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Principally Linear Partial Differential Equations of Elliptic Type

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Introduction

Our purpose in this note is to carry on investigations in a partial differential equation of the second order and of elliptic type

$$(\mathrm{I}) \qquad L(u)\equiv \sum_{i,j=1}^{m} a_{ij}(x)\partial_i\partial_j u=f(x,u,\partial_x u)^{1)}.$$

To research such an equation, especially to establish an existence theorem for solutions of the boundary value problem concerning such an equation, it is necessary that certain analytical conditions are imposed on the functions $a_{ij}(x)$ and $f(x,u,p)$; for example, Lipschitz conditions, or Hölder conditions. However we necessitate these conditions only in a part of investigation, but in the whole.

Throughout this note, we will employ generalized differential operations of elliptic type[2], with the intention to research under conditions as weakened as possible. Moreover, as these generalized operations clarify a feature of a differential operation $L(u)$ of elliptic type, one will see that the employment of these generalized operations paves the the way for research.

Thus, denoting generalized operations of elliptic type by $\mathfrak{L}(u)$, we treat of the equation

$$(\mathrm{II}) \qquad \mathfrak{L}(u)=f(x,u,\partial_x u).$$

In the case of linear elliptic partial differential equations, that is, in the case where

$$f(x,u,\partial_x u)=\sum_{i=1}^{m} b_i(x)\partial_i u+c(x)u+f(x)$$

in the equation (II), there are works issued by G. Giraud, and M. Gevrey [1], [2].

Recently, Prof. T. Satō has treated of the equation

$$\Delta z=f(x,y,z,\partial_x z,\partial_y z)$$

by using a generalized Laplacian originted by M. Zaremba, and he has obtained many important results [5]. One of our purposes in this note is to extend the results of Prof. T. Satō to the equation (II).

1) As regards notations, see § 1.

2) See § 2.

In the meanwhile, Prof. M. Nagumo has conducted investigations into the boundary value problem of the first kind concerning the equation (I) and has given out an excellent work about the existence of solutions for this problem [4]. The other of our purposes in this note is to establish the existence theorem for the equation (II), by following the method which Prof. M. Nagumo has pursued.

Chapter I is devoted to make preparations for the main subjects. In this chapter, the differential operation of elliptic type is generalized in the way given by M. Gevrey and we summarize matters appertaining to the elementary solution with respect to the generalized operation. Furthermore, establishing Poisson's formula with respect to the generalized operation, we prove a theorem concerning this operation, which is of primary importance to our research.

In Chapter II, we establish a group of theorems which are called theorems of comparison and play important roles in various branches of investigation. Moreover, considering the boundary value problem of the first kind concerning the equation (II), we impose certain conditions on the function $f(x, u, p)$, which are named conditions of uniqueness and under which the solution of the abovestated problem is determined uniquely.

The two following chapters are concerned with the existence theorem of solutions for the boundary value problem of the first kind concerning the equation (II).

In Chapter III, we first treat of the equation of Poisson's type: $\mathfrak{L}(u)=f(x)$ by making use of the results gained by J. Schauder, M. Nagumo and G. Giraud.

Next, we investigate the estimation of the derivative $\partial_x u(x)$, where $u(x)$ is a solution of the equation (II). The significance of the investigation for this subject has been well recognized and certain investigators have made researches in this subject. In the present note, this subject is investigated in the way found by M. Nagumo.

Lastly, taking up the case where the function $f(x, u, p)$ is bounded, we establish an existence theorem for solutions of the equation (II).

The purpose of Chapter IV is a deeper research of the existence theorem. In the first place, assuming the existence of solutions of equation (II) and imposing certain conditions on the function $f(x, u, p)$, we make the properties of the solutions clear, with a view of proving more general existence theorems.

Next, under the same conditions as imposed above, an existence theorem is proved in the case where the considered domain D has the sufficiently smooth boundary, and then we go forward successively.

In Chapter V, we give the definitions of the maximal solution and the minimal solution of the equation (II), and further an existence theorem of such solutions is established under the proper conditions. In the same chapter we extend Harnack's first theorem to solutions of the equation (II).

Finally, the author wishes to express his sincere thanks to Professors Tokui Satō and Mitio Nagumo for their kind guidance and helpful suggestions in the course of this research, and furthermore the same author thanks cordially Prof. Masuo Hukuhara for his kindness in affording the occasion of publishing this note.

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Chapter I. Preliminaries

§ 1. Notations and preliminary remarks.

We use the notations $\partial_{x_i} u$ or $\partial_i u$ for $\partial u/\partial x_i$, and $\partial_{x_i}\partial_{x_j} u$ or $\partial_i \partial_j u$ for $\partial^2 u/\partial x_i \partial x_j$. x and $\partial_x u$ mean m-dimensional vectors $(x_1, x_2, \cdots, x_m)$ and $(\partial_{x_1} u, \partial_{x_2} u, \cdots, \partial_{x_m} u)$ respectively. $\partial_x \partial_x u$ means an m-m matrix $(\partial_i \partial_j u)$. $\partial_i' u$ denotes $\partial u/\partial x_i'$, and the same for the other notations.

We define $|x-x'|$, $|\partial_x u|$ and $|\partial_x \partial_x u|$ as follows:

$$|x-x'|=\left\{\sum_{i=1}^m (x_i-x_i')^2\right\}^{1/2}, \quad |\partial_x u|=\left\{\sum_{i=1}^m (\partial_i u)^2\right\}^{1/2},$$

$$|\partial_x \partial_x u|=\left\{\sum_{i,j=1}^m (\partial_i \partial_j u)^2\right\}^{1/2}.$$

D denotes a bounded domain in the m-dimensional Euclidean space and C the boundary of D. Without loss of generality we can assume that D is connected. Let D_1 and D_2 be two domains, and we write $D_1 \Supset D_2$ or $D_1 \Supset D_1$, when D_1 contains completely D_2, that is, $D_1 \supset \bar{D}_2$.

Let $\{u_n(x)\}$ be a sequence of functions defined in a domain D. When the sequence $\{u_n(x)\}$ converges uniformly in any bounded closed set K which is contained in D, we say simply that *the sequence* $\{u_n(x)\}$ *converges uniformly in* D. In contrast to this terminology, when the sequence $\{u_n(x)\}$ converges uniformly in D itseif (in the old sense), we use the terminology that *the sequence*

$\{u_n(x)\}$ *converges uniformly in* D *in the strict sense.*

If D is a bounded closed domain, these two notions of uniform convergence are equivalent. Therefore, in this case we do not make any distinction between these two notions.

A function $f(x)$ is said to be H_α-continuous $(0<\alpha<1)$ in a domain D, if we have $|f(x)-f(x')|\leqq K|x-x'|^\alpha$ for any $x, x'\in D$, and the least value of K is said the Hölder constant of $f(x)$.

With L. Lichtenstein we say that the boundary C is of B_h-class and D is a domain of B_h-class, if the following condition is satisfied:

The $(m-1)$-dimensional manifold C can be, in local, represented by parameters $(s_i\,;\, i=1,2,\cdots,m-1)$ in the form $x_j=x_j(s)$, in such a manner that

$$|\partial_{s_j}\partial_{s_i}x_k(s+\varDelta s)-\partial_{s_j}\partial_{s_i}x_k(s)|\leqq H|\varDelta s|^h,\quad 0<h<1,\quad 0<H<+\infty,$$

and Rank $(\partial x_j/\partial s_i)$.

$C[D]$ denotes the family of all the continuous functions in D, and $C^k[D]$ is the family of all the functions whose partial derivatives up to k-th order are all continuous in D.

$a_{ij}(x)$ $(i,j=1,2,\cdots,n)$ are continuous in D (or $\bar{D}$) and the quadratic form $\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j$ is positive definite for any $x\in D$ (or $\in\bar{D}$).

Domains $\mathfrak{D}_1$, $\bar{\mathfrak{D}}_1$, $\mathfrak{D}_2$, $\bar{\mathfrak{D}}_2$ in the $(2m+1)$-dimensional real space are defined in the following manner:

$$\mathfrak{D}_1=\{(x,u,p);\ x\in D,\ |u|<+\infty,\ |p|<+\infty\},$$

$$\bar{\mathfrak{D}}_1=\{(x,u,p);\ x\in\bar{D},\ |u|<+\infty,\ |p|<+\infty\},$$

$$\mathfrak{D}_2=\{(x,u,p);\ x\in D,\ \underline{\omega}(x)\leqq u\leqq\bar{\omega}(x),\ |p|<+\infty\},$$

$$\bar{\mathfrak{D}}_2=\{(x,u,p);\ x\in\bar{D},\ \underline{\omega}(x)\leqq u\leqq\bar{\omega}(x),\ |p|<+\infty\},$$

where $p=(p_1,p_2,\cdots,p_m)$, and $\underline{\omega}(x)$, $\bar{\omega}(x)$ are continuous functions defined in D (or $\bar{D}$), whose full definition will be given later on ; (see § 4).

$f(x,u,p)$ is a continuous function defined in $\mathfrak{D}_1$, (or $\bar{\mathfrak{D}}_1$, or $\mathfrak{D}_2$, or $\bar{\mathfrak{D}}_2$).

For $a_{ij}(x)$, $f(x,u,p)$, we have to assume more restrictive conditions in the detailed statement and these conditions will be imposed on when the necessity arises.

In this note, we consider the equation of elliptic type

$$\mathfrak{L}(u)=f(x,u,\partial_x u), \tag{1.1}$$

where $\mathfrak{L}(u)$ is a generalized operation of elliptic type, which will be given in the next paragraph.

§ 2. Generalization of the operation of elliptic type[3)]

Let $(A_{ij}(x))$ be the inverse matrix of $(a_{ij}(x))$, and let x_0 be any fixed point in D. Consider a linear transformation $x=Tx'$ which transforms the quadratic form $\sum_{i,j=1}^{m} A_{ij}(x_0)x_i x_j$ into the form $\sum_{i=1}^{m} x_i'^2$, then this transformation changes also the forms $\sum_{i,j=1}^{m} a_{ij}(x_0)\partial_i\partial_j u$ and $\sum_{i,j=1}^{m} a_{ij}(x_0)\partial_i u\cdot\partial_j v$ into the forms $\sum_{i=1}^{m}\partial_i'\partial_j' u$ and $\sum_{j=1}^{m}\partial_i' u\cdot\partial_j' v$. The determinant of this linear transformation is equal to $a(x_0)^{-1/2}$, where $a(x)$ is the discriminant of $(a_{ij}(x))$. The linear transformation defined by the above condition is named *the linear transformation $x=Tx'$ associated with the point x_0*.

Let $u(x)$ be a function belonging to $C^2[D]$, and put $u(Tx')=u'(x')$.

The following equalities are well known :

$$(2.1)\quad \{\Delta' u'(x')\}_{x'=x_0'}\equiv\lim_{r\to 0}\frac{2m(m+2)}{\omega_m r^{m+2}}\int_{|x'-x_0'|\leq r}\{u'(x')-u'(x_0')\}\,d\omega_{x'}$$
$$=\sum_{i=1}^{m}\partial_i'\partial_j' u(x_0'),$$

$$(2.2)\quad \{\Delta' u'(x')\}_{x'=x_0'}\equiv\lim_{r\to 0}\frac{2m}{\omega_m r^{m+1}}\int_{|x'-x_0'|=r}\{u'(x')-u'(x_0')\}\,d\sigma_{x'}$$
$$=\sum_{i=1}^{m}\partial_i'\partial_j' u(x_0'),$$

where ω_m is the surface of the unit sphere in the m-dimensional Euclidean space and $d\omega_{x'}$ the volume element at a point x' and $d\sigma_{x'}$ the surface element of a sphere $|x'-x_0'|=r$ at x'.

By the transformation $x=Tx'$ associated with the point x_0, we have

$$(2.3)\quad \lim_{r\to 0}\frac{2m(m+2)}{\omega_m\sqrt{a(x_0)}\,r^{m+2}}\int_{(\Omega(x_0,r))}\{u(x)-u(x_0)\}\,d\omega_x$$
$$=\sum_{i,j=1}^{m} a_{ij}(x_0)\partial_i\partial_j u(x_0),$$

$$(2.4)\quad \lim_{r\to 0}\frac{2m}{\omega_m r^{m+1}}\int_{\Omega(x_0,r)}\{u(x)-u(x_0)\}\,d\sigma_{x'}$$
$$=\sum_{i,j=1}^{m} a_{ij}(x_0)\partial_i\partial_j u(x_0),$$

where $\Omega(x_0,r)$ is an ellipsoid defined by $\sum_{i,j=1}^{m} A_{ij}(x_0)(x_i-x_{0i})(x_j-x_{0j})=r^2$ and $(\Omega(x_0,r))$ is a domain enclosed by $\Omega(x_0,r)$. In (2.4) $d\sigma_{x'}$ means the surface element of a sphere $|x'-x_0'|=r$ at a point x' corresponding to a point

3) See [1]. Generalizations in other ways are given in [1], [2].

$x\in\Omega(x_0, r)$.

For the 2-dimensional space, we can express (2.3) and (2.4) in the following explicit forms:

$$\text{(2.5)}\qquad \lim_{r\to 0}\frac{8}{\pi r^3}\times\int_0^r rdr\int_0^{2\pi}\{u(x_{01}+r\sqrt{a_{11}(x_0)}\cos(\theta-\varphi),$$
$$x_{02}+r\sqrt{a_{22}(x_0)}\sin(\theta+\varphi))-u(x_0)\}\,d\theta=\sum_{i,j=1}^{2}a_{ij}(x_0)\partial_i\partial_j u(x_0),$$

$$\text{(2.6)}\qquad \lim_{r\to 0}\frac{2}{\pi r^2}\times\int_0^{2\pi}\{u(x_{01}+r\sqrt{a_{11}(x_0)}\cos(\theta-\varphi),$$
$$x_{02}+r\sqrt{a_{22}(x_0)}\sin(\theta+\varphi))-u(x_0)\}\,d\theta=\sum_{i,j=1}^{2}a_{ij}(x_0)\partial_i\partial_j u(x_0),$$

where $\varphi=\frac{1}{2}\sin^{-1}\{a_{12}(x_0)/\sqrt{a_{11}(x_0)a_{22}(x_0)}\}$.

Thus, we define an operation $\overline{\mathfrak{L}}(u(x))$ for any $u(x)\in C[D]$ by the expres sions

$$\text{(2.7)}\qquad \overline{\mathfrak{L}}(u(x))=\overline{\lim_{r\to 0}}\frac{2m(m+2)}{\omega_m\sqrt{a(x)}\,r^{m+2}}\int_{(\Omega(x,r))}\{u(\xi)-u(x)\}\,d\omega_\xi,$$

or

$$\text{(2.8)}\qquad \overline{\mathfrak{L}}(u(x))=\overline{\lim_{r\to 0}}\frac{2m}{\omega_m r^{m+1}}\int_{\Omega(x,r)}\{u(\xi)-u(x)\}\,d\sigma_{\xi'},$$

and an operation $\underline{\mathfrak{L}}(u(x))$ is defined by the analoguous expression which are obtained by using "$\underline{\lim}$" in stead of "$\overline{\lim}$" in (2.7) or (2.8). If $\overline{\mathfrak{L}}(u(x))=\underline{\mathfrak{L}}(u(x))$ at a point $x\in D$, then we use notation $\mathfrak{L}(u(x))$ for $\overline{\mathfrak{L}}(u(x))$ and $\underline{\mathfrak{L}}(u(x))$.

Thus, by a solution of the equation (1.1) in D we mean a function which belongs to $C^1[D]$ and satisfies the equation (1.1).

LEMMA 2.1. *Let $u(x)$ and $v(x)$ be functions belonging to $C^1[D]$, and suppose that $\mathfrak{L}(u(x))$ and $\mathfrak{L}(v(x))$ are defined at a point $x_0\in D$. Then we have*

$$\text{(2.9)}\qquad \mathfrak{L}(u(x_0)v(x_0))=v(x_0)\mathfrak{L}(u(x_0))+u(x_0)\mathfrak{L}(v(x_0))$$
$$+2\sum_{i,j=1}^{m}a_{ij}(x_0)\partial_i u(x_0)\partial_j v(x_0).$$

PROOF. By using the linear transformation $x=Tx'$ associated with the point x_0 and the generalized Laplacian Δ' defined by (2.3) or (2.4) we obtain easily

$$\{\Delta'(u'(x')v(x'))\}_{x'=x_0'}=v'(x_0')\{\Delta' u'(x')\}_{x'=x_0'}$$
$$+u'(x_0')\{\Delta' v'(x')\}_{x'=x_0'}+2\sum_{i=1}^{m}\partial_i' u'(x_0')\partial_i' v'(x_0').$$

In this expression, by changing the variable x' into x, we have (2.9), q.e.d.

§ 3. Summary about elementary solutions

In this paragraph, we give a sketch about elementary solutions and Poisson's formula relative to the operation $\mathfrak{L}(u)$.

Let the functions $a_{ij}(x)$ satisfy the following conditions:

(3.1) $$A^{-1}\leq\sum_{i,j=1}^{m}a_{ij}(x)\xi_i\xi_j\leq A \quad \text{for} \quad \sum_{i=1}^{m}\xi_i^2=1,$$

(3.2) $$\left\{\sum_{i,j=1}^{m}(a_{ij}(x')-a_{ij}(x))^2\right\}^{1/2}\leq L|x'-x|,$$

for any x, $x'\in D$, where $A(\geqq 1)$ and L are positive constants.

We put

$$V_\zeta(x,\xi)=\begin{cases}\left\{\sum_{i,j=1}^{m}A_{ij}(\zeta)(x_i-\xi_i)(x_j-\xi_j)\right\}^{\frac{2-m}{2}} & (m>2),\\ -\log\left\{\sum_{i,j=1}^{2}A_{ij}(\zeta)(x_i-\xi_i)(x_j-\xi_j)\right\} & (m=2),\end{cases}$$

and

$$H_\zeta(x,\xi)=\begin{cases}\dfrac{\Gamma(m/2-1)}{4\pi^{m/2}\sqrt{a(\zeta)}}V_\zeta(x,\xi) & (m>2),\\ \dfrac{1}{4\pi\sqrt{a(\zeta)}}V_\zeta(x,\xi) & (m=2),\end{cases}$$

where $a(\zeta)$ is the determinant of the coefficients $a_{ij}(\zeta)$.

Further we put $H_\xi(x,\xi)=H(x,\xi)$ and $\mathfrak{L}(H(x,\xi))=K(x,\xi)$[4], then it is easily seen, by an elementary calculation, that

(3.3) $$K(x,\xi)=O(|x-\xi|^{1-m})\text{[5]}.$$

Let $f(x)$ be continuous and bounded in D, and put

$$u(x)=-\int_D H(x,\xi)f(\xi)d\omega_\xi,$$

then we obtain for any $x\in D$

(3.4) $$\mathfrak{L}(u(x))=f(x)-\int_D K(x,\xi)f(\xi)d\omega_\xi\text{[6]}.$$

We call *an elementary solution relative to the operation* $\mathfrak{L}(u)$ a function $E(x,\xi)$ of two distinct points x and ξ, which possesses the following properties:

1° $\mathfrak{L}(E(x,\xi))=O$;

2° When x approaches to ξ,

$$E(x,\xi)=\{1+o(1)\}H(x,\xi).$$

The local existence of elementary solutions $E(x,\xi)$ can be easily shown; for example, Cf. [2], pp. 270–272. The operation $\mathfrak{L}(u)$ of elliptic type employed in [2] is different from ones which we employ in this note, but this difference

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4) The operation $\mathfrak{L}$ is operated with respect to the first variable for a function of several variables, if not stated otherwise.

5) See [3], p. 161.

6) See [1], pp. 55–57. In [1], this formula is shown for the case where the discriminant $a(x)$ is equal to unity, but we can easily prove it for the present general case, along the same lines as in [1].

is not essential for the proof of the local existence of elementary solutions $E(x, \xi)$. For, the proof in question is based on the formula (3.4).

Let x_0 be a point in D and let k be a positive number smaller than unity, then we can choose a sphere S containing x_0

$$\int_S |K(x, \xi)| d\omega_\xi \leq k.$$

Put $K^{(1)}(x, \xi) = K(x, \xi)$, and

$$K^{(n+1)}(x, \xi) = \int_S K(x, \zeta) K^{(n)}(\zeta, \xi) d\omega_\zeta.$$

Then we have

$$K^{(n)}(x, \xi) = \begin{cases} O(|x-\xi|^{n-m}) & (n<m), \\ O(\log(L_0/|x-\xi|) & (n=m), \\ O(1) & (n>m), \end{cases}$$

and for $n>m$, this function is continuous with respect to (x, ξ), where L_0 is a positive constant greater than the diameter of S[7].

Hence the series $\sum_{n=1}^{\infty} K^{(n)}(x, \xi)$ is convergent, and $\sum_{n=m+1}^{\infty} K^{(n)}(x, \xi)$ is continuous with respect to (x, ξ).

The function

$$E(x, \xi) = H(x, \xi) + \int_S H(x, \xi) \sum_{n=1}^{\infty} K^{(n)}(\zeta, \xi) d\omega_\zeta$$

is an elementary solution relative to the operation $\mathfrak{L}(u)$, which is defined for any two distinct points x, $\xi \in S$. It must be noticed that this elementary solution $E(x, \xi)$ depends on the choice of the sphere S.

Poisson's formula in a special form may be given as follows[8].

THEOREM 3.1. *Let S be a sphere having the above-mentioned property and let $E(x, \xi)$ be the elementary solution constructed in the above. Let $f(x)$ be a continuous and bounded function defined in S, and put*

$$u(x) = -\int_S E(x, \xi) f(\xi) d\omega_\xi.$$

Then, $u(x)$ is continuous in S and we have for any point $x \in S$

$$\mathfrak{L}(u(x)) = f(x).$$

PROOF. The continuity of $u(x)$ can be shown easily, and it follows from the definition of $E(x, \xi)$, that

$$\int_S E(x, \xi) f(\xi) d\omega_\xi$$

7) See [3], II. Theorem 3, p. 150.

8) Poisson's formula can be established in a more general form (See [2], p. 283). But Theorem 3.1 suffices for investigation.

$$=\int_S H(x,\xi)f(\xi)d\omega_\xi+\int_S f(\xi)d\omega_\xi\int_S H(x,\zeta)\sum_{n=1}^{\infty}K^{(n)}(\zeta,\xi)d\omega_\zeta$$

$$=\int_S H(x,\xi)\left\{f(\xi)+\int_S\sum_{n=1}^{\infty}K^{(n)}(\xi,\zeta)f(\zeta)d\omega_\zeta\right\}d\omega_\xi.$$

Therefore, by (3.4) we have

$$-\mathfrak{L}\left\{\int_S E(x,\xi)f(\xi)d\omega_\xi\right\}$$

$$=f(x)+\int_S\sum_{n=1}^{\infty}K^{(n)}(x,\zeta)f(\zeta)d\omega_\zeta$$

$$-\int_S K(x,\xi)\left\{f(\xi)+\int_S\sum_{n=1}^{\infty}K^{(n)}(\xi,\zeta)f(\zeta)d\omega_\zeta\right\}d\omega_\xi$$

$=f(x)$, q.e.d.

THEOREM 3.2. *Suppose that functions* $u_k(x)$ $(k=1,2,\cdots)$ *are continuous in* D *and* $\mathfrak{L}(u_k(x))$ *are defined and continuous in* D. *If two sequences* $\{u_k(x)\}$ *and* $\{\mathfrak{L}(u_k(x))\}$ *converge to two continuous functions* $u(x)$ *and* $f(x)$ *uniformly in* D, *then* $\mathfrak{L}(u(x))$ *is defined in* D, *and the equality* $\mathfrak{L}(u(x))=f(x)$ *holds for any* $x\in D$.

PROOF. For any fixed point $x_0\in D$, we can choose a small sphere S_1 with the center x_0, such that in S_1 there exists the elementary solution $E(x,\xi)$ which is constructed in the way explained above.

The functions

$$w_k(x)=-\int_{S_1}E(x,\xi)\,\mathfrak{L}(u_k(\xi))d\omega_\xi$$

are continuous in S_1 and the equalities

$$\mathfrak{L}(w_k(x))=\mathfrak{L}(u_k(x))\qquad(k=1,2,\cdots)$$

hold for any $x\in S_1$.

Let S_2 be a sphere concentric with S_1 and contained completely in $S_1: S_1\supseteq S_2$, and let $v_k(x)$ be solutions of the equation $L(v)=0$ in S_2, which attain the values $u_k(x)-w_k(x)$ on the boundary of S_2[9]. Since functions $\bar{u}_k(x)=v_k(x)+w_k(x)$ satisfy equalities $\mathfrak{L}(\bar{u}_k(x))=\mathfrak{L}(u_k(x))$ in S_2, and since $\bar{u}_k(x)=u_k(x)$ on the boundary of S_2, we have

$$\bar{u}_k(x)=u_k(x)\quad\text{in}\quad\bar{S}_2$$

by the uniqueness of solutions of an equation of elliptic type[10].

By virtue of the uniform convergence of $\{\mathfrak{L}(u_k(x))\}$, we see

$$\lim_{k\to\infty}w_k(x)=-\int_{S_1}E(x,\xi)\,f(\xi)d\,\omega_\xi=w(x)$$

uniformly in $\bar{S}_2$. From this fact and the uniform convergence of $\{u_k(x)\}$, it

9) Such solutions $v_k(x)$ certainly exist. See Theorem B' in § 6.

10) As regards the uniqueness of the solution, see Theorem 5.1 in § 5. The proof of Theorem 5.1 is free from Theorem 3.2.

follows that the sequence $\{v_k(x)\}$ converges to a continuous function $v(x)$ uniformly in $\bar{S}_2$ and we have

$$u(x)=v(x)+w(x).$$

It is easily shown that the function $v(x)$ satisfies the equation $L(v)=0$ in S_2. In fact, let $\bar{v}(x)$ be a solution of the equation $L(v)=0$ which attains the values $v(x)$ on the boundary $\dot{S}_2$ of S_2. Then by the maximum principle for solutions of the equation $L(v)=0$, we have

$$\operatorname*{Max}_{\bar{S}_2} |\bar{v}(x)-v_k(x)|=\operatorname*{Max}_{\dot{S}_2} |v(x)-v_k(x)|.$$

Therefore the sequence $\{v_k(x)\}$ converges to $\bar{v}(x)$ uniformly in $\bar{S}_2$, and hence $\bar{v}(x)=v(x)$ in S_2.

Thus, $\mathfrak{L}(u(x))$ is defined for any $x\in S_2$ and

$$\mathfrak{L}(u(x))=\mathfrak{L}(v(x))+\mathfrak{L}\left(-\int_{S_1} E(x,\xi)f(\xi)d\omega_\xi\right)=f(x),$$

q.e.d.

Chapter II. Theorems of comparison and conditions of uniqueness

In this chapter, we assume that the functions $a_{ij}(x)$ are continuous in D and the quadratic form $\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j$ is positive-definite in D (not necessarily in $\bar{D}$), except for Theorem 4.6.

§ 4. Theorems of comparison.

The function $f(x,u,p)$ is assumed to be defined in $D\times S$, where S is a set in the $(m+1)$-dimensional space (u,p), and we consider the equation

$$\mathfrak{L}(u)=f(x,u,\partial_x u). \tag{4.1}$$

THEOREM 4.1. *Let $\omega(x)$ be a function belonging to $C^1[D]$, and let the inequality*

$$\underline{\mathfrak{L}}(\omega(x))<f(x,u,\partial_x\omega(x)) \tag{4.2}$$

hold for $\omega(x)<u, x\in D$, $(u,\partial_x\omega(x))\in S$, and suppose that the equation (4.1) *has a solution $u=u(x)$, such that the inequality*

$$\lim_{x\to\dot{x}}\{\omega(x)-u(x)\}\geqq 0 \tag{4.3}$$

holds for any point $\dot{x}\in C$. Then we have

$$\omega(x)\geqq u(x) \quad \textit{in} \quad D. \tag{4.4}$$

PROOF. If (4.4) is not true, then we have $\operatorname*{Inf}_D\{\omega(x)-u(x)\}<0$ and hence there exists a sequence of points $\{x^{(n)}\}$ in D, such that

$$\lim_{n\to\infty}\{\omega(x^{(n)})-u(x^{(n)})\}=\operatorname*{Inf}_D\{\omega(x)-u(x)\}<0.$$

Without loss of generality the the sequence $\{x^{(n)}\}$ may be assumed to converge to a point $x_0 \in \bar{D}$. Furthermore, by virtue of (4.3) we see $x_0 \in D$. Hence $\omega(x)-u(x)$ attains the minimum at x_0, from which it follows that

$$\omega(x_0) < u(x_0),\ \partial_x \omega(x_0) = \partial_x u(x_0),\ \underline{\mathfrak{L}}(\omega(x_0)) \geqq \mathfrak{L}(u(x_0)).$$

Therefore we have

$$\underline{\mathfrak{L}}(\omega(x_0)) \geqq f(x, u(x_0), \partial_x \omega(x_0)),$$

which contradicts (4.2)

THEOREM 4.2. *Under the same assumptions as in Theorem* 4.1, *if the inequality*

$$\varliminf_{x \to \dot{x}} \{\omega(x) - u(x)\} > 0$$

holds for any $\dot{x} \in C$, then we have $\omega(x) > u(x)$ in D.

The proof is omitted.

Let $\bar{f}(x, u, p)$ be a function defined in $D \times S$, and let $\omega(x)$ be a solution of the equation

$$\mathfrak{L}(\bar{u}) = \bar{f}(x, \bar{u}, \partial_x \bar{u}).$$

If the inequality $\bar{f}(x, \bar{u}, p) < f(x, u, p)$ holds for $\bar{u} < u$, then we have (4.2). Thus we obtain the following:

THEOREM 4.3. *Assume that the inequality*

$$\bar{f}(x, \bar{u}, p) < f(x, u, p)$$

holds for (x, u, p), $(x, \bar{u}, p) \in D \times S$ and $\bar{u} < u$, and let $u(x)$ and $\bar{u}(x)$ be solutions of equations

$$\mathfrak{L}(u) = f(x, u, \partial_x u) \quad \textit{and} \quad \mathfrak{L}(\bar{u}) = \bar{f}(x, \bar{u}, \partial_x \bar{u})$$

respectively. If we have

$$\varliminf_{x \to \dot{x}} \{\bar{u}(x) - u(x)\} \geqq 0$$

for any $\dot{x} \in C$, then $\bar{u}(x) \geqq u(x)$ holds in D.

THEOREM 4.4. *Under the same condition as in Theorem* 4.3, *if the inequality*

$$\varliminf_{x \to \dot{x}} \{\bar{u}(x) - u(x)\} > 0$$

holds for any $\dot{x} \in C$, then we have $\bar{u}(x) > u(x)$ in D.

REMARK. In the above, by inverting the notations of inequalities we obtain the theorems which are analoguous to Theorems 4.1–4.4.

DEFINITION 4.1. *A function $\omega(x)$ is called a quasi-superfunction (a quasi-subfunction) of the equation* (4.1) *in D, if this function has the following properties:*

For any fixed point $x_0 \in D$, there exist a neighborhood U of the point x_0 and a finite number of functions $\omega_\nu(x) (\in C^1[U])$ $(\nu = 1, 2, \cdots, n)$, such that

$$\omega(x)=\min_{1\leqq\nu\leqq n}\omega_\nu(x)\ (\max_{1\leqq\nu\leqq n}\omega_\nu(x)) \quad in \quad U,$$

and

$$\underline{\mathfrak{L}}(\omega_\nu(x))\leqq f(x,\omega_\nu(x),\partial_x\omega_\nu(x)),$$
$$(\overline{\mathfrak{L}}(\omega_\nu(x))\geqq f(x,\omega_\nu(x),\partial_x\omega_\nu(x))).$$

Then, concerning the quasi-superfunction, we can prove the following theorem of comparison.

THEOREM 4.5. *Assume that $f(x,u,p)$ is non-decreasing with respect to u, and let $\bar{\omega}(x)$ be a quasi-superfunction of the equation* (4.1), *and let $v(x)$ be a function belonging to $C^1[D]$.*

If the inequality

$$\overline{\mathfrak{L}}(v(x))>f(x,v(x),\partial_x v(x)) \tag{4.5}$$

holds for the point x, at which $v(x)>\bar{\omega}(x)$, and if

$$\varliminf_{x\to\dot{x}}\{\bar{\omega}(x)-v(x)\}\geqq 0 \tag{4.6}$$

for any point $\dot{x}\in C$, then we have

$$\bar{\omega}(x)\geqq v(x) \quad in \quad D. \tag{4.7}$$

PROOF. If (4.7) is not true, then it follows the same reasoning as in the proof of Theorem 4.1, that there exists a point $x_0\in D$, having the following property:

$$\bar{\omega}(x_0)<v(x_0),\ \bar{\omega}(x_0)-v(x_0)=\operatorname*{Inf}_D\{\bar{\omega}(x)-v(x)\}.$$

There exist therefore a neighborhood U of x_0 and a function $\bar{\omega}_\nu(x)\in C^1[U]$ such that

$$\bar{\omega}_\nu(x_0)<v(x_0),\quad \bar{\omega}_\nu(x_0)-v(x_0)=\operatorname*{Inf}_U\{\bar{\omega}_\nu(x)-v(x)\}<0.$$

Hence we have

$$\partial_x\bar{\omega}_\nu(x_0)=\partial_x v(x_0) \quad \text{and} \quad \underline{\mathfrak{L}}(\bar{\omega}_\nu(x_0))-\overline{\mathfrak{L}}(v(x_0))\geqq 0,$$

from which it follows that

$$\overline{\mathfrak{L}}(v(x_0))-f(x_0,v(x_0),\partial_x v(x_0))\leqq$$
$$\underline{\mathfrak{L}}(\bar{\omega}_\nu(x_0))-f(x_0,\bar{\omega}_\nu(x_0),\partial_x\bar{\omega}_\nu(x_0))\leqq 0,$$

which contradicts the inequality (4.5), q.e.d.

REMARK. We obtain also a theorem which is analoguous to Theorem 4.5 about a quasi-subfunction $\underline{\omega}(x)$ of the equation (4.1) and $f(x,u,p)$ non-decreasing with respect to u.

We next assume that there exists a solution $\Psi(x)(\in C^1[D])$ of the equation $\mathfrak{L}(u)=-1$, which vanishes on C. If such a solution exists, then we see easily that it is unique and positive and bounded in D.

There exists certainly such a solution under the hypotheses that the functions $a_{ij}(x)$ satisfy the conditions (3.1) and (3.2) in $\bar{D}$, and that D is of B_h-class[11].

The following theorem is useful for our research.

THEOREM 4.6. *Let $f(x, u, p)$ and $F(x, u, p)$ be functions defined in $\mathfrak{D}_1$ and satisfying the conditions*

$$f(x, u, p)\begin{cases}\leqq 0 & (u \leqq 0), \\ \geqq 0 & (u \geqq 0),\end{cases}$$

and

$$|F(x, u, p)| \leqq M \qquad (M: \text{ a constant}).$$

If the equation

$$\mathfrak{L}(u) = f(x, u, \partial_x u) + F(x, u, \partial_x u)$$

has a solution $u(x)$ vanishing on C, then the inequality

$$(4.7) \qquad |u(x)| \leqq M\Psi(x)$$

holds in $\bar{D}$.

PROOF. Let M' be a constant greater than M, and put $v(x) = u(x) - M'\Psi(x)$. Then $u(x)$ satisfies the equation

$$\mathfrak{L}(v) = g(x, v, \partial_x v),$$

where

$$g(x, v, \partial_x v) = f(x, v + M'\Psi, \partial_x v + M'\partial_x \Psi) + F(x, v + M'\Psi, \partial_x v + M'\partial_x \Psi) + M'.$$

The condition of the theorem implies

$$0 < g(x, v, 0) \quad \text{for} \quad v > 0,$$

and from this fact and Theorem 4.1, it follows that

$$v(x) \leqq 0, \quad \text{hence} \quad u(x) \leqq M'\Psi(x).$$

Similarly $-M'\Psi(x) \leqq u(x)$. We have therefore

$$|u(x)| \leqq M'\Psi(x) \quad \text{in} \quad D,$$

and by letting M' tend to M, we obtain (4.8).

§ 5. Conditions of uniqueness

Let $f(x, u, p)$ be a function defined in $\mathfrak{D}_1$ and consider the boundary value problem of the first kind concerning the equation

$$(5.1) \qquad \mathfrak{L}(u) = f(x, u, \partial_x u).$$

In this paragraph, assuming the existence of the solution for the problem, we will find conditions, under which the solution is determined uniquely.

With this end in view, we first establish the following:

LEMMA 5.1. *If the following condition is satisfied:*

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11) See Theorem B' in § 6.

$$f(x,u,0)\begin{cases}<0 & for \quad u<0,\\ =0 & for \quad u=0,\\ >0 & for \quad u>0,\end{cases}$$

then a solution $u(x)$ *of the equation* (5.1), *vanishing on* C, *is equal identically to zero in* $\bar{D}$.

PROOF. Since a function $\omega(x)\equiv 0$ has the property of the function $\omega(x)$ in Theorem 4.1, we see that $u(x)\leqq\omega(x)\equiv 0$. Similarly $u(x)\geqq 0$. Therefore $u(x)\equiv 0$, q.e.d.

THEOREM 5.1. *If one of the following conditions is fulfilled, then there exists at most one solution of the equation* (5.1) *with the prescribed boundary values.*

(I) $$f(x,u,p)<f(x,\bar{u},p);$$

(II) $$f(x,u,p)\leqq f(x,\bar{u},p)$$

and

(5.2) $$-L\cdot(\bar{p}_i-p_i)\leqq f(x,u,\bar{p})-f(x,u,p),$$
$$0<\alpha<a_{ii}(x) \quad in \quad D;$$

(III) $$f(x,u,p)\leqq f(x,\bar{u},p),$$

and

(5.3) $$f(x,u,\bar{p})-f(x,u,p)\leqq L\cdot(\bar{p}_i-p_i),$$
$$0<\alpha<a_{ii}(x) \quad in \quad D,$$

where $u<\bar{u}$, $p_i<\bar{p}_i$, $p=(p_1,\cdots,p_i,\cdots,p_m)$, $\bar{p}=(p_1,\cdots,\bar{p}_i,\cdots,p_m)$, *and* i *is a fixed index, and* L, α *are positive constants.*

(IV) D *is contained in a domain*: $\alpha<x_i<\beta$, $-\infty<x_j<+\infty$ $(j\neq i)$, *where* α,β *are finite real numbers, and* i *is a fixed index. And for any* u, $\bar{u}$ *and* $x\in D$

$$|f(x,u,p)-f(x,\bar{u},\bar{p}|\leqq\frac{A(x)|u-\bar{u}|}{(x_i-\alpha)(\beta-x_i)}+\frac{B(x)|p_i-\bar{p}_i|}{|2x_i-\alpha-\beta|}$$
$$0\leqq A(x), B(x), A(x)+B(x)<2a_{ii}(x),$$

where p *and* $\bar{p}$ *are the same as above, but the condition* $p_i<\bar{p}_i$ *is omitted.*

PROOF. (I). Let $u_1(x)$ and $u_2(x)$ be two solutions with the same boundary values, and put $u(x)=u_1(x)-u_2(x)$. By applying Lemma 5.1 to the equation with respect to $u(x)$, we easily see $u(x)\equiv 0$, q.e.d.

(II). Define a function $\varphi(x)$ by $\varphi(x)=A-\exp(-Lx_i/\alpha)$, where A is a positive constant such that $\varphi(x)$ is positive in $\bar{D}$. Then we have

$$\partial_i\varphi(x)=(L/\alpha)\exp(-Lx_i/\alpha),$$
$$\partial_i\partial_i\varphi(x)=-(L/\alpha)^2\exp(-Lx_i/\alpha).$$

Put $u=\varphi(x)w$ and consider the equation with respect to w:

(5.4) $$\mathfrak{L}(w)=g(x,w,\partial_x w),$$

where

$$g(x, w, \partial_x w) = \varphi(x)^{-1}\{f(x, \varphi w, \partial_x(\varphi w)) \\ -2\sum_{j,k=1}^{m} a_{jk}(x)\partial_j\varphi\, \partial_k w - w\,\mathfrak{L}(\varphi)\}.$$

then, for $\bar{w} > w$ and $x \in D$ we obtain

$$\begin{aligned} & g(x, w, \partial_x w) - g(x, \bar{w}, \partial_x w) \\ &= \varphi(x)^{-1}\{f(x, \varphi w, w\,\partial_x\varphi + \varphi\partial_x w) - f(x, \varphi\bar{w}, \bar{w}\partial_x\varphi + \varphi\partial_x w) \\ &\qquad + \mathfrak{L}(\varphi)(\bar{w} - w)\} \\ &\leqq \varphi(x)^{-1}\{f(x, \varphi\bar{w}, w\,\partial_x\varphi + \varphi\partial_x w) - f(x, \varphi\bar{w}, \bar{w}\,\partial_x\varphi + \varphi\partial_x w) \\ &\qquad + \mathfrak{L}(\varphi)(\bar{w} - w)\} \\ &\leqq \varphi(x)^{-1}\{L\,\partial_i\varphi + \mathfrak{L}(\varphi)\}(\bar{w} - w) \\ &= \varphi(x)^{-1}\{L^2/\alpha - (L/\alpha)^2 a_{ii}(x)\}(\bar{w} - w)\exp(-Lx_i/\alpha) < 0. \end{aligned}$$

Hence

$$g(x, w, \partial_x w) < g(x, \bar{w}, \partial_x w), \tag{5.5}$$

and therefore we arrive at the case (I).

(III). The proof can be accomplished along the same lines as in (II).

(IV). Put $\varphi(x) = (x_i - \alpha)(\beta - x_i)$ and $u = \varphi(x)w$, then we have (5.4). For $\bar{w} > w$ and $x \in D$,

$$\begin{aligned} & g(w, \bar{w}, \partial_x w) - g(x, w, \partial_x w) \\ &\geqq \{2\,a_{ii}(x) - (A(x) + B(x))\}(\bar{w} - w)/\varphi(x) > 0, \end{aligned}$$

we are therefore led to the case (I), q.e.d.

One will see that the conditions (II) and (III) in the above theorem are weakened in the under-mentioned theorem.

THEOREM 5.2. *If one of the following conditions is fulfilled, then there exists at most one solution of the equation* (5.1) *with the prescribed boundary values.*

(I) $$f(x, u, p) \leqq f(x, \bar{u}, p)$$
$$-L(x, u, p\,;\, \bar{p}_i)(\bar{p}_i - p_i) \leqq f(x, u, \bar{p}) - f(x, u, p)\,;$$

(II) $$f(x, u, p) \leqq f(x, \bar{u}, p)$$
$$f(x, u, \bar{p}) - f(x, u, p) \leqq L(x, u, p\,;\, \bar{p}_i)(\bar{p}_i - p_i)$$

where $u < \bar{u}$, $p_i < \bar{p}_i$, $p = (p_1, \cdots, p_i, \cdots, p_m)$, $\bar{p} = (p_1, \cdots, \bar{p}_i, \cdots, p_m)$, *and* i *is a fixed index. Furthermore* $L(x, u, p\,;\, \bar{p}_i)$ *is a function of* $(x, u, p, \bar{p}_i)$ *which is bounded in any domain*:

$$\{(x, u, p, \bar{p}_i);\ x \in S,\ |u| \leqq M_1,\ |p| \leqq M_2,\ |\bar{p}_i| \leqq M_3\},$$

where S *is an arbitrary closed set contained in* D, *and* M_1, M_2, M_3 *are arbitrary finite positive numbers.*

PROOF. Since the theorem can be proved for the condition (II) in the same way as for the condition (I), we give a proof only for the condition (I).

Let $u_1(x)$ and $u_2(x)$ be two solutions with the same boundary values and we will show that a contradiction arises under the hypothesis that these solutions are different from each other.

Then, without loss of generality we can assume that the function $v(x)=u_1(x)-u_2(x)$ attains the negative minimum at a point of D.

Consider a sequence of domains $\{D_n\}$ such that $\bar{D}_n \subset D_{n+1}$ and $\bigcup_{n=1}^{\infty} D_n=D$, and let $\{\alpha_n\}$ and $\{L_n\}$ be two sequences of real positive numbers having the following properties:

$$\operatorname*{Min}_{\bar{D}_n} a_{ii}(x)>\alpha_n, \quad \operatorname*{Max}_{\bar{D}_n} L(x)<L_n,$$

and L_n/α_n increases monotonely to infinity for $n\to\infty$, where $L(x)=L(x, u_2(x), \partial_x u_1(x); \partial_i u_2(x))$.

The existence of these sequences can be easily shown by virtue of the conditon of the theorem.

Now put

$$\varphi_n(x)=A-\exp(-L_n(x_i-\beta)/\alpha_n),$$

where β is a real number such that $x_i-\beta$ is positive for any $x=(x_1, \cdots, x_i, \cdots, x_m)\in \bar{D}$, and A is a constant greater than unity.

Then $\varphi_n(x)$ are positive in $\bar{D}$ and hence each of functions $v_n(x)=v(x)/\varphi_n(x)$ attains the negative minimum at a point $x^{(n)}\in D$. We show that there exists a natural number n_0 such that $x^{(n_0)}\in D_{n_0}$.

In fact, if the above conclusion is not true, then each point $x^{(n)}$ belongs to $D-D_n$, and without loss of generality we can assume that the sequence $\{x^{(n)}\}$ converges to a boundary point $x^{(0)}\in C$.

On the other hand, by the definition of $x^{(n)}$, we have

$$v(x)/\varphi_n(x)\geqq v(x^{(n)})/\varphi_n(x^{(n)}) \quad \text{in} \quad D,$$

and therefore, for $n\to\infty$, the sequence $\{v(x)/\varphi_n(x)\}$ converges to $v(x)/A$ and the sequence $\{v(x^{(n)})/\varphi_n(x^{(n)})\}$ converges to zero.

Thus we see

$$v(x)\geqq 0 \quad \text{in} \quad C,$$

which contradicts the assumption that $v(x)$ attains the negative minimum at a point of D.

Eventually we obtain the following conclusion by renewing the notations:

If we choose adequately a domain D_0 and two positive numbers α and L_0, such that $\bar{D}_0\subset D$ and

$$0<\alpha<\operatorname*{Min}_{\bar{D}_0} a_{ii}(x), \quad \operatorname*{Max}_{\bar{D}_0} L(x)<L_0,$$

and if we put $\varphi(x)=A-\exp(-L_0(x_i-\beta)/\alpha)$, then $v(x)/\varphi(x)$ assumes the negative minimum in D, which is attained at a point $x^{(0)}$ belonging to D_0.

Now put $u=\varphi(x)w$ and consider the equation with respect to w:

$$\mathfrak{L}(w)=g(x, w, \partial_x w), \tag{5.6}$$

where

$$g(x, w, \partial_x w) \equiv \varphi(x)^{-1}\{f(x, \varphi w, \partial_x(\varphi w)) \\ -2\sum_{j,k=1}^{m} a_{jk}(x)\partial_j \varphi \partial_k w - w\,\mathfrak{L}(\varphi)\}.$$

Furthermore we see

$$\partial_j\varphi = 0 \ (j \neq i), \quad \partial_i\varphi = (L_0/\alpha)\exp(-L_0(x_i-\beta)/\alpha) > 0,$$
$$\partial_j(\varphi w) = \varphi\partial_j w \ (j \neq i)$$
$$\partial_i(\varphi w) = \varphi\partial_i w + w(L_0/\alpha)\exp(-L_0(x_i-\beta)/\alpha),$$
$$\mathfrak{L}(\varphi) = -a_{ii}(x)(L_0/\alpha)^2 \exp(-L_0(x_i-\beta)/\alpha).$$

On the other hand, put $w_j(x) = u_j(x)\varphi(x)^{-1}$ $(j=1, 2)$, then, as the function $w_1(x) - w_2(x)$ attains the negative minimum at the point $x^{(0)} \in D_0$, we have

(5.7) $$w_1(x^{(0)}) < w_2(x^{(0)}),\ \partial_x w_1(x^{(0)}) = \partial_x w_2(x^{(0)}),$$

(5.8) $$\mathfrak{L}(w_1(x^{(0)})) \geqq \mathfrak{L}(w_2(x^{(0)})).$$

Furthermore, put $w_j{}^{(0)} = w_j(x^{(0)})$ $(j=1, 2)$ and $\varphi^{(0)} = \varphi(x^{(0)})$, then it follows from (5.7) and the condition (I) that

$$g(x^{(0)}, w_1(x^{(0)}), \partial_x w_1(x^{(0)})) - g(x^{(0)}, w_2(x^{(0)}), \partial_x w_2(x^{(0)}))$$
$$= \{f(x^{(0)}, \varphi^{(0)} w_1{}^{(0)}, \varphi^{(0)}\partial_x w_1{}^{(0)} + w_1{}^{(0)}\partial_x \varphi^{(0)}) \\ -f(x^{(0)}, \varphi^{(0)} w_2{}^{(0)}, \varphi^{(0)}\partial_x w_2{}^{(0)} + \partial_x \varphi^{(0)}) \\ -(w_1{}^{(0)} - w_2{}^{(0)})\mathfrak{L}(\varphi^{(0)})\}/\varphi^{(0)}$$
$$\leqq \{f(x^{(0)}, \varphi^{(0)} w_2{}^{(0)}, \varphi^{(0)}\partial_x w_1{}^{(0)} + w_1{}^{(0)}\partial_x \varphi^{(0)}) \\ -f(x^{(0)}, \varphi^{(0)} w_2{}^{(0)}, \varphi^{(0)}\partial_x w_2{}^{(0)} + w_2{}^{(0)}\partial_x \varphi^{(0)}) - (w_1{}^{(0)} - w_2{}^{(0)})\mathfrak{L}(\varphi^{(0)})\}/\varphi^{(0)}$$
$$\leqq \{L(x^{(0)}, u_2{}^{(0)}, \partial_x u_1{}^{(0)}\,;\ \partial_i u_2{}^{(0)})\partial_i \varphi^{(0)} + \mathfrak{L}(\varphi^{(0)})\}\{w_2{}^{(0)} - w_1{}^{(0)}\}/\varphi^{(0)}$$
$$\leqq [\{L_0{}^2/\alpha - (L_0/\alpha)^2 a_{ii}(x^{(0)})\}(w_2{}^{(0)} - w_1{}^{(0)})/\varphi^{(0)}]\exp(-L_0(x_i{}^{(0)} - \beta)/\alpha) < 0.$$

Hence we have

$$g(x^{(0)}, w_1(x^{(0)}), \partial_x w_1(x^{(0)})) < g(x^{(0)}, w_2(x^{(0)}), \partial_x w_2(x^{(0)}))$$

which contradicts the inequality (5.8), q. e. d.

Chapter III. Existence theorems, I

§ 6. The equation $\mathfrak{L}(u) = f(x)$.

We must deal with the equation

(6.1) $$\mathfrak{L}(u) = f(x),$$

for discussing the existence theorem for the boundary value problem of the first kind concerning the equation

$$\mathfrak{L}(u) = f(x, u, \partial_x u).$$

On linear equations of more general forms, many authors have given out works, and especially, M. Gevrey and G. Giraud made researches by using generalized operations of such a sort as explained in § 2; [1], [2]. But here we touch briefly on the equation (6.1) and summarize the facts which are needed in future.

As our discussion rests on J. Schauder's and M. Nagumo's works [4], [7], we begin with explaining their results which will be used in this paragraph.

Let $f(x)$ be H_α-continuous in a bounded domain S and let $H_S^\alpha(f)$ be the Hölder constant of $f(x)$ on S. We define $||f||_S^\alpha$ by

$$||f||_S^\alpha = \operatorname{Sup}|f(x)| + H_S^\alpha(f).$$

If $f(x)$ belongs to $C^2[S]$ and if $\partial_i\partial_j f(x)$ are H_α-continuous in S, then we define $||f||_S^{\alpha, 2}$ by

$$||f||_S^{\alpha, 2} = ||f||_S^\alpha + ||\partial_x f||_S^\alpha + ||\partial_x\partial_x f||_S^\alpha.$$

In the three following theorems (Theorems A, B and B′), we assume that the function $f(x)$ is H_α-continuous in the closure $\bar{D}$ of a bounded domain D for a positive number α smaller than unity, and also we assume that functions $a_{ij}(x)$ are $H_{\alpha+\varepsilon}$-continuous in $\bar{D}$ and satisfy the condition

$$\det(a_{ij}) = 1 \quad \text{and} \quad ||a_{ij}||_D^{\alpha+\varepsilon} < \Lambda \qquad (0 < \alpha < 1, \varepsilon > 0).$$

THEOREM A. (J. Schauder). *There exists a constant C_Λ, which is determined only by α, ε and Λ, and which has the following property:*

For any compact set K contained in D and for any solution $u(x)$ of the equation

$$L(u) \equiv \sum_{i,j=1}^m a_{ij}(x)\partial_i\partial_j u = f(x) \tag{6.3}$$

such that $||u||_D^{\alpha, 2} < +\infty$, we have the inequality

$$||u(x)||_K^{\alpha, 2} \leqq C_\Lambda \delta^{-4}(||f||_D^\alpha + \operatorname{Max}|u(x)|),$$

where δ is the distance between K and the boundary C of D[12].

THEOREM B. (J. Schauder). *Let D be a bounded domain of B_h-class and let $\beta(x)$ be a function of $C^2[\bar{D}]$ such that $||\beta||_D^{\alpha, 2} < +\infty$.*

Then there exists a solution $u(x)$ of (6.3) in D with the values $\beta(x)$ on the boundary C of D, and we have

$$||u||_D^{\alpha, 2} \leqq M(||f||_D^\alpha + ||\beta||_D^{\alpha, 2}), \tag{6.4}$$

where M is a constant depending only on D, α, ε and Λ.

The following theorem may be easily proved by Theorems A and B.

THEOREM B′. *Let D be the same as in Theorem B, and let $\varphi(x)$ be a continuous function on C.*

Then there exists a solution $u(x)$ of (6.3) in D with the boundary values $\varphi(x)$ on C.

12) [7], Satz 1.

Hereafter, in the present chapter, we assume always that the functions $a_{ij}(x)$ satisfy the following conditions:

(6.5) $$A^{-1}\leqq\sum_{i,j=1}^{m}a_{ij}(x)\xi_i\xi_j\leqq A,\quad \text{for}\quad \sum_{i=1}^{m}\xi_i^2=1,$$

(6.6) $$\left\{\sum_{i,j=1}^{m}(a_{ij}(x')-a_{ij}(x))^2\right\}^{1/2}\leqq L\,|x'-x|,$$

where x and x' are any points of $\bar{D}$, and $A(\geqq 1)$, L are positive constants.

Under this assumption, it can be easily shown that there exists a constant $\varLambda$ depanding only on A and L such that the condition (6.2) is fulfilled.

With M. Nagumo, we say that a domain D possesses the property (σ), if there exists, coresponding to any point $p\in C$, a closed sphere S_p with a radius $\sigma>0$, such that $\bar{D}\cap S_p=\{p\}$.

Then it is easily seen that a domain of B_h-class has the property (σ).

The two next theorems are due to M. Nagumo.

THEOREM C. *Let D be a bounded domain having the property (σ) and let d be the diameter of D.*

Let $u(x)$ be a solution of (6.3) *vanishing on the boundary C of D. Then we obtain*

$$|u(x)|\leqq C_{A,\sigma,d}\,\rho(x)\operatorname*{Max}_{\bar{D}}|f(x)|$$

where $C_{A,\sigma,d}$ is a constant depending only on A,σ,d, and $\rho(x)=\mathrm{dist}(x,C)$[13].

THEOREM D. *Let D and d be the same as in Theorem C, and let $u(x)$ be a solution of* (6.3) *in D. Then we have*

$$|\partial_x u(x)|\leqq C^{(1)}_{A,L,d}\,\rho(x)^{-1}\operatorname*{Max}_{|x-x'|\leqq\rho(x)}|u(x')|+C^{(2)}_{A,L,d}\,\rho(x)\operatorname*{Max}_{\bar{D}}|f(x)|,$$

where $C^{(1)}_{A,L,d}$ and $C^{(2)}_{A,L,d}$ are constants depending only on A, L and d, more exactly $C^{(1)}_{A,L,d}=\sqrt{2}\,(m+6)A^3Ld$[14) *)].

Regarding Theorem D, M. Nagumo proved a more general theorem, but Theorem D will suffice for our present needs.

By making use of Theorems B′ and C, we first prove the following theorem.

THEOREM 6.1. *Let D be a bounded domain of B_h-class, and let $f(x)$ be continuous in $\bar{D}$. Then the equation* (6.1) *has a unique solution $u(x)$, which attains the boundary values $\varphi(x)$, where $\varphi(x)$ is a continuous function prescribed on the boundary C.*

Furthermore we have

13) [4], Lemma 1, p. 210.
14) [4], Corollary of Theorem 2, p. 214.
*) See Postskribo at the end of this note

$$|u(x)| \leq \underset{C}{\mathrm{Max}}\, |\varphi(x)| + C_{A,\sigma,d}\, \rho(x) \underset{\bar{D}}{\mathrm{Max}}\, |f(x)|,$$

where $\rho(x) = \mathrm{dist}(x, C)$ *and* d *is the diameter of* D, *and* $C_{A,\sigma,d}$ *is a constant depending only on* A, σ *and* d.

PROOF. Let $v(x)$ be a solution of the equation

$$L(u) = 0,$$

which attains the boundary values $\varphi(x)$ on C, and then the maximum principle implies $|v(x)| \leq \underset{C}{\mathrm{Max}}\, |\varphi(x)|$.

Next we will show that the equation (6.1) has a solution $w(x)$ vanishing on C, and that the following inequality holds :

(6.8) $$|w(x)| \leq C_{A,\sigma,d}\, \rho(x) \underset{\bar{D}}{\mathrm{Max}}\, |f(x)|.$$

Let $\{f_n(x)\}$ be a sequence of functions belonging to $C^1[\bar{D}]$ and converging to $f(x)$ uniformly in $\bar{D}$. And consider the equations

(6.9) $$L(w) = f_n(x) \qquad (n = 1, 2, \cdots),$$

then by Theorem B′, there exist solutions $w_n(x)$ of (6.9) which vanish on C, and by virtue of Theorem 4.6 we have

$$|w_n(x)| \leq \Psi(x) \underset{\bar{D}}{\mathrm{Max}}\, |f_n(x)|,$$

and

$$|w_n(x) - w_{n+k}(x)| \leq \Psi(x) \underset{\bar{D}}{\mathrm{Max}}\, |f_n(x) - f_{n+k}(x)|,$$

where $\Psi(x)$ is the solution of the equation $L(v) = -1$, which vanishes on C. From these inequalities and Theorem 3.2, it follows that the sequence $\{w_n(x)\}$ converges to a continuous function $w(x)$ uniformly in $\bar{D}$ and we obtain

$$\mathfrak{L}(w(x)) = f(x) \quad \text{and} \quad |w(x)| \leq \Psi(x) \underset{\bar{D}}{\mathrm{Max}}\, |f(x)|.$$

Theorem C in this paragraph shows $0 \leq \Psi(x) \leq C_{A,\sigma,d}\, \rho(x)$, which implies (6.8), q.e.d.

COROLLARY 1. *The solution* $u(x)$ *given in Theorem* 6.1, *belongs to* $C^1[D]$, *and we have*

(6.10) $$|\partial_x u(x)| \leq C^{(1)}_{A,L,d}\, \rho(x)^{-1} \underset{C}{\mathrm{Max}}\, |\varphi(x)| + C^{(2)}_{A,L,\sigma,d} \underset{\bar{D}}{\mathrm{Max}}\, |f(x)|$$

where $C^{(1)}_{A,L,d}$ *is a constant depending only on* A, L, d ; *more exactly* $C^{(1)}_{A,L,d} = \sqrt{2}(m+6)A^3Ld$*), *and* $C^{(2)}_{A,L,\sigma,d}$ *is a constant depending only on* A, L, σ *and* d.

PROOF. By Theorem D in this paragraph, we have

$$|\partial_x w_n(x)| \leq C^{(1)}_{A,L,d}\, \rho(x)^{-1} \underset{|x-x'| \leq \rho(x)}{\mathrm{Max}}\, |w_n(x')| + C^{(2)}_{A,L,d}\, \rho(x) \underset{\bar{D}}{\mathrm{Max}}\, |f_n(x)|,$$

and hence

*) See Postskribo at the end of this note.

$$|\partial_x w_n(x)| \leqq \{2\, C^{(1)}_{A,L,d} C_{A,\sigma,d} + C^{(2)}_{A,L,d}\, \rho(x)\} \operatorname*{Max}_{\bar{D}} |f_n(x)|,$$

$$|\partial_x w_n(x) - \partial_x w_{n+k}(x)|$$
$$\leqq \{2\, C^{(1)}_{A,L,d}\, C_{A,\sigma,d} + C^{(2)}_{A,L,d}\, \rho(x)\} \operatorname*{Max}_{\bar{D}} |f_n(x) - f_{n+k}(x)|,$$

where $C^{(1)}_{A,L,d}$, $C^{(2)}_{A,L,d}$ are constants depending only on A, L, d, and $C_{A,\sigma,d}$ is a constant depending only on A, σ, d.

The last inequality shows that the sequence $\{\partial_x w_n(x)\}$ converges to a continuous function uniformly in D, and hence $w(x)$ belonges to $C^1[D]$. Furthermore we obtain

$$|\partial_x w(x)| \leqq C^{(2)}_{A,L,\sigma,d} \operatorname*{Max}_{\bar{D}} |f(x)|,$$

where $C^{(2)}_{A,L,\sigma,d} = 2\, C^{(1)}_{A,L,d}\, C_{A,\sigma,d} + dC^{(2)}_{A,L,d}$.

On the other hand we see

$$|\partial_x v(x)| \leqq C^{(1)}_{A,L,d}\, \rho(x)^{-1} \operatorname*{Max}_{C} |\varphi(x)|,$$

we have therefore (6.10).

Corollary 2. *Let $u(x)$ be continuous in a bounded domain D, and let $\mathfrak{L}(u(x))$ be defined and continuous in D. Then $u(x)$ belongs to $C^1[D]$.*

Let D_0 be a domain such that $\bar{D}_0 \subset D$. Then we can find a function $\bar{u}(x) \in C^2[\bar{D}_0]$ for a given $\varepsilon > 0$, such that

$$|u(x) - \bar{u}(x)| < \varepsilon, \quad |\partial_x u(x) - \partial_x \bar{u}(x)| < \varepsilon, \quad |\mathfrak{L}(u(x)) - L(\bar{u}(x))| < \varepsilon,$$

in $\bar{D}_0$.

Proof. Put $\mathfrak{L}(u(x)) \equiv f(x)$. Since $u(x)$ can be regarded as a solution of the equation (6.1), we see easily that $u(x)$ belongs to $C^1[D]$.

Choose a domain D_1 of B_h-class such that $D_0 \Subset D_1 \Subset D$[15], and regard $u(x)$ as a solution of (6.1) in D_1, which attains the values $u(x)$ on the boundary of D_1.

Consider D_1 as D in Theorem 6.1 and Corollary 1, and put

$$u_n(x) = v(x) + w_n(x),$$

then $u_n(x) \in C^2[D_1]$, and the sequences $\{u_n(x)\}$, $\{\partial_x u_n(x)\}$ and $\{L(u_n(x))\}$ converge uniformly in $\bar{D}_0$ to $u(x)$, $\partial_x u(x)$ and $\mathfrak{L}(u(x))$ respectively. By these facts we obtain this corollary.

The following theorem was proved by G. Giraud.

Theorem E. *Let $\mathfrak{D}$ be a bounded domain in the m-dimensional Euclidean space and let $w(\xi)$ be a continuous function in $\mathfrak{D}$ such that $\operatorname*{Max}_{\mathfrak{D}} |w(\xi)| \leqq M < +\infty$. Furthermore let $G(x, \xi)$ and $\partial_{x_i} G(x, \xi)$ $(i = 1, 2, \cdots, m)$ be continuous functions*

15) Regarding the notation $D_1 \Subset D_0$, see § 1.

for any $x, \xi \in \mathfrak{D}$, $(x \neq \xi)$, *such that*

$$|G(x,\xi)| < N|x-\xi|^{\lambda-m}, \quad |\partial_{x_i} G(x,\xi)| < N|x-\xi|^{\lambda-m-1} \qquad (0<\lambda\leqq 1).$$

Then the function defined by

$$F(x) = \int_{\mathfrak{D}} G(x,\xi) w(\xi) d\omega_\xi$$

satisfies the following inequalities for any x, $x' \in \mathfrak{D}$:

$$|F(x)-F(x')| \leqq \begin{cases} k(m)MN\lambda^{-1}(1-\lambda)^{-1}|x-x'|^{\lambda} & \textit{for} \quad \lambda<1, \\ k(m)MN|x-x'| \log \dfrac{L_0}{|x-x'|} & \textit{for} \quad \lambda=1, \end{cases}$$

where L_0 *is a positive constant greater than the diameter of* $\mathfrak{D}$ *and* $k(m)$ *is a positive constant depending only on* m *and* L_0 [16].

THEOREM 6.2. *Let* D *be a bounded domain, and let* $\{f_\eta(x);\ \eta \in H\}$ *be a family of functions which are continuous and uniformly bounded in* D: $|f_\eta(x)| \leqq \Gamma$.

Let $u_\eta(x)$ *be solutions of the equations*

$$\mathfrak{L}(u) = f_\eta(x) \qquad (\eta \in H),$$

and suppose that $\{u_\eta(x)\}$ *is uniformly bounded in* D: $|u_\eta(x)| \leqq N$. *Then the families* $\{u_\eta(x)\}$ *and* $\{\partial_x u_\eta(x)\}$ *are normal in* D.

PROOF. Let D_0 be a domain such that $\bar{D}_0 \subset D$, and we will show that the families $\{u_\eta(x)\}$ and $\{\partial_x u_\eta(x)\}$ are normal in $\bar{D}_0$.

Choose a domain D_1 of B_h-class, in such a manner that $D_0 \Subset D_1 \Subset D$.

Then it follows from Corollary 1 of Theorem 6.1, that

$$|\partial_x u_\eta(x)| \leqq C^{(1)}_{A,L,d}\, \delta^{-1} N + C^{(2)}_{A,L,\sigma,d}\, \Gamma$$

in $\bar{D}_0$, where δ is the distance between the boundaries of D_1 and D_0. This estimation implies the uniform boundedness of $\{\partial_x u_\eta(x)\}$ in $\bar{D}_0$. Hence it is obvious that $\{u_\eta(x)\}$ is normal in $\bar{D}_0$.

Regarding the family $\{\partial_x u_\eta(x)\}$, we have only to show that $\{\partial_x u_\eta(x)\}$ is equi-continuous in a neighborhood of every point contained in D_1.

Let x_0 be any point in D_1 and choose three concentric spheres S_1, S_2 and S_3 with the center x_0, such that $D_1 \Supset S_1 \Supset S_2 \Supset S_3$, and so small that we can construct the elementary solution $E(x,\xi)$ in S_1, by the method explained in § 3.

Put

$$w_\eta(x) = -\int_{S_1} E(x,\xi) f_\eta(\xi) d\omega_\xi,$$

16) See G. Giraud's memoir [3], p. 137, Theorem 1, and p. 146, Theorem 5. To prove Giraud's theorems, the convexity of the domain $\mathfrak{D}$ must be assumed, but the hypotheses of these theorems lack this condition. When we shall make use of Theorem E later on, this condition will be fulfilled.

and let $v_\eta(x)$ be solutions of the equation $L(v)=0$, which are defined in $\bar{S}_2$, and attain the values $u_\eta(x)-w_\eta(x)$ on the boundary of S_2.

The functions $w_\eta(x)$ belong to $C^1[S_1]$, and it is easily seen that

$$u_\eta(x)=v_\eta(x)+w_\eta(x), \quad \partial_x u_\eta(x)=\partial_x v_\eta(x)+\partial_x w_\eta(x).$$

Thus we have only to show that $\{\partial_x v_\eta(x)\}$ and $\{\partial_x w_\eta(x)\}$ are equi-continuous in $\bar{S}_3$.

Consider $\{\partial_x w_\eta(x)\}$ in the first place, then in the same way as in the proof of Theorem 3.1, we easily obtain

$$w_\eta(x)=\int_{S_1} H(x,\xi)g_\eta(\xi)d\omega_\xi,$$

and

$$\partial_x w_\eta(x)=\int_{S_1} \partial_x H(x,\xi)\, g_\eta(\xi)d\omega_\xi,$$

where

$$g_\eta(\xi)=-\left\{f_\eta(\xi)+\int_{S_1} f_\eta(\zeta)\sum_{n=1}^{\infty} K^{(n)}(\xi,\zeta)d\omega_\zeta\right\}.$$

Since there exists a constant K_0 such that

$$1+\int_{S_1}\sum_{n=1}^{\infty}|K^{(n)}(\xi,\zeta)|\,d\omega_\zeta \leqq K_0 \quad \text{for} \quad \xi\in S_1,$$

the functions $g_\eta(\xi)$ have a common upper bound in such a manner that $|g_\eta(\xi)|\leqq K_0\Gamma$ for $\xi\in S_1$ and $\eta\in H$.

On the other hand, we can find a constant K_1 such that

$$|\partial_i H(x,\xi)|\leqq K_1|x-\xi|^{1-m}, \quad |\partial_j\partial_i H(x,\xi)|\leqq K_1|x-\xi|^{-m}$$

for $x,\xi\in\bar{S}_1$. Therefore, by virtue of Theorem E we obtain

$$(6.11) \qquad |\partial_x w_\eta(x)-\partial_x w_\eta(x')|\leqq k(m)K_0K_1\Gamma|x-x'|\log\frac{L_0}{|x-x'|}$$

for $x',x\in S_1$, where L_0 is a constant greater than the diameter of S_1, and $k(m)$ is a constant depending only on m, L_0.

Hence $\partial_x w_\eta(x)$ are H_λ-continuous in S_1, with a common Hölder constant for all $\partial_x w_\eta(x)$, where λ may be any positive number smaller than unity. Thus the family $\{\partial_x w_\eta(x)\}$ is equi-continuous in S_1.

Next the equi-continuity of $\{\partial_x v_\eta(x)\}$ in $\bar{S}_3$, is easily proved by showing that $\{\partial_x\partial_x v_\eta(x)\}$ is uniformly bounded in $\bar{S}_3$.

In fact, the family $\{w_\eta(x)\}$ is uniformly bounded in S_1, which derives from the inequality

$$|w_\eta(x)|\leqq\Gamma\int_{S_1}|E(x,\xi)|\,d\omega_\xi,$$

and hence there exists a positive constant K_2, such that

$$\operatorname*{Max}_{\bar{S}_2}|v_\eta(x)|\leqq\operatorname*{Max}_{\bar{S}_2}|u_\eta(x)|+\operatorname*{Max}_{\bar{S}_2}|w_\eta(x)|\leqq K_2$$

for any $\eta \in H$. Therefore it follows from Theorem A, that

$$\operatorname*{Max}_{\bar{S}_3} |\partial_x \partial_x v_\eta(x)| \leqq C_{(A,L)}\, \delta^{-4} \operatorname*{Max}_{\bar{S}_2} |v_\eta(x)| \leqq C_{(A,L)}\, \delta^{-4} K_2,$$

where δ is the distance between the boundaries of S_2 and S_3, and $C_{(A,L)}$ is a constant depending only on A and L. This shows the uniform boundedness of $\{\partial_x \partial_x v_\eta(x)\}$ in $\bar{S}_3$, q.e.d.

THEOREM 6.3. *Let D be a bounded domain of B_h-class, and let $f(x)$ be a bounded continuous function defined in D. Then the equation* (6.1) *has a unique solution $u(x)$, which attains the prescribed value $\varphi(x)$ on the boundary C of D, and we obtain*

$$|u(x)| \leqq \operatorname*{Max}_{C} |\varphi(x)| + C_{A,\sigma,d}\, \rho(x) \operatorname*{Sup}_{D} |f(x)|, \tag{6.12}$$

where $\varphi(x)$ is a continuous function defined on C and $C_{A,\sigma,d}$ has the same meaning as in Theorem 6.1.

PROOF. Without loss of generality we can assume $\varphi(x) \equiv 0$.

Let $\{D_n\}$ be a sequence of domains of B_h-class, such that $D_n \Subset D_{n+1} \Subset D$ and $\bigcup_{n=1}^{\infty} D_n = D$, and let C_n be the boundaries of D_n. Then, by virtue of Theorem 6.1 the equation (6.1) has solutions $u_n(x)$ vanishing on C_n and defined in $\bar{D}_n$. Here, we extend the domain of definition of $u_n(x)$ to D, by defining $u_n(x) \equiv 0$ in $D - \bar{D}_n$.

Let $\Psi(x)$ be the solution of the equation $L(v) = -1$, which vanishes on C. Then it follows from Theorem 4.6, that

$$|u_n(x)| \leqq M\Psi(x) \quad \text{in } D \quad (n = 1, 2, \cdots), \tag{6.13}$$

where $M > \operatorname*{Sup}_{D} |f(x)|$.

The inequalities (6.13) imply the uniform boundedness of $\{u_n(x)\}$ and hence we see, by Theorem 6.2, that $\{u_n(x)\}$ is normal in D.

Therefore we can assume, without loss of generality, that $\{u_n(x)\}$ converges to a continuous function $u(x)$ uniformly in D. $u(x)$ is, obviously, a solution of (6.1) in D, and by (6.13) we have

$$|u(x)| \leqq M\Psi(x) \quad \text{in } D. \tag{6.14}$$

This inequality shows that $u(x)$ vanishes on C. Further, since $0 \leqq \Psi(x) \leqq C_{A,\sigma,d}\, \rho(x)$, we obtain (6.12) by letting M tend to $\operatorname{Sup} |f(x)|$ in the inequality (6.14).

§ 7. Estimation of the derivatives $\partial_x u$

Let $u(x)$ be a solution of the equation (I): $L(u) = f(x, u, \partial_x u)$. M. Nagumo obtained a result for the estimation of the derivatives $\partial_x u$, and used it for proving an existence theorem of solutions for the boundary value problem concerning the equation (I).

In this paragraph, we will extend his result to the equation

$$\mathfrak{L}(u)=f(x, u, \partial_x u), \tag{7.1}$$

and modify it a little.

THEOREM 7.1. *Let D be a bounded domain with the diameter d, and let the coefficients $a_{ij}(x)$ fulfil the conditions* (6.5) *and* (6.6) *in D.*

Let $f(x, u, p)$ be defined in the domain:

$$\mathfrak{D}_2=\{(x, u, p)\ ;\ x\in D,\quad \underline{\omega}(x)\leqq u\leqq \bar{\omega}(x),\quad |p|<+\infty\},$$

and satisfy the following condition:

$$|f(x, u, p)|\leqq B|p|^2+\Gamma, \tag{7.2}$$

where B and Γ are constants.

Let $u(x)$ be a solution of the equation (7.1), *such that*

$$|u(x)|\leqq M \quad \text{in } D, \text{ and} \quad 4\gamma_m ABM<1, \tag{7.3}$$

where $\gamma_m=2\Gamma((m+2)/2)/\Gamma(1/2)\Gamma((m+1)/2)$. Then we have for any $x\in D$,

$$|\partial_x u(x)|\leqq C^{(1)}\rho(x)^{-1}\operatorname*{Max}_{|x-x'|\leqq\rho(x)}|u(x')|+C^{(2)},\ \rho(x)=\operatorname{dist}(x, C),$$

where $C^{(1)}$ and $C^{(2)}$ are constants depending only on m, A, L, B, M, d, and Γ.

PROOF. Let a be a point of D, and let Σ_κ be a closed sphere defined by

$$\Sigma_\kappa=\{x\ ;\ |x-a|\leqq\kappa\rho(a)\}\qquad(0<\kappa<1).$$

Put

$$\mu_\kappa=\operatorname*{Max}_{x\in\Sigma_\kappa}\{|\partial_x u|\,\rho_\kappa(x)\}, \tag{7.4}$$

where $\rho_\kappa(x)=\operatorname{dist}(x, \dot{\Sigma}_\kappa)$, and $\dot{\Sigma}_\kappa$ is the boundary of Σ_κ, then there exists a point $x_0\in\Sigma_\kappa$ such that

$$|\partial_x u(x_0)|\,\rho_\kappa(x_0)=\mu_\kappa. \tag{7.5}$$

On the other hand, by the linear transformation $x=Tx'$ associated with x_0[17], we obtain

$$\sum_{i,j=1}^m a_{ij}(x_0)\partial_i\partial_j v(x)=\sum_{i=1}^m\partial_i'\partial_i' v'(x')$$

for any twice differentiable function $v(x)$.

We put $u(x)=u'(x')$, $f(x, u, \partial_x u)=f'(x', u', \partial_x' u')$, and denote by $\mathfrak{L}'(u'(x'))$ the generalized operation of elliptic type after effecting the linear transformation $x=Tx'$. Then, especially we have $\mathfrak{L}'(u'(x_0'))=\Delta' u'(x_0')$.

Let S_λ be a closed sphere of a radius $\lambda\rho_\kappa(x_0)$ and having $x_0'=T^{-1}(x_0)$ as the center and contained in $T^{-1}(\Sigma_\kappa)$: $S_\lambda\subset T^{-1}(\Sigma_\kappa)$, where λ is a number such that $0<\lambda<A^{-1/2}/2$.

By virtue of Corollary 2 of Theorem 6.1, we see that there exists, for a given $\delta>0$, a function $v(x')$ belonging to $C^2[S_\lambda]$ such that inequalities

$$|u'(x')-v(x')|<\varepsilon,\quad |\partial_x' u'(x')-\partial_x' v(x')|<\varepsilon, \tag{7.6}$$
$$|\mathfrak{L}'(u'(x'))-L'(v(x'))|<\varepsilon$$

17) See § 2.

hold for any $x'\in S_\lambda$[18]. Thus, the equation (7.1) can be rewritten in such a manner that

$$(7.7)\qquad \Delta' v(x')=\sum_{i,j=1}^{m}(\delta_{ij}-a'_{ij}(x'))\partial_i'\partial_j' v(x') + \{L'(v(x'))-\mathfrak{L}'(u'(x'))\}+f'(x',u',\partial_{x'}u').$$

Let $G(x',\xi)$ be Green's function concerning the equation $\Delta' v=0$ with respect to S_λ. Then we have

$$\begin{aligned} v(x')=&-\omega_m^{-1}\int_{S_\lambda} G(x',\xi)f'(\xi,u'(\xi),\partial_\xi u'(\xi))d\omega_\xi \\ &-\omega_m^{-1}\int_{S_\lambda} G(x',\xi)\{L'(v(\xi))-\mathfrak{L}'(u'(\xi))\}\,d\omega_\xi \\ &-\omega_m^{-1}\int_{S_\lambda} G(x',\xi)\sum_{i,j=1}^{m}(\delta_{ij}-a'_{ij}(\xi))\partial_i'\partial_j' v(\xi)d\omega_\xi \\ &-h(x')-\{\bar h(x')-h(x')\}, \end{aligned}$$

where $h(x')$ and $\bar h(x')$ are harmonic in S_λ and attain the values $u'(x')$ and $v(x')$ respectively on the boundary of S_λ.

The above expression for $v(x')$ shows that

$$(7.8)\qquad |\partial_{x'} v(x_0')|\leqq (\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V}),$$

where

$$\begin{aligned} (\mathrm{I}) &= \left|\omega_m^{-1}\int_{S_\lambda}\partial_{x'} G(x_0',\xi)f'd\omega_\xi\right|, \\ (\mathrm{II}) &= \left|\omega_m^{-1}\int_{S_\lambda}\partial_{x'} G(x_0',\xi)\sum_{i,j=1}^{m}(\delta_{ij}-a'_{ij}(\xi))\partial_i'\partial_j' v(\xi)d\omega_\xi\right|, \\ (\mathrm{III}) &= |\partial_{x'} h(x_0')|. \\ (\mathrm{IV}) &= \left|\omega_m^{-1}\int_{S_\lambda}\partial_{x'} G(x_0',\xi)\{L'(v(\xi))-\mathfrak{L}'(u'(\xi))d\}\,\omega_\xi\right|, \\ (\mathrm{V}) &= |\partial_{x'}\{h(x_0)-h(x_0)\}|. \end{aligned}$$

Now, it is easily verified that

$$(7.9)\qquad |\partial_x u(x)|\leqq (1-\lambda A^{1/2})^{-1}\rho_\kappa(x_0)^{-1}\mu_\kappa \quad\text{in}\quad T(S_\lambda)\subset\Sigma_\kappa,$$

and we have

$$(7.10)\qquad |\partial_x u(x)|\leqq \sqrt{2}\,\rho_\kappa(x_0)^{-1}\mu_\kappa \quad\text{in}\quad T(S_\lambda)\subset\Sigma_\kappa,$$

by making λ so small that $(1-\lambda A^{1/2})^{-1}\leqq\sqrt{2}$.

Hence, it follows from (7.2), (7.9), that

$$\begin{aligned} (7.11)\qquad |f'(\xi,u'(\xi),\partial_\xi u'(\xi))| &= |f(x,u(x),\partial_x u(x))| \\ &\leqq (1-\lambda A^{1/2})^{-2}B\rho_\kappa(x_0)^{-2}\mu_\kappa^2+\Gamma \end{aligned}$$

for any $\xi\in S_\lambda$.

Since

18) Apply Corollary 2 of Theorem 6.1 to the new operation $\mathfrak{L}'(u'(x'))$.

$$|\partial_{x'} G(x_0', \xi)| \leqq \left\{1-\left(\frac{|\xi-x_0'|}{\lambda\rho_\kappa(x_0)}\right)^m\right\} |\xi-x_0'|^{-m+1} \leqq |\xi-x_0'|^{-m+1},$$

we obtain

(7.12) $\qquad$ (I) $\leqq \{(1-\lambda A^{1/2})^{-2} B \rho_\kappa(x_0)^{-1} \mu_\kappa^2 + \Gamma \rho_\kappa(x_0)\} \lambda.$

Furthermore, by using (7.6) and

$$|\partial_\xi \partial_{x'} G(x_0', \xi)| \leqq \left[m+\left\{1-\left(\frac{|\xi-x_0'|}{\lambda\rho_\kappa(x_0)}\right)^m\right\}\right] |\xi-x_0'|^{-m}$$
$$\leqq (m+1)|\xi-x_0'|^{-m},$$

and in the same manner as in M. Nagumo's proof[19], we have

(7.13) $\qquad$ (II) $\leqq m\sqrt{m}(m+3)A^{3/2} L \lambda\rho_\kappa(x_0) \{\sqrt{2A}\, \rho_\kappa(x_0)^{-1} \mu_\kappa + \varepsilon\},$

(7.14) $\qquad$ (III) $\leqq \lambda^{-1} \rho_\kappa(x_0)^{-1} \gamma_m \operatorname{Max}\{|u'(x')| ; |x'-x'| \leqq \lambda\rho_\kappa(x_0)\}$

$$\leqq \lambda^{-1} \rho_\kappa(x_0)^{-1} \gamma_m \sup_{|x-a| \leqq \rho(a)} \{|u(x)|\},$$

(7.15) $\qquad$ (IV) $\leqq \lambda\rho_\kappa(x_0)\varepsilon, \qquad$ (V) $\leqq \lambda^{-1} \rho_\kappa(x_0)^{-1} \gamma_m \varepsilon.$

On the other hand, the inequality

$$|\partial_{x'} u(x_0')| \geqq A^{-1/2} |\partial_x u(x_0)| = A^{-1/2} \rho_\kappa(x_0)^{-1} \mu_\kappa,$$

shows

(7.16) $\qquad |\partial_{x'} v'(x_0')| \geqq |\partial_{x'} u'(x_0')| - \varepsilon \geqq A^{-1/2} \rho_\kappa(x_0)^{-1} \mu_\kappa - \varepsilon.$

Since $\rho_\kappa(x_0) < 2\rho(a) \leqq d$, it follows from (7.12)-(7.16), that

$$\lambda C_0 \mu_\kappa^2 - (1-\lambda C_1)\mu_\kappa + \lambda^{-1} C_2 + \varepsilon C_3 \geqq 0,$$

where

$$C_0 = (1-\lambda A^{1/2})^{-2} B A^{1/2},$$
$$C_1 = m(m+3)\sqrt{2m}\, L A^{5/2} d,$$
$$C_2 = \gamma_m A^{1/2} \sup_{|x-a| \leqq \rho(a)} |u(x)| + 4 A^{1/2} \lambda^2 \rho(a)^2 \Gamma,$$
$$C_3 = m(m+3)\sqrt{m}\, L A^2 \lambda\rho_\kappa(x_0)^2 + \lambda A^{1/2} \rho_\kappa(x_0)^2 + \lambda^{-1} \gamma_m A^{1/2} + A^{1/2} \rho_\kappa(x_0).$$

Thus, by letting ε tend to zero, we have

(7.17) $\qquad \lambda C_0 \mu_\kappa^2 - (1-\lambda C_1)\mu_\kappa + \lambda^{-1} C_2 \geqq 0.$

Since $C_0 C_2 \leqq (1-\lambda A^{1/2})^{-2} AB(\gamma_m M + 4\lambda^2 \rho(a)^2 \Gamma)$, by using the condition (7.3), we can make λ so small that

$$(1-\lambda C_1)^2 > 4(\lambda C_0)(\lambda^{-1} C_2),$$

and hence an equation

$$\lambda C_0 X^2 - (1-\lambda C_1)X + \lambda^{-1} C_2 = 0$$

may have two distinct real positive roots R_1, R_2.

Therefore we obtain along the same lines as in M. Nagumo's proof

$$|\partial_x u(a)| \leqq C^{(1)} \rho(a)^{-1} \sup_{|x-a| \leqq \rho(a)} |u(x)| + C^{(2)},$$

where $C^{(1)}$, $C^{(2)}$ are constants depending only on m, A, L, B, M, Γ and d.

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19) [4], Theorem 2, pp. 211-214.

THEOREM 7.2. *If, in stead of the condition* (7.2) *the following condition is assumed*:

$$|f(x,u,p)|\leqq \Gamma,$$

and if the condition (7.3) *is dropped, then we have*

$$|\partial_x u(x)|\leqq K^{(1)}\rho(x)^{-1}\sup_{|x'-x|\leqq\rho(x)}|u(x')|+K^{(2)}\rho(x)\Gamma, \tag{7.18}$$

where $K^{(1)}$ *and* $K^{(2)}$ *are constants depending only on* m, A, L *and* d.

PROOF. Under the above-mentioned assumption, the inequality (7.17) may be replaced by

$$(1-\lambda C_1)\mu_\kappa\leqq\lambda^{-1}C_2,$$

and then by putting $\lambda=\mathrm{Min}\{(2C_1)^{-1},A^{-1/2}/4\}$, we obtain

$$\mu_\kappa\leqq 2(2C_1)C_2=4C_1\gamma_m A^{1/2}\sup_{|x-a|\leqq\rho(a)}|u(x)|+8A^{1/2}\lambda\rho(a)\Gamma.$$

Therefore by letting κ tend to unity, (7.18) can be shown.

§ 8. The case where $f(x,u,p)$ is bounded.

In this paragraph, we establish an existence theorem for solutions of the equation

$$\mathfrak{L}(u)=f(x,u,\partial_x u) \tag{8.1}$$

in the case where $f(x,u,p)$ is bounded in $\mathfrak{D}_1$.

THEOREM 8.1. *Let* D *be a bounded domain of* B_h*-class, and let the coefficients* $a_{ij}(x)$ *satisfy the conditions*:

$$A^{-1}\leqq\sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j\leqq A \quad \textit{for} \quad \sum_{i=1}^m\xi_i^2=1, \tag{8.2}$$

$$\left\{\sum_{i,j=1}^m(a_{ij}(x')-a_{ij}(x))^2\right\}^{1/2}\leqq L|x'-x|, \tag{8.3}$$

for any $x,x'\in\bar{D}$, *where* $A(\geqq 1)$ *and* L *are positive constants.*

Suppose that $f(x,u,p)$ *is continuous and bounded in the domain*

$$\mathfrak{D}_1=\{(x,u,p);\ x\in D,\ |u|<+\infty,\ |p|<+\infty\},$$

that is to say, $|f(x,u,p)|\leqq\Gamma$ *in* $\mathfrak{D}_1$.

Then there exists a solution of the equation (8.1), *which vanishes on the boundary* C *of* D.

PROOF. For a positive constant N, we denote by $\mathfrak{F}_N$ a family of all functions $v(x)(\in C^1[D])$ having the following properties:

$$v(x)\in C[\bar{D}] \quad\text{and}\quad |v(x)|\leqq\Gamma\Psi(x), \tag{8.4}$$

$$|\partial_x v(x)|\leqq N \quad\text{in}\quad D, \tag{8.5}$$

where $\Psi(x)$ is the solution of the equation $L(v)=-1$, which vanishes on C.

Therefore $\mathfrak{F}_N$ is a closed and convex set in the linear topological space $C^1[D]$. The topology in $C^1[D]$ is defined by semi-norms $\|v\|_n$:

$$\|v\|_n=\max_{D_n}|v(x)|+\max_{\bar{D}_n}|\partial_x v(x)|,$$

where $\{D_n\}$ is a sequence of domains, such that $D_n \Subset D_{n+1}$, and $\bigcup_{n=1}^{\infty} D_n = D$.

As the function $f_{[v]}(x) = f(x, v(x), \partial_x v(x))$ is continuous and bounded in D for any $v(x) \in \mathfrak{F}_N$, Theorem 6.3 shows that the equation

$$\mathfrak{L}(u) = f_{[v]}(x)$$

has a unique solution $u(x)$ vanishing on C, and moreover it follows from Theorem 6.3 and Theorem 7.2 that

$$(8.6) \qquad |\partial_x u(x)| \leqq C^* \Gamma, \quad \text{and} \quad |u(x)| \leqq \Gamma \Psi(x),$$

where $C^* = 2K^{(1)} \cdot C_{A,\sigma,d} + K^{(2)} \cdot d$ is a constant depending only on m, A, L, σ, d, and d is the diameter of D.

Choose N in such a manner that $N = C^* \Gamma$, then the solution $u(x)$ obtained above, belongs to $\mathfrak{F}_N$ and hence we can define a mapping $\mathfrak{F}_N \ni v(x) \to u(x) = \mathfrak{T}(v(x)) \in \mathfrak{F}_N$, which maps $\mathfrak{F}_N$ into itself.

The continuity of the mapping $\mathfrak{T}$ can be proved in the following way.

Without loss of generality, we can assume that all D_n are of B_h-class.

Let $u(x)$ and $u'(x)$ be the images of $v(x)$ and $v'(x)$ respectivery; then we have only to show that, for any n_0 and any $\varepsilon > 0$, there exist n_1, $\delta > 0$, such that

$$(8.7) \qquad \|v - v'\|_{n_1} < \delta \quad \text{implies} \quad \|u - u'\|_{n_0} < \varepsilon.$$

Since $w(x) = u(x) - u'(x)$ is a solution of the equation

$$\mathfrak{L}(w) = f_{[v]}(x) - f_{[v']}(x),$$

and $w(x)$ vanishes on C, we obtain by Theorem 6.3,

$$\operatorname*{Max}_{C_n} |w(x)| \leqq 2\Gamma \operatorname*{Max}_{C_n} \Psi(x) \leqq 2\Gamma C_{A,\sigma,d} \operatorname*{Max}_{C_n} \rho(x),$$

where C_n is the boundary of D_n and $\Psi(x)$ is the solution of the equation $L(v) = -1$, vanishing on C.

Hence there exists a decreasing sequence $\{\varepsilon_n\}$ of positive numbers: $\varepsilon_n \downarrow 0$ $(n \to \infty)$, such that $\operatorname*{Max}_{C_n} |w(x)| \leqq \varepsilon_n$.

It follows from Theorem 6.1 and its Corollary 1, that we have for $x \in D_n$,

$$|w(x)| \leqq \varepsilon_n + \Psi_n(x) \operatorname*{Max}_{D_n} |f_{[v]}(x) - f_{[v']}(x)|$$

$$\leqq \varepsilon_n + \Psi(x) \operatorname*{Max}_{D_n} |f_{[v]}(x) - f_{[v']}(x)|,$$

$$|\partial_x w(x)| \leqq C^{(1)}_{A,L,d(n)} \cdot \rho_n(x)^{-1} \varepsilon_n$$

$$+ C^{(2)}_{A,L,\sigma(n),d(n)} \operatorname*{Max}_{\bar{D}_n} |f_{[v]}(x) - f_{[v']}(x)|,$$

where $\rho_n(x) = \text{dist}(x, C_n)$, and $\sigma(n)$, $d(n)$ are the values of σ, d, corresponding to D_n, and $\Psi_n(x)$ is the solution of the equation $L(v) = -1$, vanishing on C_n, and further $C^{(1)}_{A,L,d(n)} = \sqrt{2}(m+6)A^3 L d(n)$*).

*) See Postskribo at the end of this note.

Since $C^{(1)}_{A,L,d(n)} \leqq C^{(1)}_{A,L,d}$, we can replace $C^{(1)}_{A,L,d(n)}$ by $C^{(1)}_{A,L,d}$ in the above estimation of $|\partial_x w(x)|$.

Now we can find a natural number n_1, such that

$$C^{(1)}_{A,L,d}\, \rho_{n_1}(x)^{-1}\, \varepsilon_{n_1} < \varepsilon/4 \quad \text{for} \quad x \in D_{n_0}, \quad \text{and} \quad \varepsilon_{n_1} < \varepsilon/4,$$

which follows from the fact that there exists a positive number ρ_0 having the following property: $\rho_n(x) \geqq \rho_0 > 0$ for any $x \in D_{n_0}$ and any $n > n_0$.

On the other hand, the continuity of $f(x, u, p)$ shows that there exists a positive number δ, such that $\|v - v'\|_{n_1} < \delta$ implies

$$K \cdot \operatorname*{Max}_{\bar{D}_{n_1}} |f_{[v]}(x) - f_{[v']}(x)| < \varepsilon/4,$$

$$C^{(2)}_{A,L,\sigma(n_1),d(n_1)} \operatorname*{Max}_{\bar{D}_{n_1}} |f_{[v]}(x) - f_{[v']}(x)| < \varepsilon/4,$$

where K is a constant such that $\Psi(x) \leqq K$ for $x \in \bar{D}$.

Hence we obtain (8.7)

Furthermore, by virtue of Theorem 6.2, the families

$$\{u(x)\,;\, u(x) \in \mathfrak{T}(\mathfrak{F}_N)\} \quad \text{and} \quad \{\partial_x u(x)\,;\, u(x) \in \mathfrak{T}(\mathfrak{F}_N)\}$$

are normal in D.

Therefore, it follows from the existence theorem of fixed points in a functional space[20], that there exists, in $\mathfrak{F}_N$, a function $v(x)$ satisfying

$$\mathfrak{L}(v(x)) = f_{[v]}(x) \equiv f(x, v(x), \partial_x v(x)),$$

which shows that $v(x)$ is a required solution of (8.1), q.e.d.

THEOREM 8.2. *Let D and $a_{ij}(x)$ be the same as in Theorem 8.1, and suppose that $f(x, u, p)$ is bounded in the wider sense with respect u in $\mathfrak{D}_1$[21], and $f(x, u, p) \leqq f(x, \bar{u}, p)$ for $u < \bar{u}$.*

Then there exists a solution of the equation (8.1), which attains the boundary value $\beta(x)$ on C, where $\beta(x)$ is a continuous function prescribed on the boundary C of D.

PROOF. Without loss of generality we can assume $\beta(x) \equiv 0$. By the hypothesis of the theorem, there exists a positive constant M such that $|f(x, 0, p)| \leqq M$.

We see easily that, if the equation (8.1) has a solution $u(x)$ vanishing on C, then the inequality

$$|u(x)| \leqq M\Psi(x) \leqq \Gamma < +\infty$$

holds in $\bar{D}$, where $\Psi(x)$ is the solution of the equation $L(v) = -1$, vanishing on C.

20) See [6].

21) We say that $f(x, u, p)$ is bounded in the wider sense with respect to u in $\mathfrak{D}_1$, if $f(x, u, p)$ is bounded in a domain

$$\{(x, u, p)\,;\, x \in D,\ |u| < N,\ |p| < +\infty\}$$

for any positive real number $N < +\infty$.

This follows from Theorem 4.6, and the equality

$$f(x, u, p) = \{f(x, u, p) - f(x, 0, p)\} + f(x, 0, p).$$

If we define a function $g(x, u, p)$ in the following way:

$$g(x, u, p) = \begin{cases} f(x, -\Gamma, p) & (u < -\Gamma), \\ f(x, u, p) & (-\Gamma \leqq u \leqq \Gamma), \\ f(x, \Gamma, p) & (\Gamma < u); \end{cases}$$

then $g(x, u, p)$ is bounded in $\mathfrak{D}_1$.

Therefore, by virtue of Theorem 8.1, there exists a solution $u(x)$ of the equation

$$\mathfrak{L}(u) = g(x, u, \partial_x u),$$

which vanishes on C. Since $|u(x)| \leqq \Gamma$, $u(x)$ is a solution of the equation (8.1), q.e.d.

Chapter IV. Existence theorems, II

§ 9. Introductive remarks.

In this chapter, we establish an existence theorem of solutions for the boundary value problem of the first kind concerning the equation

$$\mathfrak{L}(u) = f(x, u, \partial_x u), \tag{9.1}$$

by making use of results gained in the preceding chapter.

Our investigations are conducted by following the way pursued by M. Nagumo. But, as we make researches by employing the generalized differential operation of elliptic type and by extending the sence of solution of the equation, the proof of the theorems given in the present chapter, are more simplified than in M. Nagumo's work.

On the other hand, the generalized solutions which we treat of in this note, do not generally belong to $C^2[D]$, however it may be easily shown that our generalized solutions belong to $C^2[D]$ under the condition[22] which M. Nagumo imposed on the function $f(x, u, p)$ in his work.

In the present chapter, we assume also that the functions $a_{ij}(x)$ fulfil the following conditions in a bounded domain $\bar{D}$:

$$A^{-1} \leqq \sum_{i,j=1}^{m} a_{ij}(x)\xi_i \xi_j \leqq A \quad \text{for} \quad \sum_{i=1}^{m} \xi_i^2 = 1, \tag{9.2}$$

$$\left\{\sum_{i,j=1}^{m} (a_{ij}(x) - a_{ij}(x'))^2\right\}^{1/2} \leqq L\,|x - x'|, \tag{9.3}$$

where $A(\geqq 1)$ and L are positive constants.

22) The condition that the function $f(x, u, p)$ is H_α-continuous with respect to (x, u, p) for a positive number $\alpha: 0 < \alpha < 1$. For details, see Theorem 11.3 in § 11.

§ 10. Lemmas and a preliminary existence theorem.

In this paragraph, we give lemmas which show some characters of solutions of the equation (9.1), and assuming that the considered domain D is B_h-class, we establish a preliminary existence theorem for solutions of the equation (9.1).

LEMMA 10.1. *Let D be a bounded domain having the property (σ) and let d be the diameter of D.*

Suppose furthermore that the function $f(x,u,p)$ satisfies the following condition:

(10.1) $$|f(x,u,p)| \leqq B|p|^2+\Gamma$$ [23]

in the domain

$$\mathfrak{D}_2=\{(x,u,p)\,;\, x\in D,\ \underline{\omega}(x)\leqq u\leqq \bar{\omega}(x),\ |p|<+\infty\},$$

where B and Γ are positive constants, and $\underline{\omega}(x)$, $\bar{\omega}(x)$ are continuous functions defined in D.

Let $u(x)$ be a solution of the equation (9.1) *vanishing on the boundary C of D, such that*

(10.2) $$|u|\leqq M\ \ (x\in D)\ \ \text{and}\ \ 4\gamma_m ABM<1$$

where $\gamma_m=2\Gamma((m+2)/2)/\Gamma(1/2)\Gamma((m+1)/2)$. Then there exists a positive constant C^* depending only on m, L, A, B, M, σ and Γ, such that

$$|u(x)|\leqq C^*\rho(x),$$

where $\rho(x)=\mathrm{dist}(x,C)$.

LEMMA 10.2. *Under the same conditions as in Lemma* 10.1, *we have*

$$|\partial_x u(x)|\leqq C^{\#}\ \ (x\in D),$$

where $C^{\#}$ is a positive constant depending only on m, L, A, B, M, d, σ and Γ.

These lemmas for the operation $L(u)$ were proved by M. Nagumo in the case in which the following condition is fulfilled in stead of the condition (10.2):

(10.3) $$|u(x)|\leqq M\ \ (x\in D)\ \ \text{and}\ \ 16ABM<1$$ [24]

Under the condition stated in this note, also we can prove Lemma 10.1 in the same way as in M. Nagumo's proof[25] and by using Theorem 4.5. Hence we omit the detailed proof of Lemma 10.1.

Lemma 10.2 follows from Lemma 10.1 and Theorem 7.1.

LEMMA 10.3. *Let D be a bounded domain and suppose that the function $f(x,u,p)$ fulfils the condition* (10.1) *in the domain:*

$$\mathfrak{D}_2=\{(x,u,p)\,;\, x\in D, \underline{\omega}(x)\leqq u\leqq \bar{\omega}(x), |p|<+\infty\},$$

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23) See Theorem 7.1.

24) See [4], Lemma 2, pp. 218–219. As a matter of convenience we divide, in this note, M. Nagumo's Lemma 2 into Lemmas 10.1 and 10.2.

25) Calculating in the same manner as given by M. Nagumo, it is easily seen that the same result is obtained also under the condition (10.2).

where $\underline{\omega}(x)$ *and* $\bar{\omega}(x)$ *are continuous functions in* D, *having the following properties*:

$$|\underline{\omega}(x)| \leqq M, \quad |\bar{\omega}(x)| \leqq M \quad in \quad D \quad and \quad 4ABM < 1.$$

Let $\mathfrak{F}$ *be a family of solutions* $u(x)$ *of the equation* (9.1), *such that* $\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x)$ *in* D.

Then the families $\mathfrak{F}$ *and* $\mathfrak{F}' = \{\partial_x u(x)\,;\ u(x) \in \mathfrak{F}\}$ *are both normal in* D.

Proof. Let D_0 be a domain such that $\bar{D}_0 \subset D$. Then we get by Theorem 7.1,

$$|\partial_x u(x)| \leqq C^{(1)} M \operatorname{dist}(D_0, C)^{-1} + C^{(2)}$$

for any $x \in \bar{D}_0$ and any $u(x) \in \mathfrak{F}$, where C is the boundary of D, and $C^{(1)}$, $C^{(2)}$ are positive constants depending only on m, L, A, B, M, d and Γ.

The above estimation of $\partial_x u(x)$ shows the uniform boundedness of $\mathfrak{F}'$ in $\bar{D}_0$, and hence $\mathfrak{F}$ is normal in D, because D_0 can be any domain such that $\bar{D}_0 \subset D$.

Put $f_{[u]}(x) \equiv f(x, u(x), \partial_x u(x))$ for any $u(x) \in \mathfrak{F}$, then the functions $f_{[u]}(x)$ are uniformly bounded in $\bar{D}_0$, and hence in the same way as in the proof of Theorem 6.2, it is easily proved that $\mathfrak{F}'$ is equi-continuous in D_0. Therefore $\mathfrak{F}'$ is seen to be normal in D for the same reason as in the case of $\mathfrak{F}$, q.e.d.

Here we repeat the definitions of a quasi-superfunction $\bar{\omega}(x)$ and a quasi-subfunction $\underline{\omega}(x)$ of the equation (9.1) which were given in § 4.

A function $\bar{\omega}(x)$ ($\underline{\omega}(x)$) is called a quasi-superfunction (a quasi-subfunction) of the equation (9.1) in a domain D, if this function has the following properties.

For any point $x_0 \in D$, there exist a neighborhood U of the points x_0 and a finit number of functions $\bar{\omega}_\nu(x) \in C^1[U]$ ($\underline{\omega}_\nu(x) \in C^1[U]$), $\nu = 1, 2, \cdots, n$, such that

$$\text{(10.4)} \qquad \bar{\omega}(x) = \operatorname*{Min}_{1 \leqq \nu \leqq n} \bar{\omega}_\nu(x) \quad \text{in} \quad U,$$
$$(\underline{\omega}(x) = \operatorname*{Max}_{1 \leqq \nu \leqq n} \underline{\omega}_\nu(x) \quad \text{in} \quad U),$$

and

$$\text{(10.5)} \qquad \underline{\mathfrak{L}}(\bar{\omega}_\nu(x)) \leqq f(x, \bar{\omega}_\nu(x), \partial_x \bar{\omega}_\nu(x)) \quad \text{in} \quad U,$$
$$(\bar{\mathfrak{L}}(\underline{\omega}_\nu(x)) \leqq f(x, \underline{\omega}_\nu(x), \partial_x \underline{\omega}_\nu(x)) \quad \text{in} \quad U).$$

Theorem 10.1. *Let* D *be a bounded domain of* B_h*-class and let* D_0 *be a bounded domain containing* D. *Let* $\underline{\omega}(x)$ *and* $\bar{\omega}(x)$ *be respectively a quasi-superfunction of the equation* (9.1) *in* D_0, *and assume that the following inequalities are satisfied*:

$$|\underline{\omega}(x)| \leqq M, \quad |\bar{\omega}(x)| \leqq M, \quad \underline{\omega}(x) < \bar{\omega}(x) \quad in \quad D.$$

Suppose that the function $f(x, u, p)$ *is continuous in the domain*

$$\{(x, u, p)\,;\ x \in D_0, \underline{\omega}(x) \leqq u \leqq \bar{\omega}(x), |p| < +\infty\},$$

and satisfies the condition (10.1) *in* $\bar{\mathfrak{D}}_2$ *and*

$$4\gamma_m ABM<1,$$

where $\gamma_m=2\Gamma((m+2)/2)/\Gamma(1/2)\Gamma((m+1)/2)$.

Now let $\beta(x)$ *be a function belonging to* $C^3[\bar{D}]$ *such that* $\underline{\omega}(x)\leqq\beta(x)\leqq\bar{\omega}(x)$ *in* D. *Then there exists a solution* $u(x)$ *of the equation* (9.1) *in* D, *which attains the values* $\beta(x)$ *on the boundary* C *of* D *and satisfies*

$$\underline{\omega}(x)\leqq u(x)\leqq\bar{\omega}(x)\quad in\quad D.$$

PROOF. Without loss of generality, we can assume $\beta(x)\equiv 0$.

It follows from the assumption, that there exists a finite number of closed sets $\{U_j\}_{i=1}^n$ such that $\bigcup_{i=1}^n U_j\supset\bar{D}$, and the relations (10.4) and (10.5) hold for every U_j.

Let $u(x)$ be a solution of the equation (9.1) such that $\underline{\omega}(x)\leqq u(x)\leqq\bar{\omega}(x)$ in D. Then we obtain

$$|\partial_x u(x)|\leqq C^{\#}\quad \text{in}\quad D,$$

where $C^{\#}$ is the constant given in Lemma 10.2.

Let $C^{\#}_{(\Gamma+1)}$ be the constant given in Lemma 10.2, provided that Γ is replaced by $\Gamma+1$, and put

$$N=\text{Max}\{\max_{x\in U_j,\nu}|\partial_x\bar{\omega}_\nu(x)|,\ \max_{x\in U_j,\nu}|\partial_x\underline{\omega}_\nu(x)|, C^{\#}_{(\Gamma+1)}\}.$$

Define a function $f_1(x,u,p)$ by

$$f_1(x,u,p)=\begin{cases}f(x,u,p) & \text{for}\quad |p|\leqq N,\\ f(x,u,N|p|^{-1}p) & \text{for}\quad |p|>N,\end{cases}$$

and define a function $f_2(x,u,p)$ by

$$f_2(x,u,p)=\begin{cases}f_1(x,\bar{\omega}(x),p)+\dfrac{u-\bar{\omega}(x)}{1+u-\bar{\omega}(x)} & \text{for}\quad u>\bar{\omega}(x),\\ f_1(x,u,p) & \text{for}\quad \underline{\omega}(x)\leqq u\leqq\bar{\omega}(x),\\ f_1(x,\underline{\omega}(x),p)+\dfrac{u-\underline{\omega}(x)}{1+\underline{\omega}(x)-u} & \text{for}\quad u<\underline{\omega}(x).\end{cases}$$

Since $f_2(x,u,p)$ is continuous and bounded in the domain

$$\bar{\mathfrak{D}}_1=\{(x,u,p);\ x\in\bar{D}, |u|<+\infty, |p|<+\infty\},$$

it follows from Theorem 8.1, that there exists a solution $u(x)$ of the equation

$$\mathfrak{L}(u)=f_2(x,u,\partial_x u),$$

which vanishes on the boundary C of D.

We can prove that $u(x)$ is a solution of the equation (9.1), by showing

$$(10.7)\qquad \underline{\omega}(x)\leqq u(x)\leqq\bar{\omega}(x)\quad \text{in}\quad D,$$

$$(10.8)\qquad |\partial_x u(x)|\leqq C^{\#}_{(\Gamma+1)}\leqq N\quad \text{in}\quad D.$$

We first prove the inequality

$$u(x)\leqq\bar{\omega}(x)\qquad \text{in}\quad D.$$

If the above inequality did not hold, there would exist a point $x_0 \in D$ such that

$$\bar{\omega}(x_0)-u(x_0)=\operatorname*{Inf}_{D}\{\bar{\omega}(x)-u(x)\}<0,$$

which follows from the fact that $u(x)=0\leqq\bar{\omega}(x)$ on C.

Put

$$\Phi(u(x))\equiv\mathfrak{L}(u(x))-f_2(x,\bar{\omega}(x),\partial_x u(x)).$$

Then for the point x_0 we have

$$\begin{aligned}\Phi(u(x_0))&=\mathfrak{L}(u(x_0))-f_2(x_0,\bar{\omega}(x_0),\partial_x u(x_0))\\&=f_2(x_0,u(x_0),\partial_x u(x_0))-f_2(x_0,\bar{\omega}(x_0),\partial_x u(x_0))\\&=\frac{u(x_0)-\bar{\omega}(x_0)}{1+u(x_0)-\bar{\omega}(x_0)}>0.\end{aligned}$$

On the other hand, there exist, by the definition of $\bar{\omega}(x)$, a natural number ν_0 and a neighborhood U of the point x_0, such that $\bar{\omega}(x_0)=\bar{\omega}_{\nu_0}(x_0)$, and $\bar{\omega}(x)\leqq\bar{\omega}_{\nu_0}(x)$ in U.

Therefore we have

$$\operatorname*{Inf}_{U}\{\bar{\omega}_{\nu_0}(x)-u(x)\}=\operatorname*{Inf}_{D}\{\bar{\omega}(x)-u(x)\}=\bar{\omega}(x_0)-u(x_0)<0.$$

which implies that the minimum of the function $\bar{\omega}_{\nu_0}(x)-u(x)$ in U is attained at the point x_0. Hence we get

$$\partial_x\bar{\omega}_{\nu_0}(x_0)=\partial_x u(x_0) \tag{10.9}$$

and

$$\begin{aligned}0&\leqq\underline{\mathfrak{L}}(\bar{\omega}_{\nu_0}(x_0)-u(x_0))=\underline{\mathfrak{L}}(\bar{\omega}_{\nu_0}(x_0))-\mathfrak{L}(u(x_0))\\&\leqq f(x_0,\bar{\omega}_{\nu_0}(x_0),\partial_x\bar{\omega}_{\nu_0}(x_0))-\mathfrak{L}(u(x_0)).\end{aligned} \tag{10.10}$$

If follows from (10.9) and (10.10), that

$$\begin{aligned}0&\geqq\mathfrak{L}(u(x_0))-f(x_0,\bar{\omega}_{\nu_0}(x_0),\partial_x\bar{\omega}_{\nu_0}(x_0))\\&=\mathfrak{L}(u(x_0))-f_2(x_0,\bar{\omega}_{\nu_0}(x_0),\partial_x\bar{\omega}_{\nu_0}(x_0))\\&=\mathfrak{L}(u(x_0))-f_2(x_0,\bar{\omega}_{\nu_0}(x_0),\partial_x u(x_0))\equiv\Phi(u(x_0)),\end{aligned}$$

which contradicts the inequality $\Phi(u(x_0))>0$.

We therefore obtain the inequality

$$u(x)\leqq\bar{\omega}(x)\qquad\text{in}\quad D,$$

and similarly $u(x)\geqq\underline{\omega}(x)$ in D. Thus we have (10.7).

Now we see that the function $f_2(x,u,p)$ satisfies the condition

$$|f_2(x,u,p)|\leqq B\,|p|^2+\Gamma+1,$$

and furthermore $|u(x)|\leqq M$, $4\gamma_m ABM<1$. Therefore the inequality (10.8) follows from Lemma 10.2, q.e.d.

§ 11. Existence theorem for the general case.

Let D be a bounded domain and let $\{D_n\}$ be a sequence of domains of B_h-class, which converges to D in such a manner that $\bar{D}_n\subset D_{n+1}$ and $\bigcup_{n=1}^{\infty}D_n=D$.

It is well known that we can find a sequence $\{D_n\}$ of domains having the above-mentioned properties for any bounded domain D.

THEOREM 11.1. *Suppose that $f(x,u,p)$ is continuous in the domain*

$$\mathfrak{D}_2=\{(x,u,p)\,;\ x\in D, \underline{\omega}(x)\leqq \bar{\omega}(x), |p|<+\infty\},$$

and satisfies the condition

$$|f(x,u,p)|\leqq B\,|p|^2+\Gamma_n$$

in the domains

$$\mathfrak{D}^{(n)}=\{(x,u,p)\,;\ x\in \bar{D}_n,\quad \underline{\omega}(x)\leqq u\leqq \bar{\omega}(x),\quad |p|<+\infty\}\qquad (n=1,2,\cdots),$$

where $\underline{\omega}(x)$, $\bar{\omega}(x)$ are continuous and bounded functions defined in D: $|\underline{\omega}(x)|$, $|\bar{\omega}(x)|\leqq M$, and B, Γ_n are positive constants such that $4\gamma_m ABM<1$, $\gamma_m=2\Gamma((m+2)/2)/\Gamma(1/2)\Gamma((m+1)/2)$.

Let $\{\bar{\omega}_\alpha(x)\}$ and $\{\underline{\omega}_\alpha(x)\}$ $(\alpha\in\Omega)$ be systems of quasi-superfunctions and quasi-subfunctions of the equation (9.1) *respectively, which satisfy the following inequality*:

$$\underline{\omega}(x)\leqq \underline{\omega}_{\alpha'}(x)<\bar{\omega}_\alpha(x)\leqq\bar{\omega}(x)\quad \text{in}\quad D,\ (\alpha,\alpha'\in\Omega).$$

Then there exists a solution $u(x)$ of the equation (9.1), *such that*

$$\operatorname*{Sup}_{\alpha\in\Omega}\underline{\omega}_\alpha(x)\leqq u(x)\leqq \operatorname*{Inf}_{\alpha\in\Omega}\bar{\omega}_\alpha(x)\quad \text{in}\quad D.$$

PROOF. Fix an index $\alpha(\in\Omega)$ and let $\beta_n(x)$ be a function belonging to $C^3[\bar{D}_n]$ such that $\underline{\omega}_\alpha(x)\leqq\beta_n(x)\leqq\bar{\omega}_\alpha(x)$ in D_n. Then, by virtue of Theorem 10.1, there exists a solution of the equation (9.1), which assumes the values $\beta_n(x)$ on the boundary of D_n, and satisfies the inequality

$$\underline{\omega}_\alpha(x)\leqq u_n(x)\leqq\bar{\omega}_\alpha(x)\quad \text{in}\quad D_n.$$

Consider a closed domain S such that $S\subset D$, and then we see $S\subset D_n$ for sufficiently large n, and hence it follows from Lemma 10.3 in § 10, that the sequences $\{u_n(x)\}$ and $\{\partial_x u_n(x)\}$ are normal in S.

Now, Since S may be an arbitrary closed domain such that $S\subset D$, we can choose a sequence $n(\nu)$ of natural numbers, for which the sequence $\{u_{n(\nu)}(x)\}$ and $\{\partial_x u_{n(\nu)}(x)\}$ converge uniformly in D.

Thus, if we put

$$\lim_{\nu\to\infty}u_{n(\nu)}(x)=u(x)\quad\text{and}\quad \lim_{\nu\to\infty}\partial_x u_{n(\nu)}(x)=\partial_x u(x),$$

then the sequence $\{f(x,u_{n(\nu)}(x),\partial_x u_{n(\nu)}(x))\}$ also converges to $f(x,u(x),\partial_x u(x))$ uniformly in D. Therefore, by Theorem 3.2 in § 3, $u(x)$ is a solution of the equation (9.1) and satisfies the inequality

$$\underline{\omega}_\alpha(x)\leqq u(x)\leqq\bar{\omega}_\alpha(x)\quad \text{in}\quad D.$$

Denote by $\mathfrak{F}_\alpha$ the family of all solutions $u(x)$ of the equation (9.1), satisfying the following inequality:

$$\underline{\omega}_\alpha(x)\leqq u(x)\leqq\bar{\omega}_\alpha(x)\quad \text{in}\quad D,$$

and introduce a topology into $C^1[D]$ by means of semi-norms $||u||_n$:

$$\|u\|_n = \underset{\bar{D}_n}{\mathrm{Max}} |u(x)| + \underset{\bar{D}_n}{\mathrm{Max}} |\partial_x u(x)| \qquad \text{for} \quad u(x) \in C^1[D].$$

Then, it follows from Lemma 10.3 in § 10, that $\mathfrak{F}_\alpha$ is a compact set in the linear topological space $C^1[D]$.

Now let $\{\alpha_i\}_{i=1}^n$ be a finite number of indices $\alpha_i \in \Omega$, and put

$$\bar{\omega}_*(x) = \underset{1\leqq i\leqq n}{\mathrm{Min}} \{\bar{\omega}_{\alpha_i}(x)\} \quad \text{and} \quad \underline{\omega}_*(x) = \underset{1\leqq i\leqq n}{\mathrm{Max}} \{\underline{\omega}_{\alpha_i}(x)\}.$$

Then it is easily seen that $\bar{\omega}_*(x)$ and $\underline{\omega}_*(x)$ are a quasi-superfunction and a quasi-subfunction of the equation (9.1) respectively, which satisfy the following inequality:

$$\underline{\omega}(x) \leqq \underline{\omega}_*(x) < \bar{\omega}_*(x) \leqq \bar{\omega}(x) \qquad \text{in} \quad D.$$

Furthermore, the intersection $\bigcap_{i=1}^n \mathfrak{F}_{\alpha_i}$ of the families $\{\mathfrak{F}_{\alpha_i}\}_{i=1}^n$ is the family of all solutions $u(x)$ of the equation (9.1) satisfying the following inequality

$$\underline{\omega}_*(x) \leqq u(x) \leqq \bar{\omega}_*(x) \qquad \text{in} \quad D,$$

and this family $\bigcap_{i=1}^n \mathfrak{F}_{\alpha_i}$ is non-void by virtue of the first part of the proof.

Therefore, it follows from the intersection property of compact sets, that the intersection $\bigcap_{\alpha\in\Omega} \mathfrak{F}_\alpha$ is non-void, q. e. d.

THEOREM 11.2. *Let D be a bounded domain satisfying Poincaré's condition*[26]*, and let $a_{ij}(x)$ fulfil the conditions* (9.2) *and* (9.3).

Let $\bar{\omega}(x)$ and $\underline{\omega}(x)$ be a quasi-superfunction and a quasi-subfunction of the equation (9.1) *respectively, such that $|\bar{\omega}(x)|$, $|\underline{\omega}(x) \leqq M$ and $\underline{\omega}(x) < \bar{\omega}(x)$ in D.*

Suppose that the function $f(x, u, p)$ is continuous in the domain:

$$\mathfrak{D}_2 = \{(x, u, p);\ x \in \bar{D}, \underline{\omega}(x) \leqq u \leqq \bar{\omega}(x), |p| < +\infty\},$$

and satisfies the following condition

$$(11.1) \qquad |f(x, u, p)| \leqq B|p|^2 + \Gamma \qquad \text{in} \quad \mathfrak{D}_2,$$

where B and Γ are positive constants such that $4r_m ABM < 1$, $\gamma_m = 2\Gamma((m+2)/2)/\Gamma(1/2)\Gamma((m+1)/2)$.

Let $\beta(x)$ be a continuous function defined in $\bar{D}$ such that $\underline{\omega}(x) < \beta(x) < \bar{\omega}(x)$ in D. Then there exists a solution $u(x)$ of the equation (9.1), *which assumes the boundary values $\beta(x)$ on C, and satisfies the inequality*

$$(11.2) \qquad \underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x) \qquad \text{in} \quad D.$$

This theorem can be proved in the same way as M. Nagumo's theorem[27].

DEFINITION 11.1. *We say that the function $f(x, u, p)$ is H_α-continuous $(0 < \alpha \leqq 1)$ in the finite part of the domain $\mathfrak{D}_1$, if the following condition is*

26) We say that a bounded domain D satisfies Poincaré's condition, if every boundary point of D can be touched from the outside by the vertex of a cone.

27) [4], Theorem 6, p. 225.

satisfied.

For any positive numbers M, N, there exists a positive constant $H_{M,N}$ such that an inequality

$$|f(x,u,p)-f(x,',u',p')|\leqq H_{M,N}\{|x-x'|^{\alpha}+|u-u'|^{\alpha}+|p-p'|^{\alpha}\}$$

holds for $x, x'\in D$, $|u|, |u'|\leqq M$ *and* $|p|, |p'|\leqq N$.

Then we can prove the following:

THEOREM 11.3. *Let the function $f(x,u,p)$ be H_{α}-continuous in the finite part of the domain $\mathfrak{D}_1$ and let $u(x)$ be a generalized solution of the equation* (9.1) *in D. Then $u(x)$ belongs to $C^2[D]$, that is, $u(x)$ is a solution of the equation*

$$L(u)\equiv\sum_{i,j=1}^{m}a_{ij}(x)\partial_i\partial_j u=f(x,u,\partial_x u).$$

PROOF. Let x_0 be any point in D. Then we have only to show that $u(x)$ has the continuous derivatives of the second order in a neighbourhood of x_0.

Let S be an open sphere with the center x_0 and a sufficiently small radius. Then, along the same lines as in the proof of Theorem 6.2, we can easily prove that $u(x)$ and $\partial_x u(x)$ are H_{λ}-continuous in $\bar{S}$ for a positive number $\lambda<1$, and hence we have

$$|u(x)-u(x')|\leqq K|x-x'|^{\lambda},\quad |\partial_x u(x)-\partial_x u(x')|\leqq K|x-x'|^{\lambda}$$

for $x, x'\in\bar{S}$, where K is a positive constant.

On the other hand, since there exist positive numbers M, N such that $|u(x)|\leqq M$ and $|\partial_x u(x)|\leqq N$ for $x\in\bar{S}$, we see

$$|f_{[u]}(x)-f_{[u]}(x')|$$
$$\leqq H_{M,N}\{|x-x'|^{\alpha}+|u(x)-u(x')|^{\alpha}+|\partial_x u(x)-\partial_x u(x')|^{\alpha}\}$$

where $f_{[u]}(x)=f(x,u(x),\partial_x u(x))$.

Therefore, by assuming $|x-x'|<1$ for $x, x'\in\bar{S}$, we obtain

$$|f_{[u]}(x)-f_{[u]}(x')|\leqq H_{M,N}\{1+2K^{\alpha}\}|x-x'|^{\lambda\alpha}\qquad\text{for}\quad x, x'\in\bar{S},$$

and hence $f_{[u]}(x)$ is $H_{\lambda\alpha}$-continuous in $\bar{S}$.

Consider an equation

$$L(v)=f_{[u]}(x), \tag{11.3}$$

then, by Theorem B′ of § 6 we see that there exists a solution $v(x)$ of the equation (11.3), which attains values $u(x)$ on the boundary $\dot{S}$ of S.

The solutions $u(x)$ and $v(x)$ may be regarded as solutions of the equation

$$\mathfrak{L}(v)=f_{[u]}(x), \tag{11.4}$$

and these solutions assume the same boundary values on $\dot{S}$. Hence the identity $u(x)\equiv v(x)$ holds in $\bar{S}$, and $u(x)$ belongs to $C^2[S]$, q. e. d.

Chapter V. The maximal solution, the minimal solution, and Harnack's theorems

In this chapter, we assume also that the coefficients $a_{ij}(x)$ satisfy the conditions (9.2) and (9.3) in $\bar{D}$.

§ 12. The maximal solution and the minimal solution.

Consider the equation

$$\mathfrak{L}(u)=f(x, u, \partial_x u), \tag{12.1}$$

and suppose that there exist solutions of the equation (12.1), which attain the prescribed boundary values $\varphi(x)$ on the boundary C of D.

A solution $\omega(x)$ is named *the maximal solution of the equation* (12.1) *for the prescribed boundary values* $\varphi(x)$ (or *the maximal solution of the equation* (12.1) *attaining the prescribed boundary values* $\varphi(x)$ *on* C), if the following condition is satisfied:

Let $u(x)$ be any solution of the equation (12.1), such that

$$u(x)=\omega(x)=\varphi(x) \qquad \text{on} \quad C.$$

Then we have the inequality $u(x) \leqq \omega(x)$ in D.

The minimal solution of the equation (12.1) *for the prescribed values* $\varphi(x)$, can be defined in the analoguous way.

In the following theorem, imposing certain conditions on the function $f(x, u, p)$, we prove the existence of the maximal solution and the minimal solution of the equation (12.1).

THEOREM 12.1. *Let D be a bounded domain of B_h-class, and let $f(x, u, p)$ be continuous in the domain*

$$\mathfrak{D}_1=\{(x, u, p)\,;\, x\in D, |u|<+\infty, |p|<+\infty\}.$$

Suppose further that $f(x, u, p)$ is bounded in the wider sense with respect to u in $\mathfrak{D}_1$[28] *and $f(x, u, p) \leqq f(x, \bar{u}, p)$ for $u<\bar{u}$.*

Then there exist the maximal solution and the minimal solution of the equation (12.1) *vanishing on C.*

PROOF. By virtue of Theorem 8.2, we see that the equation (12.1) has a solution $u(x)$ vanishing on C.

Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers: $\varepsilon_n \downarrow 0$, and consider the equations

$$\mathfrak{L}(u)=f(x, u, \partial_x u)-\varepsilon_n. \tag{12.2}$$

Then it follows from Theorem 8.2 and Theorem 4.3, that the equations (12.2) have solutions $u_n(x)$ vanishing on C and we obtain

28) For the definition of boundedness in the wider sense with respect to u, see the footnote 21).

(12.3) $$u_n(x) \geqq u_{n+1}(x) \geqq u(x) \qquad (x \in \bar{D}).$$

The sequence $\{u_n(x)\}$ is therefore uniformly bounded in $\bar{D}$ and hence the hypothesis of the theorem implies that there exists a positive constant Γ, such that

$$|f(x, u_n(x), \partial_x u_n(x)) - \varepsilon_n| \leqq \Gamma, \qquad (n=1, 2, \cdots).$$

Consequently we have

$$|u_n(x)| \leqq \Gamma \Psi(x) \qquad (n=1, 2, \cdots),$$

where $\Psi(x)$ is the solution of the equation $L(v) = -1$, vanishing on C.

Thus, along the same lines as in the proof of Theorem 6.2, it is easily shown that the sequences $\{u_n(x)\}$ and $\{\partial_x u_n(x)\}$ are normal in D, and therefore we can assume, without loss of generality, that the sequences $\{u_n(x)\}$ and $\{\partial_x u_n(x)\}$ converge uniformly in D.

Thus we have

$$\lim_{n\to\infty} u_n(x) = \omega(x) \quad \text{and} \quad \lim_{n\to\infty} \partial_x u_n(x) = \partial_x \omega(x),$$

and

$$|\omega(x)| \leqq \Gamma \Psi(x).$$

Hence, it follows from Theorem 3.2, that $\omega(x)$ is a solution of the equation (12.1), which vanishes on C.

Since the inequality (12.3) implies $\omega(x) \geqq u(x)$ in D, $\omega(x)$ is the maximal solution of the equation (12.1) vanishing on C.

Concerning the minimal solution we can prove the theorem in the same way as above.

§ 13. Harnack's theorem.

Here we generalize Harnack's first theorem for harmonic functions to solutions of the equation (12.1).

In this paragraph, D is a bounded domain without any more restrictions, and we suppose that the function $f(x, u, p)$ is continuous in $\mathfrak{D}_1$.

THEOREM 13.1. *Suppose that the function $f(x, u, p)$ fulfils the condition*

(13.1) $$f(x, u, p) < f(x, \bar{u}, p)$$

for $(x, u, p), (x, \bar{u}, p) \in \mathfrak{D}_1$ and $u < \bar{u}$.

Let $\{u_n(x)\}$ be a sequence of solutions of the eqvation (12.1), *which are continuous in $\bar{D}$.*

If the sequence $\{u_n(x)\}$ converges uniformly on the boundary C of D, then the sequence $\{u_n(x)\}$ converges to a continuous function $u(x)$ uniformly in $\bar{D}$.

PROOF. Put $v(x) = u_{n+k}(x) - u_n(x)$ and consider the equation with respect to v:

$$\mathfrak{L}(v) = g(x, v, \partial_x v),$$

where

$$g(x, v, \partial_x v) \equiv f(x, v+u_n(x), \partial_x v+\partial_x u_n(x)) - f(x, u_n(x), \partial_x u_n(x)).$$

The condition (13.1) implies

$$g(x, v, 0)\begin{cases} <0 & (v<0), \\ >0 & (v>0), \end{cases}$$

and it follows from the uniform convergence of $\{u_n(x)\}$ on C, that for a given $\varepsilon>0$, there exists a natural number N, such that

$$-\varepsilon \leqq u_{n+k}(x)-u_n(x) \leqq +\varepsilon$$

for $n>N$, $k=1,2,\cdots$, and $x \in C$.

Therefore, by Theorem 4.1 and the theorem analoguous to Theorem 4.1, we obtain

$$|u_{n+k}(x)-u_n(x)| \leqq \varepsilon$$

for $n>N$, $k=1,2,\cdots$, and $x \in \bar{D}$, which shows that $\{u_n(x)\}$ converges to a continuous function $u(x)$ uniformly in $\bar{D}$, q. e. d.

It follows from the condition (9.2) that there exists a positive number α, such that $0<\alpha<a_{ii}(x)$ in D, and hence, by means of the transformation used in the proof of Theorem 5.1, we can prove the following theorem, which is stated without proof.

THEOREM 13.2. *Suppose that the function $f(x,u,p)$ fulfils one of the following conditions:*

(I) $$f(x,u,p) \leqq f(x,\bar{u},p) \quad \text{for} \quad u<\bar{u},$$

and

$$-K(\bar{p}_i-p_i) \leqq f(x,u,\bar{p})-f(x,u,p),$$
$$(K:\ \text{a positive constant});$$

(II) $$f(x,u,p) \leqq f(x,\bar{u},p) \quad \text{for} \quad u<\bar{u},$$

and

$$f(x,u,\bar{p})-f(x,u,p) \leqq K(\bar{p}_i-p_i),$$
$$(K:\ \text{a positive constant}),$$

where $p=(p_1,\cdots,p_i,\cdots,p_m)$, $\bar{p}=(p_1,\cdots,\bar{p}_i,\cdots,p_m)$, $p_i<\bar{p}_i$ and i is a fixed index.

Let $\{u_n(x)\}$ be a sequence of solutions of the equation (12.1) *which are continuous in $\bar{D}$, and suppose that the sequence $\{u_n(x)\}$ converges uniformly on the boundary C of D. Then the sequence $\{u_n(x)\}$ converges to a continuous function $u(x)$ uniformly in $\bar{D}$.*

Regarding the convergence of the sequence $\{\partial_x u_n(x)\}$, we can prove the following:

THEOREM 13.3. *Suppose that the limit function $u(x)$ of the sequence $\{u_n(x)\}$ in Theorems* 13.1 *and* 13.2, *satisfies the inequality $|u(x)|<M$. Furthermore, let the function $f(x,u,p)$ fulfil the condition*

(13.2) $$|f(x,u,p)| \leqq B|p|^2+\Gamma_M$$

in the domain

$$\mathfrak{D}_M=\{(x, u, p)\,;\ x\in D, |u|\leqq M, |p|<+\infty\},$$

where Γ_M is a positive constant depending only on M, and B is a positive constant such that $4\gamma_m ABM<1$, $\gamma_m=2\Gamma((m+2)/2)/\Gamma(1/2)\Gamma((m+1)/2)$.

Then the sequence $\{\partial_x u_n(x)\}$ converges to $\partial_x u(x)$ uniformly in D, and $u(x)$ is a solution of the equation (12.1).

PROOF. Since the sequence $\{u_n(x)\}$ converges to $u(x)$ uniformly in $\bar{D}$, there exists a natural number N such that $|u_n(x)|\leqq M$ for $n>N$.

Therefore we have by Theorem 7.1,

$$|\partial_x u_n(x)|\leqq C^{(1)}\rho(x)^{-1}\operatorname*{Max}_{|x-x'|\leqq\rho(x)}|u_n(x')|+C^{(2)}\leqq C^{(1)}M\rho(x)^{-1}+C^{(2)},$$

for $n>N$, where $C^{(1)}$, $C^{(2)}$ are positive constants depending only on m, A, L, B, M, d and Γ_M.

Let D_0 be any bounded domain such that $\bar{D}_0\subset D$. Then the above estimation of the derivatives $\partial_x u_n(x)$ implies that $\{\partial_x u_n(x)\}$ is uniformly bounded in $\bar{D}_0$, and hence the sequence of functions $\{f(x, u_n(x), \partial_x u_n(x))\}$ is uniformly bounded in $\bar{D}_0$. Consequently we see in the way analoguous to the proof of Theorem 6.2, that $\{\partial_x u_n(x)\}$ is normal in D_0.

Furthermore $\{\partial_x u_n(x)\}$ is normal in D, for D_0 is arbitrary provided that D_0 is bounded and $\bar{D}_0\subset D$.

By the above conclusion and the uniform convergence of $\{u_n(x)\}$ in $\bar{D}$, it is easily seen that $\{\partial_x u_n(x)\}$ converges to $\partial_x u(x)$ uniformly in D.

Thus, since the sequence of functions $\{f(x, u_n(x), \partial_x u_n(x))\}$ converges to $f(x, u(x), \partial_x u(x))$ uniformly in D, it follows from Theorem 3.2 that $u(x)$ is a solution of the equation (12.1).

COROLLARY. *If, in Theorems* 13.1 *and* 13.2, *the function $f(x, u, p)$ satisfies the condition*

(13.3) $$|f(x, u, p)|\leqq B_M|p|^{\alpha}+\Gamma_M \qquad (0\leqq\alpha<2)$$

in any domain

$$\mathfrak{D}_M=\{(x, u, p)\,;\ x\in D, |u|\leqq M, |p|<+\infty\},$$

where M is any positive number, and B_M, Γ_M are positive constants depending only on M.

Then the sequence $\{\partial_x u_n(x)\}$ converges to $\partial_x u(x)$ uniformly in D, and $u(x)$ is a solution of the equation (12.1).

PROOF. By the uniform convergence of the sequence $\{u_n(x)\}$ on $\bar{D}$, we see that there exists a positive number M' such that $|u_n(x)|\leqq M'$ for $n=1, 2, \cdots$, and further the condition (13.3) implies that the function $f(x, u, p)$ fulfils the condition

$$|f(x, u, p)|\leqq B|p|^2+\Gamma_{M'}$$

in the domain $\mathfrak{D}_{M'}$, where $\Gamma_{M'}$ is a positive constant depending only on M' and

B is a positive constant such that $4\gamma_m ABM' < 1$, $\gamma_m = 2\Gamma((m+2)/2)/\Gamma(1/2)$ $\Gamma((m+1)/2)$.

Therefore this corollary derives from Theorem 13.3.

REMARK. The condition (13.3) for $\alpha=1$ is fulfilled, when the equation (12.1) is linear, that is when

$$f(x, u, \partial_x u) = \sum_{i=1}^{m} b_i(x)\partial_i u + c(x)u + f(x)$$

in the equation (12.1), where $b_i(x)$, $c(x)$ and $f(x)$ are continuous functions defined in $\bar{D}$.

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(Ricevita la 28-an de Marto, 1959)

Precipe Liniaj Partaj Diferencialaj Ekvacioj ed Tipo Elipsa

de Yoshikazu HIRASAWA

Resumo. Tiu ĉi artikolo estas koncernata kun unu parta diferenciala ekvacio de l'dua ordo kaj de tipo elipsa

$$L(u) \equiv \sum_{i,j=1}^{m} a_{ij}(x)\partial_i\partial_j u = f(x, u, \partial_x u).$$

Tra la tuta artikolo, ni enkondukas la ĝeneraligitan diferencialan operacion de tipo elipsa, elpensitan de M. Gevrey, intencante la faciligon de studadoj por diversaj subjektoj pri tiela ekvacio kaj la diskuton sub la kondico kiel eble plej malforta. Tiam, montrante la ĝeneraligitan diferencialan operacion per $\mathfrak{L}(u)$, ni traktas pri unu ekvacio

(II) $\mathfrak{L}(u)=f(x, u, \partial_x u).$

En Ĉapitro I, ni diras mallonge pri la elementa solvo rilata al la ĝeneraligita operacio kaj unu formulo de Poisson.

En Ĉapitro II, ni demonstracias unu grupon de teoremoj, kiuj estas nomataj teoremoj de komparo. Krom tio, konsiderante la problemon al la limo de l'unua speco koncernantan la ekvacion (II), ni donas kondiĉojn de unikeco por la solvo de tiu ci problemo.

La du sekvantaj ĉapitroj (Ĉapitroj III kaj IV) estas rilatataj kun la ekzisto teoremo de solvoj por la problemo al la limo de l'unua speco koncernanta la ekvacion (II).

En Ĉapitro V, donante la difiniciojn de l'maksimuma solvo kaj de l'minimuma solvo de l'ekvacio (II), ni pruvas la ekziston de tielaj solvoj sub taŭgaj kondiĉoj. Krom tio, la unua teoremo de Harnack estas plivastigata al solvoj de l'ekvacio (II).

Postskribo

De la de supre 19-a linio de la paĝo 45 ĝis la de malsupre 10-a de le sama pago: Teoremo D estas citata de la traktato de M. Nagumo. Sed post la publiko eraroj estis eltrovitaj en la demonstracio de la teoremo de M. Nagumo kaj li korektis tiun demonstracion (en unu privata komuniko sendita al la aŭtoro). Lau lia korekto ni korektas la konstanton $C^{(1)}_{A,L,d}=\sqrt{2}(m+6)A^3Ld$ jene.

$$C^{(1)}_{A,L,d}=\mathrm{Max}\{4\gamma_m A,\ \sqrt{8}\,\gamma_m k_m A^3 Ld\},$$

kie $\gamma_m=2\Gamma((m+2)/2)/\Gamma(1/2)\Gamma((m+1)/2)$, kaj k_m estas ankaŭ unu konstanto dependa nur de la dimensio m.

Krome ni korektas same la konstanton $C^{(1)}_{A,L,d}$ en la de malsupre 5-a linio de la paĝo 46 kaj la konstanton $C^{(1)}_{A,L,d(n)}$ en la de malsupre 1-a linio de la paĝo 55.

(Ricevita la 13-an de marto, 1961)