One-dimensional Shape Memory Alloy Problems Including a Hysteresis Operator

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1 Introduction

In this paper we are concerned with the global existence and uniqueness of a solution to a one-dimensional model of thermomechanical evolution of shape memory alloys. First, the following two differential equations are derived from the conservation laws of linear momentum and energy:

$$u_{tt} + \gamma u_{xxxx} = \hat{\sigma}_x \quad \text{in } Q(T) := (0,T) \times (0,1), \tag{1.1}$$

$$U_t + q_x = \hat{\sigma}\varepsilon_t \quad \text{in } Q(T), \tag{1.2}$$

where u denotes the displacement, $\varepsilon := u_x$ is the strain, $\hat{\sigma}$ is the stress, U is the internal energy, q is the heat flux and γ is a positive constant. Here, we refer Brokate-Sprekels [4, Section 5] and Pawlow [10] for the physical background of these laws. Now, we use the classical Fourier law and an elementary approximation $U_t = \theta_t$ where $\theta := \theta(t,x)$ is the temperature field. Therefore, (1.2) can be written by

$$\theta_t - \kappa\theta_{xx} = \hat{\sigma}\varepsilon_t \quad \text{in } Q(T), \tag{1.3}$$

where κ is a positive constant depending on the specific heat and the heat conductivity. By some mathematical reasons we assume that there are interior frictions in the form of viscous stresses in the material. Then we can apply Hooke's-like law so that we have

$$\hat{\sigma} = \sigma + \mu\varepsilon_t, \tag{1.4}$$

where $\mu > 0$ is the constant viscosity. The above composition of the stress was investigated by many mathematicians (cf. [9, 6, 4]). Falk's model [5] is well known as the system describing the dynamics of one-dimensional shape memory alloys. Falk's model is based on the Landau-Devonshire theory. This means that σ is decided by the derivative of the Helmholtz energy $\Psi := \Psi(\theta, \varepsilon)$, that is,

$$\sigma = \frac{\partial \Psi(\theta,\varepsilon)}{\partial \varepsilon}.$$

However, by some experiments we know that the relationship between the stress and the strain is described by the hysteresis loop depending on the temperature. In our previous

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works [1, 2] we have already pointed out that the relationship can be represented by the ordinary differential equations including the subdifferentials of the indicator function of the closed interval as follows:

$$\sigma_t + \partial I(\theta, \varepsilon; \sigma) \ni c\varepsilon_t, \tag{1.5}$$

where c is a positive constant depending on the hysteresis loops and I is the indicator function of the closed interval $[f_a(\theta,\varepsilon), f_d(\theta,\varepsilon)]$ for given continuous functions f_a and f_d on $R \times R$ with $f_a \le f_d$ on $R \times R$, that is,

$$I(\theta,\varepsilon;\sigma) = \begin{cases} 0 & \text{if } f_a(\theta,\varepsilon) \le \sigma \le f_d(\theta,\varepsilon), \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore the following system $\mathrm{P}_0 := \mathrm{P}_0(u_0, v_0, \theta_0, \sigma_0)$ is derived from (1.1) and (1.3) $\sim$ (1.5).

$$u_{tt} + \gamma u_{xxxx} - \mu u_{xxt} - \sigma_x = 0 \quad \text{in } Q(T), \tag{1.6}$$

$$\theta_t - \kappa\theta_{xx} = \sigma u_{xt} + \mu|u_{xt}|^2 \quad \text{in } Q(T), \tag{1.7}$$

$$\sigma_t + \partial I(\theta,\varepsilon;\sigma) \ni c u_{xt} \quad \text{in } Q(T), \tag{1.8}$$

$$u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0 \quad \text{for } 0 < t < T, \tag{1.9}$$

$$\theta_x(t,0) = \theta_x(t,1) = 0 \quad \text{for } 0 < t < T, \tag{1.10}$$

$$\sigma_x(t,0) = \sigma_x(t,1) = 0 \quad \text{for } 0 < t < T, \tag{1.11}$$

$$u(0) = u_0, u_t(0) = v_0, \theta(0) = \theta_0, \sigma(0) = \sigma_0 \quad \text{on } (0,1), \tag{1.12}$$

where u_0, v_0, θ_0 and σ_0 are given initial functions.

Our formulation does not require the monotonicity for f_a and f_d, but needs the boundedness of f_a and f_d on $R \times R$. Moreover, it can cover the special case where

$$f_a(\theta,\varepsilon) = f_d(\theta,\varepsilon) = \frac{\partial\Psi(\theta,\varepsilon)}{\partial\varepsilon},$$

which gives a Falk-type model with σ bounded. The main purpose of this paper is to give the existence and uniqueness theorem for P_0. In [1] we have already proved the wellposedness for P_0 with (1.13) and (1.14) instead of (1.7) and (1.8), respectively.

$$\theta_t - \kappa\theta_{xx} = \sigma u_{xt} \quad \text{in } Q(T), \tag{1.13}$$

$$\sigma_t - \nu\sigma_{xx} + \partial I(\theta,\varepsilon;\sigma) \ni c u_{xt} \quad \text{in } Q(T), \tag{1.14}$$

where $\nu > 0$ is a positive constant. Also, in [2] we studied P_0 with (1.14) instead of (1.8), which is denoted by $\mathrm{P}_\nu = \mathrm{P}_\nu(u_0, v_0, \theta_0, \sigma_0)$ for $\nu > 0$.

In this paper we refer the book [3] and [7] for the theory on maximal monotone operators and subdifferentials of convex functions in a Hilbert space.

2 Main result

Throughout this paper we shall use the following notations. $H := L^2(0,1)$, $V := H_0^1(0,1)$, V^* is the dual space of V, and $< \cdot, \cdot >$ is the duality pair on $V \times V^*$. Also, for given $\theta \in H$