Structure of positive solutions to a semilinear initial value problem

Jong-Shenq Guo*, Je-Chiang Tsai

Department of Mathematics, National Taiwan Normal University
88, S-4 Ting Chou Road, Taipei 117, Taiwan

Abstract. We study an initial value problem for a semilinear ordinary differential equation. This problem is closely related to the blow-up behaviour of a semilinear parabolic problem. Under some restrictions, we characterize the structure of solutions and derive the uniqueness of positive global solution of this initial value problem.

Keywords: initial value problem, semilinear, blow-up behavior, global solution

1 Introduction

In this paper, we are concerned with the positive solutions of the initial value problem (P):

$$w'' - \frac{y}{2}w' - \alpha w + w^p = 0, \quad y > 0, \tag{1.1}$$

$$w(0) > 0, w'(0) = -w^q(0), \tag{1.2}$$

where $w = w(y), q = (p+1)/2$, and

$$\alpha = \frac{1}{p-1}. \tag{1.3}$$

Hereafter the prime denotes the differentiation with respect to y. We always assume that $p > 1$.

The problem (P) is closely related to the following semilinear parabolic problem

$$u_t = u_{xx} + u^p, \quad x \in (0,1), t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, u_x(1,t) = u^q(1,t), \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.6}$$

where $u_0(x)$ is a positive smooth function. We say that the solution u of the problem (1.4)–(1.6) blows up if there is a finite time T such that $\max_{x\in[0,1]} u(x,t) \to \infty$ as $t \uparrow T$. It has been shown by Lin and Wang [3] that the solution $u(x,t)$ of the problem (1.4)-(1.6) always blows up, since $p > 1$. Also, under some conditions (for example, $u_0' \geq 0$), $x = 1$ is the only blow-up point. See [3] and [1].

In order to understand the time asymptotic behaviour of $u(x,t)$ as $t \uparrow T$, we make the following well-known Giga-Kohn transformation [2]

$$y = \frac{1-x}{\sqrt{T-t}}, \quad s = -\ln(T-t),$$

$$w(y,s) = (T-t)^\alpha u(x,t).$$

*Corresponding author. Fax: 886-2-29332342. *Email address:* jsguo@math.ntnu.edu.tw (J.-S. Guo)

Then the function w satisfies

$$w_s = w_{yy} - \frac{y}{2}w_y - \alpha w + w^p, \quad 0 < y < e^{s/2}, \quad s > -\ln T,$$
$$w_y(0,s) = -w^q(0,s), \quad w_y(e^{s/2},s) = 0, \quad s > -\ln T,$$
$$w(y, -\ln T) = T^\alpha u_0(1 - y\sqrt{T}), \quad 0 \le y \le 1/\sqrt{T}.$$

It is nature to expect that, as $s \to \infty$ (or, $t \uparrow T$), $w(y,s)$ tends to a global positive solution of (P). Therefore, the existence and uniqueness of global positive (monotone decreasing) solution of (P) plays an important role in studying the time asymptotic behaviour of $u(x,t)$ as $t \uparrow T$. In fact, if $W(y)$ is the unique global positive monotone decreasing solution of (P), then we have

$$(T-t)^\alpha u(1 - y\sqrt{T-t}, t) \to W(y)$$

as $t \uparrow T$ uniformly for $y \in [0, C]$ for any $C > 0$. See [1] for more detail.

It has been proved in [4] (See also [1]) that there is a global positive monotone decreasing solution of (P) for any $p > 1$. Also, the uniqueness of global positive monotone decreasing solutions of (P) for $p \in (1, 2]$ was proved in [1]. The main purpose of this paper is to show the following theorem on the structure of positive solutions of (P).

Theorem 1.1 *Suppose that $1 < p \le 2$. Let $w(y;\eta)$ be the solution of* (P) *with the initial value $w(0;\eta) = \eta > 0$. Then there exists a unique $\bar\eta > 0$ such that*

(i) if $\eta > \bar\eta$, then $w(y;\eta)$ is decreasing to zero at some finite R;

(ii) if $\eta = \bar\eta$, then $w(y;\eta)$ is a global positive monotone decreasing solution;

(iii) if $\eta < \bar\eta$, then there exist $y_0, y_1, y_2 > 0$ such that $w'(y) < 0$ for $y \in [0, y_0)$, $w'(y) > 0$ for $y \in (y_0, y_1)$, $w'(y) < 0$ for $y \in (y_1, y_2)$, and $w(y_2) = 0$.

We emphasize here that it follows from Theorem 1.1 that any solution of (P) must vanish at some finite R except the solution starting with $\bar\eta$.

This paper is organized as follows. We first recall some facts from [1] in §2 and derive the assertions (i) and (ii) of Theorem 1.1. Then in §3 we prove the assertion (iii) of Theorem 1.1. Hence the only global positive solution of (P) is the monotone decreasing solution $w(y;\bar\eta)$.

2 Preliminary

Let us recall some facts from [1]. Given any $\eta > 0$, there is a unique local solution $w := w(y;\eta)$ of (P) with $w(0;\eta) = \eta$. Let $\rho(y) = \exp(-y^2/4)$ and $f(w) = w^p - \alpha w$. Then w satisfies

$$(\rho w')(y) = -\eta^q - \int_0^y \rho(s) f(w(s))ds. \tag{2.1}$$

Next, we define the energy functional E_w by

$$E_w[y] = \frac{1}{2}[w'(y)]^2 + F(w(y)), \tag{2.2}$$

where $F(w) = \int_\kappa^w f(s)ds$ and κ is the unique positive solution of $f(w) = 0$. Note that

$$E'_w[y] = \frac{1}{2} y [w'(y)]^2$$

which implies that $E_w[y]$ is increasing in y as long as $w(y)$ is defined. Finally, we define

$$I_1 = \{\eta > 0 \mid w(y;\eta) \text{ is decreasing to zero at some finite } R\},$$
$$I_2 = \{\eta > 0 \mid w'(y;\eta) \text{ vanishes before } w(y;\eta) \text{ vanishes }\}.$$