# Counterexample to global existence for systems of nonlinear wave equations with different propagation speeds

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## 1 Introduction and main result

We consider the Cauchy problem for systems of semilinear wave equations with different propagation speeds in three space dimensions of the form

$$\Box_{c_i} u_i = F_i(u, \partial_t u), \quad (x,t) \in \mathbb{R}^3 \times [0,\infty),\ i = 1,2, \tag{1.1}$$

$$u_i(x,0) = \varepsilon\varphi_i(x), \quad \partial_t u_i(x,0) = \varepsilon\psi_i(x), \quad x \in \mathbb{R}^3,\ i = 1,2, \tag{1.2}$$

where $\Box_c = \partial_t^2 - c^2\Delta$, $c_1$, $c_2$, $\varepsilon$ are positive constants, $c_1 \neq c_2$, and $u = (u_1, u_2)$ is an $\mathbb{R}^2$-valued unknown function of $(x,t)$. We assume that the nonlinear functions $F_1$ and $F_2$ are quadratic with respect to $(u, \partial_t u)$, and study the small data global existence and blowup for (1.1). Here, we say that the small data global existence holds for (1.1) if for any $\varphi_i$, $\psi_i \in C_0^\infty(\mathbb{R}^3)$ $(i = 1,2)$ there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the Cauchy problem (1.1)–(1.2) admits a unique global classical solution $u \in C^\infty(\mathbb{R}^3 \times [0,\infty), \mathbb{R}^2)$. Moreover, we say that the small data blowup occurs if the small data global existence does not hold. In the present paper, we do not consider the case where the nonlinear terms $F_i$ depend only on $u$ (for that case, see Kubo and Ohta [10]), and we put

$$F_i(u, \partial_t u) = \sum_{j,k=1,2} (A_i^{j,k} u_j \partial_t u_k + B_i^{j,k} \partial_t u_j \partial_t u_k), \tag{1.3}$$

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where $A_i^{j,k}$, $B_i^{j,k} \in \mathbb{R}$, $i = 1, 2$. In what follows, we always assume that

$$A_i^{i,i} = B_i^{i,i} = 0, \quad i = 1, 2, \tag{1.4}$$

because it is proved by F. John [4] that the small data blowup occurs for the single equations $\Box u = u\partial_t u$ and $\Box u = (\partial_t u)^2$ in three space dimensions (see Klainerman [8] and Christodoulou [2] for the small data global existence when $c_1 = c_2$).

For the case $c_1 \neq c_2$ and (1.4), the small data global existence for (1.1) has been studied by many authors (see, e.g., [1, 3, 5, 6, 7, 9, 11, 12, 13]). Yokoyama [13] proved that the small data global existence holds for (1.1) with (1.3) if $c_1 \neq c_2$ and $A_i^{j,k} = B_i^{i,i} = 0$ for $i, j, k = 1, 2$. For the case where both $F_1$ and $F_2$ can be written in the divergent form

$$F_i = \partial_t \left( \sum_{j,k=1,2} D_i^{j,k} u_j u_k \right), \quad D_i^{j,k} \in \mathbb{R}, \ i = 1, 2,$$

it is proved in [5] that the small data global existence holds for (1.1) if $c_1 \neq c_2$ and $D_i^{i,i} = 0$ for $i = 1, 2$. Moreover, Katayama [7] proved that the small data global existence holds for (1.1) with (1.3) if $c_1 \neq c_2$ and $A_i^{j,j} = B_i^{j,j} = 0$ for $i, j = 1, 2$.

However, to our knowledge, no results on the small data blowup have been obtained for (1.1) with (1.3) when $c_1 \neq c_2$ and (1.4). The purpose in the present paper is to show that the condition (1.4) is not sufficient to prove the small data global existence for (1.1) with (1.3) when $c_1 \neq c_2$. More precisely, we consider

$$\begin{cases} \Box_{c_1} u_1 = u_2 \partial_t u_1, & (x,t) \in \mathbb{R}^3 \times [0, \infty), \\ \Box_{c_2} u_2 = (\partial_t u_1)^2, & (x,t) \in \mathbb{R}^3 \times [0, \infty), \\ u_1(x,0) = 0, \ \partial_t u_1(x,0) = \varepsilon \psi_1(|x|), & x \in \mathbb{R}^3, \\ u_2(x,0) = 0, \ \partial_t u_2(x,0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

The main result in the present paper is as follows.

**Theorem 1.1** *Let $0 < c_1 < c_2$, $\varepsilon \in (0, 1]$, $\psi_1(|x|) \in C_0^\infty(\mathbb{R}^3)$, and we assume that there exists a constant $\delta > 0$ such that*

$$\psi_1(r) > 0 \ \text{for} \ r \in [0, \delta), \quad \psi_1(r) = 0 \ \text{for} \ r \in [\delta, \infty). \tag{1.6}$$

*Then, the classical solution $(u_1, u_2)$ of (1.5) blows up in a finite time $T^*(\varepsilon)$. Moreover, there exists a positive constant $C^*$, which is independent of $\varepsilon$, such that*

$$T^*(\varepsilon) \leq \exp(C^* \varepsilon^{-2}), \quad \varepsilon \in (0, 1].$$

In the next section, we will give the proof of Theorem 1.1.