underline{\omega}(x)$ *and* $\bar{\omega}(x)$ *satisfy the hypothesis in Lemma* 6.1. *Let* $f(x,u)$ *be continuous in* $\mathscr{D}$ *and non-decreasing with respect to* u.

Assume that the inequality (6.2) *holds in* $\mathscr{D}$. *Then there exist maximal and minimal solutions in* D *of the equation* (6.1) *with vanishing initial condition. Denote them by* $\bar{u}(x)$ *and* $\underline{u}(x)$ *respectively.*

Then, for any x^1 *in* D *and any* u_1 *such that* $\underline{u}(x^1)\leq u_1\leq\bar{u}(x^1)$, *there exists a solution* $u(x)$ *of* (6.1) *satisfying the condition*

$$u(x^1) = u_1$$

and the inequality

$$\underline{u}(x)\leq u(x)\leq\bar{u}(x)$$

in $\bar{D}$.

Proof: The existence of maximal and minimal solutions follows from Theorem 4.1 and its corollary.

Now we choose a sequence of numbers $\{\varepsilon_n\}$ with ε_1 sufficiently small such that $\varepsilon_n \downarrow 0$. Then, from Lemma 6.1, there exists a sequence of functions $\{F_n(x, u)\}$ continuous in $\mathscr{D}$, non-decreasing and Lipschitz continuous with respect to u which satisfy the inequalities

$$|f(x, u)-F_n(x, u)| < \varepsilon_n,$$
$$\Box\underline{\omega}(x) < F_n(x, u) < \Box\overline{\omega}(x). \qquad (n=1, 2, \cdots)$$

Hence we get

$$F_n(x, \underline{u})-\varepsilon_n < f(x, u) < F_n(x, \overline{u})+\varepsilon_n$$

for $(x, \underline{u}), (x, \overline{u}) \in \mathscr{D}$ and $\underline{u} \leq u \leq \overline{u}$.

Then it follows from Theorems 3.2 and 5.2 that there exist unique solutions $\underline{u}_n(x)$ and $\overline{u}_n(x)$ of the equations

$$\Box u = F_n(x, u)-\varepsilon_n,$$
$$\Box u = F_n(x, u)+\varepsilon_n$$

with vanishing condition. By Corollary 2.2, we have

$$\underline{u}_n(x) \leq \underline{u}(x) \leq \overline{u}(x) \leq \overline{u}_n(x)$$

in $\overline{D}$.

Consider the equation

$$\Box u = F_n(x, u)+\lambda, \qquad |\lambda| \leq \varepsilon_1$$

and denote by $u_n(x, \lambda)$ the unique solution in D of this equation with vanishing condition. Then, by Lemma 6.2, $u_n(x, \lambda)$ is continuous with respect to x and λ. Hence, if $|\lambda| \leq \varepsilon_n$, we have in virtue of Corollary 3.2 and Theorem 5.2

$$\underline{u}_n(x) \leq u_n(x, \lambda) \leq \overline{u}_n(x)$$

in $\overline{D}$. Therefore we see that there exists a number λ_n such that

$$u_n(x^1, \lambda_n) = u_1, \qquad |\lambda_n| \leq \varepsilon_n.$$

Denoting by $u_n(x)$ a unique solution in D of the equation

$$\Box u = F_n(x, u)+\lambda_n$$

with vanishing condition, it is evident from the proof of Theorem 4.1 that the sequence $\{u_n(x)\}$ converges uniformly in $\overline{D}$ to a function $u(x)$. By definition, λ_n tends to 0 as n tends to infinity. Hence we have

$$u(x^1) = u_1.$$

Moreover, by the definition of $u_n(x)$, we have in $D \cup \Sigma$

$$u_n(x) = \frac{1}{2\pi}\int_{D_S^x} \frac{F_n(\xi, u_n(\xi))+\lambda_n}{r} d\xi.$$

Hence the uniform convergence of $F_n(x, u)$ and $u_n(x)$ yields

$$u(x) = \frac{1}{2\pi}\int_{D_S^x} \frac{f(\xi, u(\xi))}{r} d\xi.$$

Therefore $u(x)$ is a solution in D of (6.1) with vanishing condition.

q. e. d.

Bibliography

[1] A. Alexiewicz and W. Orlicz, Some remarks on the existence and uniqueness of solutions of the hyperbolic equations $s=f(x, y, z, p, q)$, Studia Math., **15** (1956), pp. 201～215.

[2] R. Conti, Sul problema di Darboux per l'equazione $s=f(x, y, z, p, q)$, Ann. Univ. Ferrara, **2** (1953), pp. 129～140.

[3] C. Corduneanu, L'approximation et la stabilité des solutions des équations hyperboliques, Com. Acad. R. P. Romîne, **5** (1955), pp. 21～26.

[4] C. Corduneanu, La dépendance des solutions des équations hyperboliques par rapport aux coefficients et aux données sur les caractéristiques, Revue de Math., **1** (1956), pp. 41～44.

[5] J. B. Diaz, On an analogue of the Euler-Cauchy polygon method for the numerical solution of $u_{xy}=f(x, y, u, u_x, u_y)$, Arch. Rational Mechanics and Analysis, **1** (1958), pp. 357～390.

[6] C. Foias, Une condition d'unicité pour l'équation $s=f(x, y, z)$, Com. Acad. R. P. Romîne, **4** (1954), pp. 565～572.

[7] C. Foias, G. Gussi et V. Poenaru, Une méthode directe dans l'étude du problème de Cauchy pour les équations aux dérivées partielles, hyperboliques, du second ordre, Revue de Math., **1** (1956), pp. 61～98.

[8] P. Hartman and A. Wintner, On hyperbolic partial differential equations, Amer. Journ. Math., **74** (1952), pp. 834～864.

[9] M. Hukuhara, Le problème de Darboux pour l'équation $s=f(x, y, z, p, q)$, Ann. Mat. Pura Appl., **51** (1960), pp. 39～54.

[10] M. Hukuhara and T. Satō, Theory of differential equations (in Japanese), Tokyo, Kyoritsu Shuppan K. K., (1957).

[11] J. B. Keller, On solutions of nonlinear wave equations, Comm. Pure Appl. Math., **10** (1957), pp. 523～530.

[12] J. Kisyński, Sur l'existence et l'unicité des solutions des problèmes classiques relatifs à l'équation $s=F(x, y, z, p, q)$, Ann. Univ. Mariae Curie-Sklodowska, A **11** (1957), pp. 73～108.

[13] P. Leeney, On the existence of not necessarily unique solutions of the classical hyperbolic boundary value problem for non-linear second order partial differential equations in two independent variables, Ph. D. thesis, Brown Univ., (1950).

[14] I. G. Petrovsky, Lectures on partial differential equations, Interscience Publishers, INC., New York, (1954).

[15] M. Riesz, L'intégrale de Riemann-Liouville et le problème de Cauchy, Acta Math., **81** (1949), pp. 1～223.

[16] P. Santoro, Sul problema di Darboux per l'equazione $s=f(x, y, z, p, q)$ e il fenomeno di Peano, Rend. dell'Accademia Nazionale dei XL, IV **10** (1959), pp. 3～17.

[17] T. Satō, Sur l'équation aux dérivées partielles hyperboliques $s=f(x, y, z, p, q)$, Mem. Fac. Sci., Kyushu Imp. Univ., A **2** (1941), pp. 107～123.

[18] T. Satō, On hyperbolic partial differential equations (in Japanese), Kyudai Rigakubu Kenkyū Hōkoku, **1** (1945), pp. 203～249.

[19] H. Schaefer, Eine Bemerkung über hyperbolische System partieller Differentialgleichungen zweiter Ordnung, Jber. Deutsc. Math. Verein., **58** (1955), pp. 39～42.

[20] S. Vasilach, Un nouveau problème aux limites pour les équations à dérivées partielles de type hyperbolique, Com. Acad. R. P. Romîne, **1** (1951), pp. 35～39.

[21] W. Walter, Über die Differentialgleichung $u_{xy}=f(x, y, u, u_x, u_y)$ I, II, III, Math.

Zeitschr., **71** (1959), pp. 308~324, pp. 436~453, **73** (1960), pp. 268~279.

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